Main Results
C_0 Sum Rule
The Jost Function

Spectral Theory of Orthogonal Polynomials

Barry Simon
IBM Professor of Mathematics and Theoretical Physics
California Institute of Technology
Pasadena, CA, U.S.A.

Lecture 10: Fuchsian Groups and Finite Gaps, II

- Lecture 8: Finite Gap Isospectral Torus
- Lecture 9: Fuchsian Groups and Finite Gaps, I
- Lecture 10: Fuchsian Groups and Finite Gaps, II
- Lecture 11: Selected Additional Topics, I

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Main Results

This lecture will differ from the earlier ones which had more or less complete proofs or proofs an astute listener might be expected to fill in. This lecture will discuss results about finite gap sets, $\mathfrak{e} \subset \mathbb{R}$, namely analogs of the Shohat–Nevai and Szegő asymptotics theorems, and it will discuss some ideas in the proofs but full proofs would require several more lectures. Details can be found in the roughly 50 pages of [SzThm] on the subject or the first two CSZ papers

Main Results

Theorem (Shohat–Nevai type theorem for finite gap). *Let J be a half-line Jacobi matrix with Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ and $\sigma_{\text{ess}}(J) = \mathfrak{e}$. Let $\{E_j\}_{j=1}^\infty$ be a labelling of eigenvalues in $\mathbb{R} \setminus \mathfrak{e}$. Suppose*

$$\sum_j \operatorname{dist}(E_j, \mathfrak{e})^{1/2} < \infty$$

Then, with $d\mu = f(x)\,dx + d\mu_s$, we have that

$$\overline{\lim_{n\to\infty}}\left[\left(\prod_{j=1}^n a_j\right)/C(\mathfrak{e})^n\right] > 0 \Leftrightarrow \int_{\mathfrak{e}} \operatorname{dist}(x, \mathbb{R}\setminus\mathfrak{e})^{-1/2} \log(f(x))\,dx > -\infty$$

and if those conditions hold, then

$$0 < \liminf \frac{\prod_{j=1}^n a_j}{C(\mathfrak{e})^n} \le \limsup \frac{\prod_{j=1}^n a_j}{C(\mathfrak{e})^n} < \infty$$

Main Results

Remarks. 1. Since a Szegő condition implies μ is regular for $\mathfrak{e}$, we expect $\lim(\prod_{j=1}^n a_j)^{1/n} = C(\mathfrak{e})$, so the division by $C(\mathfrak{e})^n$ is needed for the product to possibly be bounded above and away from 0.

2. At first the surprise might be that $\prod_{j=1}^n a_j/C(\mathfrak{e})^n$ has no limit but elements of the isospectral torus obey the Blaschke and Szegő conditions and for them $a_j/C(\mathfrak{e})$ does not go to 1.

3. In fact, $\lim_{n\to\infty} \prod_{j=1}^n a_j/C(\mathfrak{e})^n$ is asymptotically almost periodic.

Main Results

For $\mathfrak{e} = [-2, 2]$, $a_j \to 1$, $b_j \to 0$, if Blaschke and Szegő conditions hold. We can't expect approach to constants if $\ell > 1$ as seen by the isospectral torus. Rather $\{a_n, b_n\}$ approach a moving target!

Theorem. *Let J be a Jacobi matrix with Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ and $\sigma_{\mathrm{ess}}(J) = \mathfrak{e}$. Suppose J obeys the three conditions of the Shohat–Nevai type theorem. Then, there exists $\{a_n^\sharp, b_n^\sharp\}_{n=-\infty}^{\infty}$, a two-sided Jacobi matrix in the isospectral torus for $\mathfrak{e}$ so that*

$$|a_n - a_n^\sharp| + |b_n - b_n^\sharp| \to 0 \text{ as } n \to \infty$$

Main Results

Theorem (Nevai Conjecture for Finite Gap Case). *Let $\{a_n^\sharp, b_n^\sharp\}_{n=-\infty}^{\infty}$ be an element of the isospectral torus for $\mathfrak{e}$. Let $\{a_n, b_n\}_{n=1}^{\infty}$ be a set of Jacobi parameters obeying*

$$\sum_{n=1}^{\infty} |a_n - a_n^\sharp| + |b_n - b_n^\sharp| < \infty$$

Then J obeys the Blaschke and Szegő conditions.

Main Results

Remarks. 1. This result is due to Frank–Simon [Duke Math. J. **157** (2011), 461–493].

2. The key is to prove a Lieb–Thirring type inequality

$$\sum_{j=1}^{\infty} \mathrm{dist}(E_j, \mathfrak{e})^{1/2} \le C\Big(\sum_{n=1}^{\infty} |a_n - a_n^\sharp| + |b_n - b_n^\sharp|\Big)$$

for the ℓ^1 condition and $0 < \inf a_n^\sharp < \sup a_n^\sharp < \infty$ imply $\lim_{n\to\infty} \prod_{j=1}^{n} (a_j/a_j^\sharp)$ exists and is in $(0, \infty)$.

3. Frank–Simon needed to develop new techniques to obtain these Lieb–Thirring type inequalities for eigenvalues in the gap.

Main Results

Theorem (Szegő Asymptotics). *Let J obey the Szegő and Blaschke conditions for $\mathfrak{e}$. Let $J^\sharp$ be the element of the isospectral torus to which J is asymptotic. Let $p_n^\sharp$ be the orthonormal OPs for the half-line Jacobi matrix associated to $J^\sharp$ and p_n for J. Then, for all $x \in \mathbb{C} \setminus [\alpha_1, \beta_{\ell+1}]$, $p_n(x)/p_n^\sharp(x)$ has a finite limit $q(x)$ which is non-zero on $\mathbb{C} \setminus \{[\alpha_1, \beta_{\ell+1}] \cup \sigma(J)\}$.*

Main Results

Corollary 1. $\dfrac{a_1^\sharp \cdots a_n^\sharp}{a_1 \cdots a_n}$ *has a non-zero limit.*

This is just Szegő asymptotics at $x = \infty$.

Corollary 2. $\dfrac{a_1 \cdots a_n}{C(\mathfrak{e})^n}$ *is asymptotically almost periodic.*

Corollary 3. For $x \in \mathbb{C} \setminus [\alpha_1, \beta_{\ell+1}]$, $p_n(\mathbf{z})B(\mathbf{z}(x))^n$ is asymptotically almost periodic where $\mathbf{z}(x)$ is the unique point in $\overset{\circ}{D}(\Gamma)$ where $\mathbf{x}(\mathbf{z}(x)) = x$.

C_0 Sum Rule

The key to proving the Shohat–Nevai rule is a step-by-step C_0 Sum Rule. One first writes a non-local step-by-step sum rule and looks at its 0th Taylor coefficients. Thus far, no one has made use of higher order sum rules.

C_0 Sum Rule

One defines $M(z)$ on $\mathbb{D}$ by

$$M(z) = -m(\mathbf{x}(z))$$

The minus sign is there so that $\operatorname{Im} M > 0$ on $\mathring{D} \cap \mathbb{C}_+$.
Recall $\mathbf{x}$ maps $\mathring{D} \cap \mathbb{C}_+$ to $\mathbb{C}_-$.
The poles and zeros of $m(z)$ (which are eigenvalues of J and J_1) can be decomposed into sequences that converge to one of the points $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ so that each sequence interlaces. If p_j and z_j are the points in $D \cap \bar{\mathbb{C}}_+$ which map to these poles and zeros, one proves that $B_\infty(z) = \prod B(z, z_j) / \prod B(z, p_j)$ is given as a conditionally convergent product.

C_0 Sum Rule

One then gets a Poisson–Jensen representation

$$\log\left(\frac{\operatorname{Im} M(e^{i\theta})}{\operatorname{Im} M_1(e^{i\theta})}\right) \in \bigcap_{p<\infty} L^p(\partial\mathbb{D}, \frac{d\theta}{2\pi})$$

$$a_1 M(z) = B(z) B_\infty(z) \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(\frac{\operatorname{Im} M(e^{i\theta})}{\operatorname{Im} M_1(e^{i\theta})}\right) \frac{d\theta}{2\pi}\right)$$

$\arg\log(M/BB_\infty)$ is not bounded but is in h^p, $p<\infty$ because of the exponential bound in Beardon's theorem. By taking logs and evaluating at $z=0$ (for M/B), one obtains

$$-\log\left(\frac{a_1}{C(\mathfrak{e})}\right) = Z(J_1 \mid J) + \sum \left[G_{\mathfrak{e}}(\mathbf{x}(z_j)) - G_{\mathfrak{e}}(\mathbf{x}(p_j))\right]$$

C_0 Sum Rule

$C(\mathfrak{e})$ occurs since $B(z) = \frac{C(\mathfrak{e})}{x_\infty} z + O(z^2)$,

$m(x) = -\frac{1}{x}$ near $x = \infty$ and $\mathbf{x}(z) = \frac{x_\infty}{z} + O(1)$.

Thus, $M(z) = \frac{z}{x_\infty} + O(z^2)$ so $\frac{a_1 M(z)}{B(z)} = \frac{a_1}{C(\mathfrak{e})}$.

The C_0 sum rule and (complicated) modifications of the $[-2, 2]$ arguments lead to the Shohat–Nevai type theroem.

The Jost Function

A key tool of the analysis is the Jost function defined for $\mu \in \mathrm{Sz}(\mathfrak{e})$, the measures obeying a Szegő and a Blaschke condition. We need a reference measure, $w_0(x)dx$ which we take to be the measure of that point in the isospectral torus where each "gap pole" is not only on the second sheet but with $\mathbf{z}(x)$ as far from $z = 0$ as possible.

$$u(z;\mu) = [\prod_j B(z,p_j)] \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left[\frac{w_0(\mathbf{x}e^{i\theta})}{w(\mathbf{x}e^{i\theta})}\right]\right)$$

The Jost Function

An important fact is that each Jost function is character automorphic. One can state the representation theorem for M as

$$a_1 M(z;\mu) = \frac{B(z)u(z;\mu_1)}{u(z;\mu)}$$

which implies a useful relation between the characters of $u(\,\cdot\,;\mu_1)$, $u(\,\cdot\,;\mu)$ and B (M is automorphic).

In particular, the character of $u(\,\cdot\,;\mu)$ determines the orbit of the character of $u(\,\cdot\,;\mu_n)$.

The Jost Function

The abelianization of the Fuchsian group is $\mathbb{Z}^\ell$ so the character group is $(\partial\mathbb{D})^\ell$ which has the same topology as the isospectral torus. A deep and important fact is that the map of a point in the isospectral torus to the character of its Jost function is an isomorphism which we call the Jost isomorphism.

The proof uses Abel's theorem on meromorphic functions on hyperelliptic surfaces and an explicit formula for m's in the isospectral torus in terms of theta functions.

The Jost Function

We claimed a basic result is that any $\mu \in \mathrm{Sz}(\mathfrak{e})$ had Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ with $|a_n - a_n^\sharp| + |b_n - b_n^\sharp| \to 0$ for a $J^\sharp$ in the isospectral torus.

How are J and $J^\sharp$ related? $J^\sharp$ is precisely the unique element of the torus whose Jost character is the same as the Jost character of J !

This can be understood in terms of the coefficient stripping character relations.