A description of Alexander Pushnitski's research interests written for a non-specialist.

Spectral perturbation theory of self-adjoint operators in a Hilbert space. This heading describes my research interests in broadest terms. First consider a Hermitian matrix A in a Euclidean space $\mathbb{C}^N$. To perform a *spectral analysis* of this matrix means to find its eigenvalues λ_j and the corresponding eigenvectors $\psi_j \in \mathbb{C}^N$, $j = 1, \dots, N$:

$$A\psi_j = \lambda_j \psi_j. \tag{1}$$

Now suppose we change the matrix A by adding to it another Hermitian matrix V; so the "new" matrix is $\widetilde{A} = A + V$. The matrix $\widetilde{A}$ will have its own set of eigenvalues $\widetilde{\lambda}_j$ and eigenvectors $\widetilde{\psi}_j$. Thus, changing the matrix, we change its spectral decomposition. When we go from A to $\widetilde{A}$, the eigenvalues *shift* and the eigenvectors *rotate*. One of the main questions addressed by spectral perturbation theory is the description of these shifts and rotations depending on the perturbation V.

Of course, the answer depends on our assumptions on A and V. Usually, one assumes that V is in some sense "small" or "weak". "Small" usually means that an appropriate norm of V is small. "Weak" is a more subtle notion and can mean many things, but in this context it can mean, for example, that the matrix V is sparse (i.e. the matrix elements of V contain many zeros).

Of course, the more advanced version of the above set-up is the situation when instead of Hermitian matrices in the finite dimensional Euclidean space $\mathbb{C}^N$ one considers self-adjoint operators in an infinite dimensional Hilbert space. In this case, the situation is much more complicated, mainly due to the fact that instead of a sequence of eigenvalues separated from each other, a much more complex spectrum can occur: one can have intervals of continuous spectrum, eigenvalues that are dense on some intervals, etc. A good background reading on spectral theory of self-adjoint operators is given by Sections VI and VII of the book [4].

Spectral shift function theory. Let us return to the situation with two Hermitian matrices A, A'. Suppose we would like to compare the two sets of eigenvalues $\{\lambda_j\}_{j=1}^N$ and $\{\widetilde{\lambda}_j\}_{j=1}^N$. One way of doing this is to introduce the *eigenvalue counting function*

$$N(\lambda; A) = \operatorname{card}\{j : \lambda_j < \lambda\},$$
$$N(\lambda; \widetilde{A}) = \operatorname{card}\{j : \widetilde{\lambda}_j < \lambda\}$$

(here $\operatorname{card} X$ stands for the number of elements of X), and then look at the difference

$$\xi(\lambda; \widetilde{A}, A) = N(\lambda; A) - N(\lambda; \widetilde{A}).$$

The function $\xi(\lambda; \widetilde{A}, A)$ gives information about the shifts of eigenvalues of $\widetilde{A}$ relatively to the eigenvalues of A. Thus, it is called the *spectral shift function*.

Again, a more sophisticated definition of spectral shift function involves pairs of self-adjoint operators in a Hilbert space rather than pairs of matrices. A standard survey in the spectral shift function theory is [3]; see also [2] for an easy introduction.

My main contribution to this field is a certain representation for the spectral shift function, see [2].

Scattering theory. Scattering theory is a multi-faceted subject. One facet is the study of wave propagation phenomena in physics. The waves could be, for example, acoustic waves, water waves, or electromagnetic waves. My own interests are mainly related to the quantum mechanical waves which are not really waves but mathematical objects (wave functions ψ) describing the motion of subatomic particles. These waves are governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = -\Delta\psi + V\psi,$$

where V is the potential of the external electric field. This external electric field is assumed to be localised in space, i.e. $V(x) = 0$ for all sufficiently large $|x|$. This corresponds to waves-particles coming from infinity, propagating in an empty space, hitting some obstacle (represented by V) and being scattered away to infinity by this obstacle.

One of the methods of analysing solutions to this equation is to look at wave functions with a certain fixed energy λ. These wave functions satisfy the stationary Schrodinger equation

$$-\Delta\widetilde{\psi} + V\widetilde{\psi} = \lambda\widetilde{\psi}. \tag{2}$$

In the absence of an electric field, we have $V = 0$ and the equation becomes

$$-\Delta\psi = \lambda\psi. \tag{3}$$

This is the equation of a quantum particle with energy λ moving in an empty space. The analysis usually involves comparison of the solutions ψ and $\widetilde{\psi}$ corresponding to *the same value of* λ.

This equations (2) and (3) look like the eigenvalue equation (1) with the "matrices" $A = -\Delta$ and $\widetilde{A} = -\Delta + V$. This connection leads to another facet of scattering theory. It turns out that the values of λ for which the above problem makes sense lie in the *continuous spectrum* of the operators A and $\widetilde{A}$. One of the tasks of mathematical scattering theory is the comparison of eigenvectors of two self-adjoint operators A, $\widetilde{A}$ corresponding to the same eigenvalue λ from the continuous spectrum. Here one assumes that $\widetilde{A} = A + V$, where V is in some sense a "weak" perturbation of A. Informally speaking, mathematical scattering theory studies the rotations (in Hilbert space) of eigenvectors corresponding to the continuous spectrum. In rigourous terms, this is performed through the analysis of *wave operators,* *scattering operator, scattering matrix,* and *spectral shift function.* See, for example, [5] for an introduction to the main concepts of scattering theory.

My own interests in this field are in establishing a transparent relationship between the scattering matrix the geometry of Hilbert space (rotation of eigenvectors).

Schrödinger operators with magnetic fields. Consider a quantum particle in an external magnetic field. The magnetic field in quantum mechanics is described by the *magnetic vector potential* $A(x)$, which is a vector valued function of $x \in \mathbb{R}^3$. Mathematically, the behaviour of this quantum particle can be described through the spectral analysis of the *magnetic Schrödinger operator*

$$H = \sum_{j=1}^{3} \left(-i \frac{\partial}{\partial x_j} - A_j \right)^2 .$$

Here A_j are the coordinates of the vector-potential A. Spectral analysis of magnetic Schrodinger operators is (besides being physically relevant) full of interesting and difficult mathematical problems; see e.g. the survey [1].

My own interests in this area are mostly related to the perturbations of the Schrödinger operator with a constant magnetic field. Most of the "hard" analysis I've learnt in the recent years is due to my interest in this problem; it involves a fascinating mix of complex analysis, special functions, orthogonal polynomials, embedding theorems, functional analysis, geometry, etc.

References

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