

THE BIRMAN-SCHWINGER PRINCIPLE ON THE ESSENTIAL SPECTRUM

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ABSTRACT. Let H_0 and H be self-adjoint operators in a Hilbert space. We consider the spectral projections of H_0 and H corresponding to a semi-infinite interval of the real line. We discuss the index of this pair of spectral projections and prove an identity which extends the Birman-Schwinger principle onto the essential spectrum. We also relate this index to the spectrum of the scattering matrix for the pair H_0, H .

1. INTRODUCTION

1.1. **The function $\Xi(\lambda; H, H_0)$.** For a self-adjoint operator H in a Hilbert space we denote by $E(\Lambda; H)$ the spectral projection of H associated with a Borel set $\Lambda \subset \mathbb{R}$ and let

$$N(\Lambda; H) = \text{rank } E(\Lambda; H) \leq \infty.$$

Let H_0 and H be two self-adjoint operators in a Hilbert space \mathcal{H} ; we wish to compare the eigenvalue distribution functions of H_0 and H . If our Hilbert space is finite dimensional, then the difference

$$(1.1) \quad N((-\infty, \lambda); H_0) - N((-\infty, \lambda); H)$$

describes the shifts of the eigenvalues of H relatively to the eigenvalues of H_0 . Below we discuss a certain analogue of this difference in the infinite dimensional case.

Throughout this paper, we assume that

$$(1.2) \quad \begin{aligned} &H_0 \text{ and } H \text{ are semi-bounded from below with the same form domain} \\ &\text{and the operator } V = H - H_0 \text{ is } H_0\text{-form compact.} \end{aligned}$$

This, in particular, ensures that the essential spectra of H_0 and H coincide: $\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H)$. Under these assumptions, the difference (1.1) is of course still well defined for $\lambda < \inf \sigma_{\text{ess}}(H_0)$. The difficulty arises when the interval $(-\infty, \lambda)$ contains points of the essential spectrum; then (1.1) formally gives $\infty - \infty$.

In this paper, we discuss the function

$$(1.3) \quad \Xi(\lambda; H, H_0) = \text{index}(E((-\infty, \lambda); H_0), E((-\infty, \lambda); H)),$$

where the r.h.s. is the Fredholm index of a pair of projections, the notion which is recalled in Section 2.1 below. As it will be clear from the discussion in Section 2, for $\lambda < \inf \sigma_{\text{ess}}(H_0)$ we have

$$(1.4) \quad \Xi(\lambda; H, H_0) = N((-\infty, \lambda); H_0) - N((-\infty, \lambda); H)$$

2000 *Mathematics Subject Classification.* Primary 47A40; Secondary 35P25, 47B25, 47F05.

Key words and phrases. Birman-Schwinger principle, essential spectrum, spectral projections, scattering matrix.

and thus the definition (1.3) provides a natural regularisation of the difference (1.1). For $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H_0)$, the index function $\Xi(\lambda; H, H_0)$ defined by (1.3) has appeared before in the literature in various guises; we briefly discuss this in Section 2.2.

1.2. Birman-Schwinger principle in terms of $\Xi(\lambda; H, H_0)$. To the best of our knowledge, the index $\Xi(\lambda; H, H_0)$ for λ on the essential spectrum of H_0 has not been studied before. The purpose of this paper is to present a step in this direction. Our main result (Theorem 2.4 below) is an explicit formula for $\Xi(\lambda; H, H_0)$, $\lambda \in \sigma_{\text{ess}}(H_0)$, in terms of the “sandwiched resolvent” of H_0 . This formula can be interpreted as an extension of the Birman-Schwinger principle onto the essential spectrum.

To give the general flavour of our main result, let us assume that $V \leq 0$ in the quadratic form sense and suppose that the limit

$$T_0(\lambda + i0) = \lim_{\varepsilon \rightarrow +0} |V|^{1/2} (H_0 - \lambda - i\varepsilon)^{-1} |V|^{1/2}$$

exists in the operator norm. Then, denoting $\text{Re } T_0 = (T_0 + T_0^*)/2$, under the appropriate assumptions we prove that

$$(1.5) \quad \Xi(\lambda; H, H_0) = -N((1, \infty); \text{Re } T_0(\lambda + i0)), \quad V \leq 0,$$

as long as 1 is not an eigenvalue of $\text{Re } T_0(\lambda + i0)$. For $\lambda < \inf \sigma(H_0)$, by virtue of (1.4) this formula simplifies to

$$(1.6) \quad N((-\infty, \lambda); H) = N((1, \infty); T_0(\lambda + i0)), \quad V \leq 0, \quad \lambda < \inf \sigma(H_0),$$

which is the Birman-Schwinger principle in its usual form.

1.3. $\Xi(\lambda; H, H_0)$ and the scattering matrix. In the scattering theory framework we point out the following connection between $\Xi(\lambda; H, H_0)$ and the spectrum of the scattering matrix $S(\lambda)$ corresponding to the pair H_0, H . Recall that since $S(\lambda)$ is a unitary operator, the eigenvalues of $S(\lambda)$ are located on the unit circle in \mathbb{C} . Suppose that λ is monotonically increasing, moving through an interval of the absolutely continuous spectrum of H_0 . Then every time that an eigenvalue of $S(\lambda)$ of multiplicity n crosses the point -1 on the unit circle, the index $\Xi(\lambda; H, H_0)$ acquires a jump of $+n$ or $-n$. The jump of $+n$ occurs if the eigenvalue of $S(\lambda)$ crosses -1 by rotating in a clockwise direction, and $-n$ corresponds to the anti-clockwise rotation. See Theorem 3.1.

1.4. The structure of the paper. In Sections 2.1 and 2.2 we recall the definition of the index of a pair of projections and collect the basic properties of the index function $\Xi(\lambda; H, H_0)$ for $\lambda \notin \sigma_{\text{ess}}(H_0)$. In Sections 2.3 and 2.4, we recall the Birman-Schwinger principle for $\lambda \notin \sigma_{\text{ess}}(H_0)$ and state it in terms of the index function $\Xi(\lambda; H, H_0)$. In Section 2.5 we state our main result: the extension of the Birman-Schwinger principle to the case $\lambda \in \sigma_{\text{ess}}(H_0)$. Application to the Schrödinger operator is discussed in Section 2.7. In Section 3, we discuss the connection between the index function $\Xi(\lambda; H, H_0)$ and the spectrum of the scattering matrix $S(\lambda)$. The proof of the main result is given in Sections 4–6.

2. MAIN RESULTS

2.1. The index of a pair of projections. Let P, Q be orthogonal projections in a Hilbert space. By using some simple algebra (see e.g. [2, Theorem 4.2]) it is not difficult to see that $\sigma(P - Q) \subset [-1, 1]$ and

$$(2.1) \quad \dim \text{Ker}(P - Q - \lambda I) = \dim \text{Ker}(P - Q + \lambda I), \quad \lambda \neq \pm 1;$$

the proof of (2.1) is based on the identity

$$(P - Q)W = W(Q - P), \quad W = I - P - Q.$$

A pair P, Q is called Fredholm, if

$$(2.2) \quad \{1, -1\} \cap \sigma_{\text{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q$ is compact, then the pair P, Q is Fredholm. The index of a Fredholm pair is defined by the formula

$$(2.3) \quad \text{index}(P, Q) = \dim \text{Ker}(P - Q - I) - \dim \text{Ker}(P - Q + I).$$

We note that $\text{index}(P, Q)$ coincides with the Fredholm index of the operator QP viewed as a map from $\text{Ran } P$ to $\text{Ran } Q$, see [2, Proposition 3.1].

If $P - Q$ is a trace class operator, then

$$(2.4) \quad \text{index}(P, Q) = \text{Tr}(P - Q),$$

since all the eigenvalues of $P - Q$ apart from 1 and -1 in the series $\text{Tr}(P - Q) = \sum_k \lambda_k(P - Q)$ cancel out by (2.1). In the simplest case of finite rank projections P, Q we have

$$\text{index}(P, Q) = \text{rank } P - \text{rank } Q.$$

2.2. Definition and basic properties of Ξ . Let us accept the following

Definition. Let H_0 and H be self-adjoint operators in a Hilbert space. Suppose that $E((-\infty, \lambda); H), E((-\infty, \lambda); H_0)$ is a Fredholm pair. Then we will say that the index $\Xi(\lambda; H, H_0)$ exists and define it by

$$\Xi(\lambda; H, H_0) = \text{index}(E((-\infty, \lambda); H_0), E((-\infty, \lambda); H)).$$

Note that by this definition, $\Xi(\lambda; H, H_0)$ is integer valued. We need a simple existence statement for Ξ :

Proposition 2.1. Assume (1.2). Then for all $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H_0)$ the difference of projections $E((-\infty, \lambda); H) - E((-\infty, \lambda); H_0)$ is compact and therefore the index $\Xi(\lambda; H, H_0)$ exists.

This proposition is almost obvious, but for the sake of completeness we give the proof at the end of Section 2.3.

Below, assuming (1.2), we briefly recall the basic properties of $\Xi(\lambda; H, H_0)$. Most of these properties have appeared before in the literature in various guises (see e.g. [14, 1, 11, 12, 27, 10, 9, 7, 4, 3, 17, 13, 15]) and can be regarded as folklore; they were reviewed and proven in a systematic fashion in [20].

For any $\lambda \in \mathbb{R}$, the index $\Xi(\lambda; H, H_0)$ exists if and only if $\Xi(\lambda; H_0, H)$ exists and if both of these indices exist, we have

$$(2.5) \quad \Xi(\lambda; H, H_0) = -\Xi(\lambda; H_0, H).$$

If $[a, b] \cap \sigma_{\text{ess}}(H_0) = \emptyset$, then

$$(2.6) \quad \Xi(b; H, H_0) - \Xi(a; H, H_0) = N([a, b]; H_0) - N([a, b]; H).$$

In particular, we get (1.4) for $\lambda < \inf \sigma_{\text{ess}}(H_0)$. For any $\lambda \in \mathbb{R}$, if $\Xi(\lambda; H, H_0)$ exists then the estimates

$$(2.7) \quad -\text{rank } V_- \leq \Xi(\lambda; H, H_0) \leq \text{rank } V_+, \quad V_{\pm} = \frac{1}{2}(|V| \pm V)$$

hold true. In particular,

$$\pm V \geq 0 \implies \pm \Xi(\lambda; H, H_0) \geq 0.$$

The estimates (2.7) can be improved if λ is not in the spectrum of H_0 . Suppose that for some $a > 0$, one has $[\lambda - a, \lambda + a] \cap \sigma(H_0) = \emptyset$. Then [20, Corollary 3.3] one has

$$(2.8) \quad -N((-\infty, -a); V) \leq \Xi(\lambda; H, H_0) \leq N((a, \infty); V).$$

Next, if V is a trace class operator, then

$$(2.9) \quad \Xi(\lambda; H, H_0) = \xi(\lambda; H, H_0), \quad \lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H_0),$$

where $\xi(\lambda; H, H_0)$ is M. G. Krein's spectral shift function. See e.g. [29, Chapter 8] for a survey of the spectral shift function theory. Note that (2.9) is in general false for $\lambda \in \sigma_{\text{ess}}(H_0)$, since Ξ is integer valued and ξ is real valued.

Remark 2.2. $\xi(\lambda; H, H_0)$ and $\Xi(\lambda; H, H_0)$ are, in fact, two different regularisations of

$$(2.10) \quad \text{Tr}(E((-\infty, \lambda); H_0) - E((-\infty, \lambda); H)).$$

By an example due to M. G. Krein [18] (see also Section 2.6 below), the difference of spectral projections in (2.10) may fail to belong to the trace class if $\lambda \in \sigma_{\text{ess}}(H_0)$. Thus, the trace in (2.10) may not exist. The spectral shift function is the regularisation of (2.10) obtained by replacing the difference of spectral projections by $\varphi(H) - \varphi(H_0)$, where φ is a smooth approximation of the characteristic function of $(-\infty, \lambda)$. The index $\Xi(\lambda; H, H_0)$ is obtained by replacing Tr by index in (2.10). These two regularisations coincide in simplest cases but in general are distinct.

Finally, for $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H_0)$, the index $\Xi(\lambda; H, H_0)$ coincides with the spectral flow (i.e. the net flux of eigenvalues) of the operator family $\{H_0 + \alpha V\}_{\alpha \in [0,1]}$ through λ as α increases monotonically from 0 to 1; see e.g. [20, Section 2.6]. The spectral flow is particularly easy to define when $V \geq 0$ or $V \leq 0$; in this case the eigenvalues of $H_0 + \alpha V$ are monotone in α and the spectral flow is simply the total number of eigenvalues that cross the point λ as α increases from 0 to 1. In general, one has to count the eigenvalues with the sign plus or minus depending on whether they cross λ to the right or to the left. See [13] for a comprehensive survey of the spectral flow in perturbation theory. We will return to the subject of spectral flow in Section 3 in the context of unitary operators.

2.3. The sandwiched resolvents and the resolvent identities. Here we set up some notation and recall the resolvent identity in the form convenient for us.

The Birman-Schwinger principle is most conveniently stated if the perturbation V is factorised. Let us assume that V is represented as $V = G^* JG$, where G is an operator from \mathcal{H} to an auxiliary Hilbert space \mathcal{K} and J is an operator in \mathcal{K} . We assume that

$$(2.11) \quad \begin{aligned} & J = J^*, J \text{ is bounded and has a bounded inverse,} \\ & \text{Dom}(H_0 - aI)^{1/2} \subset \text{Dom } G \text{ and } G(H_0 - aI)^{-1/2} \text{ is compact, } \forall a < \inf \sigma(H_0). \end{aligned}$$

These assumptions ensure (by the ‘‘KLMN Theorem’’, see e.g. [23, Theorem X.17]) that V is H_0 -form compact and H coincides with the form sum $H_0 + V$. Thus, (1.2) follows from (2.11). In fact, (2.11) is just another way of stating the assumption (1.2). Indeed, assuming (1.2), one can always take $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$ and¹ $J = \text{sign}(V)$ and then (2.11) holds true. In applications, the factorisation $V = G^* JG$ often arises naturally due to the structure of the problem.

Note that since H_0 and H have the same form domain, under the assumption (2.11) we also have

$$(2.12) \quad \text{Dom}(H - aI)^{1/2} \subset \text{Dom } G \text{ and } G(H - aI)^{-1/2} \text{ is compact}$$

for any $a < \inf \sigma(H)$.

Let us denote $R_0(z) = (H_0 - zI)^{-1}$ and $R(z) = (H - zI)^{-1}$ whenever the inverse operators exist. Let us define the operators $T_0(z)$, $T(z)$ (sandwiched resolvents) formally by setting

$$T_0(z) = GR_0(z)G^*, \quad T(z) = GR(z)G^*.$$

More precisely, this means

$$(2.13) \quad T_0(z) = G(H_0 - aI)^{-1/2}(H_0 - aI)R_0(z)(G(H_0 - aI)^{-1/2})^*, \quad a < \inf \sigma(H_0),$$

$$(2.14) \quad T(z) = G(H - aI)^{-1/2}(H - aI)R(z)(G(H - aI)^{-1/2})^*, \quad a < \inf \sigma(H).$$

By (2.11), (2.12), the operators $T_0(z)$, $T(z)$ are compact. The operator $T_0(z)$ is self-adjoint for all $z \in \mathbb{R} \setminus \sigma(H_0)$ and $T(z)$ is self-adjoint for all $z \in \mathbb{R} \setminus \sigma(H)$.

For future reference, let us display the iterated resolvent identity for the operators H_0 and H :

$$(2.15) \quad R(z) - R_0(z) = -(GR_0(\bar{z}))^* J(GR(z)) = -(GR_0(\bar{z}))^* (J - JT(z)J)(GR_0(z))$$

and its direct consequence

$$(2.16) \quad (J^{-1} + T_0(z))(J - JT(z)J) = (J - JT(z)J)(J^{-1} + T_0(z)) = I.$$

From (2.15), in particular, we easily obtain

Proof of Proposition 2.1. Let Γ be a compact positively oriented contour in $\mathbb{C} \setminus (\sigma(H_0) \cup \sigma(H))$ such that the bounded set $(\sigma(H) \cup \sigma(H_0)) \cap (-\infty, \lambda)$ is contained inside Γ . Then

$$E((-\infty, \lambda); H) - E((-\infty, \lambda); H_0) = \frac{1}{2\pi i} \int_{\Gamma} ((R_0(z) - R(z)) dz.$$

¹Here and in what follows $\text{sign}(x) = 1$ for $x \geq 0$ and $\text{sign}(x) = -1$ for $x < 0$. In particular, $\text{sign}(V)$ has a bounded inverse.

From (2.15) and (2.11), (2.12) it is easy to see that the operator in the r.h.s. is compact, as required. ■

2.4. The Birman-Schwinger principle. In what follows, we assume (2.11). We first note that by Proposition 2.1, for all $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ the indices $\Xi(\lambda; H, H_0)$ and $\Xi(0; J^{-1} + T_0(\lambda), J^{-1})$ exist.

Proposition 2.3. *Assume (2.11). Then*

$$(2.17) \quad \dim \text{Ker}(H - \lambda I) = \dim \text{Ker}(J^{-1} + T_0(\lambda)), \quad \forall \lambda \in \mathbb{R} \setminus \sigma(H_0),$$

$$(2.18) \quad \Xi(\lambda; H, H_0) = -\Xi(0; J^{-1} + T_0(\lambda); J^{-1}), \quad \forall \lambda \in \mathbb{R} \setminus (\sigma(H_0) \cup \sigma(H)).$$

In particular, in the cases $J = I$ or $J = -I$, the identity (2.18) can be written as

$$(2.19) \quad \Xi(\lambda; H_0 + G^*G, H_0) = N((-\infty, -1); T_0(\lambda)),$$

$$(2.20) \quad \Xi(\lambda; H_0 - G^*G, H_0) = -N((1, \infty); T_0(\lambda)).$$

Note that for $\lambda < \inf \sigma(H_0)$, formula (2.20) is equivalent to (1.6).

Formula (2.18) has a long history starting from the celebrated papers by M. Sh. Birman [5] and J. Schwinger [26] where it was stated in the form equivalent to (1.6). The identities (2.19), (2.20) were extensively used (see e.g. [14, 8, 11, 1, 12]) in the context of the spectral flow and also in [27, Theorem 3.5] in the context of the spectral shift function theory (see (2.9)). The identity (2.18) as stated above, i.e. in terms of the index of a pair of projections, was proven in [9] in the context of the spectral shift function theory for trace class perturbations V . It was extended to the general case in [20].

Remark. The right hand side of (2.18) is not symmetric with respect to the interchange of H_0 and H . However, under the assumptions of Proposition 2.3 by writing $H_0 = H - V$ and using (2.5), one also obtains

$$\Xi(\lambda; H, H_0) = \Xi(0; J^{-1} - T(\lambda); J^{-1}), \quad \forall \lambda \in \mathbb{R} \setminus (\sigma(H_0) \cup \sigma(H)).$$

Our main result below is an extension of (2.18) to the case when λ belongs to the essential spectrum of H_0 .

2.5. Main result. As above, we assume that the perturbation V is factorised as $V = G^*JG$ with the properties (2.11) and use the notation $T_0(z)$ for the sandwiched resolvent. Let $\Delta \subset \mathbb{R}$ be an open interval. We assume that the following version of the limiting absorption principle holds true:

$$(2.21) \quad \begin{aligned} &T_0(z) \text{ is uniformly continuous in the operator norm} \\ &\text{in the rectangle } \text{Re } z \in \Delta, \text{Im } z \in (0, 1). \end{aligned}$$

Of course, this trivially implies that the limit $T_0(\lambda + i0)$ exists in the operator norm and is continuous in $\lambda \in \Delta$. The operator $T_0(\lambda + i0)$ is compact and in general non-selfadjoint. We denote

$$(2.22) \quad A_0(\lambda) = \text{Re } T_0(\lambda + i0), \quad B_0(\lambda) = \text{Im } T_0(\lambda + i0),$$

where $\text{Re } X = (X + X^*)/2$, $\text{Im } X = (X - X^*)/2i$. We also set

$$(2.23) \quad \mathcal{N} = \{\lambda \in \Delta \mid 0 \in \sigma(J^{-1} + A_0(\lambda))\}.$$

Below is our main result. For the purposes of future reference, we break up the statement of this theorem into several parts.

Theorem 2.4. *Assume (2.11) and (2.21). Then:*

- (i) *the set \mathcal{N} defined by (2.23) is closed in Δ (i.e. $\Delta \setminus \mathcal{N}$ is open);*
- (ii) *for all $\lambda \in \Delta \setminus \mathcal{N}$, the index $\Xi(\lambda; H, H_0)$ exists;*
- (iii) *for all $\lambda \in \Delta \setminus \mathcal{N}$, the identity*

$$(2.24) \quad \Xi(\lambda; H, H_0) = -\Xi(0; J^{-1} + A_0(\lambda); J^{-1})$$

holds true;

- (iv) *the index $\Xi(\lambda; H, H_0)$ is constant on every connected component of the set $\Delta \setminus \mathcal{N}$.*

The proof is given in Sections 4–6. The proof uses Proposition 2.3 and a certain continuous deformation argument. Roughly speaking, we reduce Theorem 2.4 to Proposition 2.3 by making an “infinitesimal spectral gap” in the spectrum of H_0 near λ .

- Remarks.** 1. The most important statement in Theorem 2.4 is part (iii). Part (i) is trivial, part (ii) follows from the results of [21], and part (iv) is an easy consequence of part (iii).
2. The existence of $\Xi(0; J^{-1} + A_0(\lambda), J^{-1})$ in the r.h.s. of (2.24) follows from Proposition 2.1 and from the fact that $A_0(\lambda)$ is compact.
3. If $\lambda \in \mathbb{R} \setminus \sigma(H_0)$, then the hypothesis of Theorem 2.4 is trivially satisfied (with Δ being a sufficiently small neighbourhood of λ) and $T_0(\lambda + i0)$ is self-adjoint. Thus, in this case (2.24) coincides with (2.18).
4. If $J = I$ or $J = -I$, then (2.24) becomes

$$\begin{aligned} \Xi(\lambda; H_0 + G^*G, H_0) &= N((-\infty, -1); A_0(\lambda)), \\ \Xi(\lambda; H_0 - G^*G, H_0) &= -N((1, \infty); A_0(\lambda)). \end{aligned}$$

In particular, we obtain (1.5).

5. Let $\Delta \subset \mathbb{R} \setminus \sigma(H_0)$. Then, by (2.17), $\mathcal{N} = \sigma(H) \cap \Delta$. Equivalently, \mathcal{N} is the set of all discontinuities (jumps) of $\Xi(\lambda; H, H_0)$ on Δ .

According to (2.6), away from $\sigma_{\text{ess}}(H_0)$ the jumps of the function $\Xi(\lambda; H, H_0)$ occur at the eigenvalues of H_0 and H . Thus, one is tempted to interpret the jumps of $\Xi(\lambda; H, H_0)$ on the essential spectrum as certain “pseudo-eigenvalues” of H_0 or H , depending on the sign of the jump. In the framework of Theorem 2.4 we see that these “pseudo-eigenvalues” can occur only at the points of the set \mathcal{N} . In Section 3, we give an alternative description of these “pseudo-eigenvalues” in terms of the scattering matrix $S(\lambda)$ for the pair H_0, H .

2.6. The set \mathcal{N} : example. The following example shows that the set \mathcal{N} can be quite large: $\mathcal{N} = \Delta$. In [18], M. G. Krein considered the operator H_0 in $L^2(0, \infty)$ with the integral kernel $H_0(x, y)$ given by

$$H_0(x, y) = \begin{cases} \sinh(x)e^{-y}, & x \leq y, \\ \sinh(y)e^{-x}, & x \geq y \end{cases}$$

and the operator H in the same Hilbert space with the integral kernel $H(x, y) = H_0(x, y) + e^{-x}e^{-y}$. Thus, $V = H - H_0$ is a rank one operator. In fact, $H_0 = (h_0 + I)^{-1}$ and $H = (h + I)^{-1}$, where h_0 (resp. h) is the self-adjoint realisation of the operator $-\frac{d^2}{dx^2}$ in $L^2(0, \infty)$

with the Dirichlet (resp. Neumann) boundary condition at zero. In this example, $\sigma(H_0) = \sigma(H) = [0, 1]$.

M. G. Krein showed that for any $\lambda \in (0, 1)$, the difference

$$E((-\infty, \lambda); H) - E((-\infty, \lambda); H_0)$$

does not belong to the Hilbert-Schmidt class. The more detailed analysis of [16] shows that for any $\lambda \in (0, 1)$,

$$\sigma_{\text{ess}}(E((-\infty, \lambda); H) - E((-\infty, \lambda); H_0)) = [-1, 1]$$

and so $\Xi(\lambda; H, H_0)$ does not exist for any $\lambda \in (0, 1)$.

In this example, the rank one perturbation V can be factorised as $V = G^*G$, with $G : L^2(0, \infty) \rightarrow \mathbb{C}$, $Gf = \int_0^\infty f(x)e^{-x}dx$. Thus, the operator $T_0(z)$ reduces to a multiplication by a scalar in \mathbb{C} . Using the explicit formula for the resolvent of h_0 , one easily checks that

$$T_0(\lambda + i0) = -1 + i\sqrt{\lambda^{-1} - 1}, \quad A_0(\lambda) = -1, \quad \forall \lambda \in (0, 1),$$

and therefore $\mathcal{N} = \Delta$.

Considering rank one perturbations, it is not difficult to construct examples when the set \mathcal{N} has a more complex structure. We shall not pursue this direction here. On the other hand, Theorem 2.6 in the next subsection shows that in some situations of applied interest, the set \mathcal{N} consists of isolated points.

2.7. Application: Schrödinger operator. Let $H_0 = -\Delta$ in $\mathcal{H} = L^2(\mathbb{R}^d)$ with $d \geq 1$ and let $H = H_0 + V$ where V is the operator of multiplication by a function (potential in physical terminology) $V : \mathbb{R}^d \rightarrow \mathbb{R}$. We assume that V is a short range potential, i.e.

$$(2.25) \quad |V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1.$$

Let us discuss the index function $\Xi(\lambda; H, H_0)$. For $\lambda < 0$, this function reduces to the eigenvalue counting function, see (1.4). In order to analyse the index function for $\lambda > 0$, let us apply Theorem 2.4. Let $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$, $J = \text{sign } V$. Under the assumption (2.25), the hypotheses (2.11) and (2.21) are satisfied with $\Delta = (\lambda_1, \lambda_2)$ for any $0 < \lambda_1 < \lambda_2 < \infty$; see e.g. [24, Theorem XIII.33]. Thus, for any $\lambda > 0$ formula (2.24) holds true. The operator $A_0(\lambda)$ in this case is the self-adjoint integral operator in $L^2(\mathbb{R}^d)$ with the kernel

$$(2.26) \quad |V(x)|^{1/2}|V(y)|^{1/2} \frac{1}{4}(2\pi)^{-\nu} k^{d-2} \frac{J_\nu(k|x-y|)}{(k|x-y|)^\nu}, \quad x, y \in \mathbb{R}^d,$$

where $\nu = (d-2)/2$, $k = \sqrt{\lambda} > 0$, and J_ν is the Bessel function. We have

Theorem 2.5. *Assume (2.25). For any $\lambda > 0$, if $\Xi(\lambda; H, H_0)$ exists then it satisfies the estimates*

$$(2.27) \quad -N([1, \infty); A_0(\lambda)) \leq \Xi(\lambda; H, H_0) \leq N((-\infty, -1]; A_0(\lambda)).$$

Moreover, for all sufficiently large $\lambda > 0$ the index $\Xi(\lambda; H, H_0)$ exists and equals zero.

Proof. Since $\sigma(J^{-1}) = \{-1, 1\}$, we can apply (2.8) to the r.h.s. of (2.24) with any $a \in (0, 1)$. This yields

$$-N((a, \infty); A_0(\lambda)) \leq \Xi(\lambda; H, H_0) \leq N((-\infty, -a); A_0(\lambda)).$$

Taking $a \rightarrow 1$, we obtain (2.27).

Next, under the assumption (2.25), one has (see e.g. [24, Problem 60, page 390]):

$$(2.28) \quad \|T_0(\lambda + i0)\| \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Thus, for all sufficiently large $\lambda > 0$ one has $\|A_0(\lambda)\| < 1$. For such λ , the operator $J^{-1} + A_0(\lambda) = J^{-1}(I + JA_0(\lambda))$ is invertible. Thus by Theorem 2.4(ii) the index $\Xi(\lambda; H, H_0)$ exists. For such λ we have

$$N((-\infty, -1]; A_0(\lambda)) = N([1, \infty); A_0(\lambda)) = 0$$

and therefore by (2.27) we get $\Xi(\lambda; H, H_0) = 0$, as required. ■

Theorem 2.5 can be combined with spectral estimates for $A_0(\lambda)$ to yield explicit bounds for $\Xi(\lambda; H, H_0)$ in terms of V . Let us give a simple example of such a bound. Let $d = 3$. Then the integral kernel of $A_0(\lambda)$, $\lambda = k^2 > 0$, is

$$|V(x)|^{1/2}|V(y)|^{1/2} \frac{\cos k|x-y|}{4\pi|x-y|}.$$

Using the estimate

$$N([1, \infty); \pm A_0(\lambda)) \leq \|A_0(\lambda)\|_2^2$$

in terms of the Hilbert-Schmidt norm $\|\cdot\|_2$, we obtain

$$|\Xi(\lambda; H, H_0)| \leq \frac{1}{16\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|V(x)||V(y)|}{|x-y|^2} dx dy,$$

whenever the integral in the r.h.s. converges.

Under additional assumptions on the potential V , one can ensure that the set \mathcal{N} is finite:

Theorem 2.6. *Assume that $|V(x)| \leq \exp(-\gamma|x|)$ with some $\gamma > 0$. Then the index $\Xi(\lambda; H, H_0)$ exists for all $\lambda \in \mathbb{R} \setminus \mathcal{N}_0$, where \mathcal{N}_0 is a finite set.*

Proof. By Proposition 2.1, the index $\Xi(\lambda; H, H_0)$ exists for all $\lambda < 0$. By Theorem 2.4, it suffices to prove that $I + JA_0(\lambda)$ is invertible for all $\lambda > 0$ apart from a finite set. Let us use formula (2.26). It is well known that $z^{-\nu}J_\nu(z)$ is an entire function of z which obeys

$$|z^{-\nu}J_\nu(z)| \leq \frac{\exp(|\operatorname{Im} z|)}{2^\nu \Gamma(\nu + 1)}, \quad \nu \geq -1/2.$$

It follows that the operator $A_0(k^2)$ is analytic in k for $|\operatorname{Im} k| < \gamma/2$ and $d \geq 2$. For $d = 1$, the operator $A_0(k^2)$ is analytic in k for $|\operatorname{Im} k| < \gamma/2$, $k \neq 0$ and has a single pole at $k = 0$. By (2.28), the operator $I + JA_0(\lambda)$ is invertible for all sufficiently large λ . By the analytic Fredholm alternative, we see that $I + JA_0(\lambda)$ is invertible for all but finitely many $\lambda > 0$. ■

3. Ξ AND THE SCATTERING MATRIX

Below we recall the definition of the scattering matrix $S(\lambda)$ for the pair H_0, H and define the *spectral flow* $\mu(e^{i\theta}; \lambda)$ of the scattering matrix. Next, we establish a formula (3.8) which relates $\Xi(\lambda; H, H_0)$ and the spectral flow. This formula allows one to describe the jumps of $\Xi(\lambda; H, H_0)$ in terms of the spectrum of the scattering matrix. The proof of Theorem 2.4 does not use the material of this section.

3.1. The scattering matrix. Let $E^{\text{ac}}(\cdot; H_0)$ be the absolutely continuous part of the spectral measure of the operator H_0 , and let $\hat{\sigma}(H_0)$ be the core of the spectrum of H_0 , i.e. a minimal Borel support of the measure $E^{\text{ac}}(\cdot; H_0)$. Let $\Delta = (a, b)$ be an open interval such that the closure $\bar{\Delta} = [a, b]$ belongs to $\hat{\sigma}(H_0)$. Suppose that the local wave operators

$$W_{\pm} = W_{\pm}(H, H_0; \Delta) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} E^{\text{ac}}(\Delta; H_0)$$

exist and are complete, i.e. $\text{Ran } W_+ = \text{Ran } W_- = \text{Ran } E^{\text{ac}}(\Delta; H)$. Then the local scattering operator $S = W_+^* W_-$ is unitary in $\text{Ran } E^{\text{ac}}(\Delta; H_0)$ and commutes with H_0 . Consider the direct integral decomposition

$$(3.1) \quad \text{Ran } E^{\text{ac}}(\Delta; H_0) = \int_{\Delta}^{\oplus} \mathfrak{h}(\lambda) d\lambda$$

which diagonalises H_0 . Since S commutes with H_0 , this decomposition also diagonalises S , i.e. in the direct integral (3.1) the operator S can be represented as the operator of multiplication by the unitary operator valued function

$$S(\lambda) : \mathfrak{h}(\lambda) \rightarrow \mathfrak{h}(\lambda), \quad \lambda \in \Delta.$$

The operator $S(\lambda)$ is called the *scattering matrix* for the pair H, H_0 .

Let us give a sufficient condition for the existence and completeness of the wave operators. The condition below is typical for smooth scattering theory. Assume that for $\bar{\Delta} \subset \hat{\sigma}(H_0)$, we have

$$(3.2) \quad \begin{aligned} T_0(z) \text{ and } T(z) \text{ are uniformly continuous in the operator norm} \\ \text{in the rectangle } \text{Re } z \in \bar{\Delta}, \text{Im } z \in (0, 1). \end{aligned}$$

Under these assumptions, the wave operators $W_{\pm}(H, H_0; \Delta)$ exist and are complete (see e.g. [25, Theorem XIII.24]). Moreover, $S(\lambda) - I$ is compact and there exists a unitary ‘‘model operator’’ (see [29, Section 7.7] for the details)

$$\tilde{S}(\lambda) = I - 2iB_0(\lambda)^{1/2}(J - JT(\lambda + i0)J)B_0(\lambda)^{1/2}$$

in \mathcal{K} such that $S(\lambda)$ is unitarily equivalent to $\tilde{S}(\lambda)$, or, more precisely,

$$(3.3) \quad S(\lambda)|_{(\text{Ker}(S(\lambda)-I))^{\perp}} \text{ is unitarily equivalent to } \tilde{S}(\lambda)|_{(\text{Ker}(\tilde{S}(\lambda)-I))^{\perp}}.$$

Thus, all the eigenvalues $e^{i\theta} \neq 1$ of $S(\lambda)$ and $\tilde{S}(\lambda)$ coincide, counting multiplicities. It is easy to see that $\tilde{S}(\lambda)$ is norm-continuous in $\lambda \in \Delta$ and therefore the eigenvalues of $S(\lambda)$ depend continuously on $\lambda \in \Delta$. Below we introduce the eigenvalue counting function, or spectral flow, of $S(\lambda)$.

3.2. The spectral flow of $S(\lambda)$. Let, as above, $\Delta = (a, b)$ be an interval such that (3.3) holds true. Let us define the spectral flow of the family $\{S(\lambda)\}_{\lambda \in [a, b]}$. The spectral flow is an integer valued function μ on $\mathbb{T} \setminus \{1\}$. The naive definition of the spectral flow is

$$(3.4) \quad \begin{aligned} \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) = \\ \langle \text{the number of eigenvalues of } S(\lambda) \text{ which cross } e^{i\theta} \text{ in the anti-clockwise direction} \\ - \langle \text{the number of eigenvalues of } S(\lambda) \text{ which cross } e^{i\theta} \text{ in the clockwise direction} \rangle, \end{aligned}$$

as λ increases monotonically from a to b . Here $\theta \in (0, 2\pi)$ and the eigenvalues are counted with multiplicities taken into account. The eigenvalues of $S(\lambda)$ may cross $e^{i\theta}$ infinitely many times, and thus the above naive definition needs to be replaced by a more robust one. Below we describe one of such possible regularisations.

Let us introduce the notation for the eigenvalue counting function of $S(\lambda)$. For $\theta_1, \theta_2 \in (0, 2\pi)$ denote

$$N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = \sum_{\theta \in [\theta_1, \theta_2)} \dim \text{Ker}(S(\lambda) - e^{i\theta} I)$$

if $\theta_1 < \theta_2$ and

$$N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = -N(e^{i\theta_2}, e^{i\theta_1}; S(\lambda))$$

if $\theta_1 > \theta_2$. We note that by (3.3), one has

$$(3.5) \quad N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = N(e^{i\theta_1}, e^{i\theta_2}; \tilde{S}(\lambda)).$$

Assume first that there exists $\theta_0 \in (0, 2\pi)$ such that $e^{i\theta_0} \notin \sigma(S(\lambda))$ for all $\lambda \in [a, b]$. Then one can define the spectral flow of the family $\{S(\lambda)\}_{\lambda \in [a, b]}$ by

$$(3.6) \quad \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) = N(e^{i\theta}, e^{i\theta_0}; S(b)) - N(e^{i\theta}, e^{i\theta_0}; S(a)).$$

It is evident that this definition is independent of the choice of θ_0 and agrees with the naive definition (3.4) whenever the latter makes sense.

In general, θ_0 as above may not exist. However, using formula (3.5) and the norm continuity of $\tilde{S}(\lambda)$ in λ , one can always find the values $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$ such that for each of the subintervals $\Delta_i = [\lambda_{i-1}, \lambda_i]$, a point θ_0 with the required properties can be found. Thus, the spectral flow of each of the corresponding families $\{S(\lambda)\}_{\lambda \in \Delta_i}$ is well defined. Now one can set

$$(3.7) \quad \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) = \sum_{i=1}^n \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in \Delta_i}).$$

It is not difficult to see that the above definition is independent on the choice of the subintervals Δ_i and agrees with the naive definition (3.4).

3.3. Ξ and the spectrum of $S(\lambda)$.

Theorem 3.1. *Let H_0 and H be as above; assume (1.2). Let $\Delta = (a, b) \subset \hat{\sigma}(H_0)$ be such that (3.2) holds true. Then:*

- (i) *for any $\lambda \in (a, b)$, the index $\Xi(\lambda; H, H_0)$ exists if and only if $-1 \notin \sigma(S(\lambda))$;*
- (ii) *let $\lambda_1, \lambda_2 \in (a, b)$, $\lambda_1 < \lambda_2$; assume that $-1 \notin \sigma(S(\lambda_j))$, $j = 1, 2$. Then*

$$(3.8) \quad \Xi(\lambda_2; H, H_0) - \Xi(\lambda_1; H, H_0) = -\mu(-1; \{S(\lambda)\}_{\lambda \in [\lambda_1, \lambda_2]}),$$

where the quantity in the r.h.s. is the spectral flow defined in Section 3.2.

In fact, the set \mathcal{N} (see (2.23)) in this example can be alternatively described as the set of points $\lambda \in (a, b)$ where $-1 \in \sigma(S(\lambda))$; see (3.13) below.

Suppose that $\lambda \in (a, b)$ is monotonically increasing and as λ passes through λ_0 , an eigenvalue of $S(\lambda)$ crosses -1 . Formula (3.8) shows that the index function $\Xi(\lambda) = \Xi(\lambda; H, H_0)$ has a jump at $\lambda = \lambda_0$, i.e. $\Xi(\lambda_0 + 0) - \Xi(\lambda_0 - 0) = n$. The absolute value $|n|$ of this jump equals the multiplicity of the eigenvalue of $S(\lambda)$ which crosses -1 . The value of n is

positive if the eigenvalue of $S(\lambda)$ crosses -1 in the clockwise direction and it is negative for the anti-clockwise direction.

3.4. Example: Schrödinger operator. Let H_0, H be as in Section 2.7. Then as it is well known, the hypothesis (3.3) is satisfied with $\Delta = (a, b)$ for any $0 < a < b < \infty$. Moreover, $\|S(\lambda) - I\| \rightarrow 0$ as $\lambda \rightarrow \infty$. It follows that in (3.8) one can take $\lambda_2 \rightarrow \infty$. Combining this with Theorem 2.5, we obtain

$$(3.9) \quad \Xi(\lambda; H, H_0) = \mu(-1; \{S(\lambda')\}_{\lambda' \in [\lambda, \infty)}).$$

Remark 3.2. In view of Remark 2.2, one can argue that (3.9) has some similarity to the Birman-Krein formula [6]

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda; H, H_0)}.$$

Indeed, both identities relate some regularisation of (2.10) to the spectrum of the scattering matrix. This similarity becomes more transparent if the Birman-Krein formula is written as

$$(3.10) \quad \xi(\lambda; H, H_0) = -\frac{1}{2\pi} \arg \det S(\lambda) = -\frac{1}{2\pi} \sum_n \theta_n(\lambda) \pmod{1},$$

where $e^{i\theta_n(\lambda)}$ are the eigenvalues of the scattering matrix $S(\lambda)$. Informally speaking, (3.9) is an integer valued version of (3.10).

3.5. Proof of Theorem 3.1. (i) In [21, Theorem 2.2] it is proven that under the assumption (3.3), for all $\lambda \in (a, b)$ we have

$$\sigma_{\text{ess}}(E((-\infty, \lambda); H) - E((-\infty, \lambda); H_0)) = [-\alpha(\lambda), \alpha(\lambda)], \quad \alpha(\lambda) = \frac{1}{2} \|S(\lambda) - I\|.$$

Thus, $\Xi(\lambda; H, H_0)$ exists if and only if $\alpha(\lambda) < 1$. Since $S(\lambda)$ is unitary, this means that $\Xi(\lambda; H, H_0)$ exists if and only if $-1 \notin \sigma(S(\lambda))$, as required.

(ii) We use the notation (2.22). By Theorem 2.4, it suffices to prove that

$$(3.11) \quad \mu(-1; \{S(\lambda)\}_{\lambda \in [\lambda_1, \lambda_2]}) = \Xi(0; J^{-1} + A_0(\lambda_2), J^{-1}) - \Xi(0; J^{-1} + A_0(\lambda_1), J^{-1}),$$

whenever $-1 \notin \sigma(S(\lambda_j))$, $j = 1, 2$. In fact, we will prove a more general statement: if $\theta \in (0, 2\pi)$, then

$$(3.12) \quad \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [\lambda_1, \lambda_2]}) = \Xi(0; J^{-1} + A_0(\lambda_2) + \cot(\theta/2)B_0(\lambda_2), J^{-1}) \\ - \Xi(0; J^{-1} + A_0(\lambda_1) + \cot(\theta/2)B_0(\lambda_1), J^{-1}),$$

whenever $e^{i\theta} \notin \sigma(S(\lambda_j))$, $j = 1, 2$. The proof of this given below relies on the results of [19]. Denote

$$F(\lambda, \theta) = \Xi(0; J^{-1} + A_0(\lambda) + \cot(\theta/2)B_0(\lambda), J^{-1}).$$

In [19, Lemma 5.1], it has been proven that

$$(3.13) \quad \dim \text{Ker}(S(\lambda) - e^{i\theta}I) = \dim \text{Ker}(J^{-1} + A_0(\lambda) + \cot(\theta/2)B_0(\lambda))$$

for all $\lambda \in (a, b)$ and $\theta \in (0, 2\pi)$. It follows [19, Lemma 5.3] that

$$(3.14) \quad N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = F(\lambda, \theta_1) - F(\lambda, \theta_2),$$

if $e^{i\theta_j} \notin \sigma(S(\lambda))$, $j = 1, 2$.

Suppose that the interval $[\lambda_1, \lambda_2] \subset (a, b)$ is chosen such that for some $\theta_0 \in (0, 2\pi)$ and all $\lambda \in [\lambda_1, \lambda_2]$ one has $e^{i\theta_0} \notin \sigma(S(\lambda))$. Then, by (3.13), we have

$$0 \notin \sigma(J^{-1} + A_0(\lambda) + \cot(\theta_0/2)B_0(\lambda))$$

for all $\lambda \in [\lambda_1, \lambda_2]$. From here by Proposition 4.1(ii) and Lemma 4.2 of the next section it follows that $F(\lambda, \theta_0)$ is constant in the interval $\lambda \in [\lambda_1, \lambda_2]$ and thus $F(\lambda_1, \theta_0) = F(\lambda_2, \theta_0)$. From here and (3.14) we get

$$N(e^{i\theta}, e^{i\theta_0}; S(\lambda_2)) - N(e^{i\theta}, e^{i\theta_0}; S(\lambda_1)) = F(\lambda_2, \theta) - F(\lambda_1, \theta).$$

By the definition (3.6) of the spectral flow, it follows

$$(3.15) \quad \mu(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [\lambda_1, \lambda_2]}) = F(\lambda_2, \theta) - F(\lambda_1, \theta).$$

Finally, let $[\lambda_1, \lambda_2] \subset (a, b)$ be an arbitrary interval. According to the definition (3.7), we need to split $[\lambda_1, \lambda_2]$ into subintervals Δ_i and add the expressions in the r.h.s of (3.15) corresponding to these subintervals. This leads to a telescoping sum, and so we see that formula (3.15) extends to an arbitrary interval $[\lambda_1, \lambda_2] \subset (a, b)$. ■

4. PROOF OF THEOREM 2.4

In this section we prove parts (i), (ii), and (iv) of Theorem 2.4.

4.1. Stability of index. Recall the following statement, see e.g. [22, Theorem VIII.20(i)] and [22, Theorem VIII.23(b)]:

Proposition 4.1. *Let A_n and A be selfadjoint operators and suppose that $A_n \rightarrow A$ as $n \rightarrow \infty$ in the norm resolvent sense. Then:*

- (i) *If f is a continuous function on \mathbb{R} with $\lim_{|x| \rightarrow \infty} f(x) = 0$, then $\|f(A_n) - f(A)\| \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *Let $a, b \in \mathbb{R}$, $a < b$, and suppose that $a \notin \sigma(A)$, $b \notin \sigma(A)$. Then*

$$\|E((a, b); A_n) - E((a, b); A)\| \rightarrow 0$$

as $n \rightarrow \infty$.

Next, we need a stability theorem for the index of a pair of projections. Variants of this statement appeared before, see e.g. [9, Theorem 3.12].

Lemma 4.2. *Let P, Q be a Fredholm pair of orthogonal projections in a Hilbert space. Let P_n, Q_n , $n \geq 1$, be orthogonal projections such that*

$$(4.1) \quad \|(P_n - Q_n) - (P - Q)\| \rightarrow 0$$

as $n \rightarrow \infty$. Then for all sufficiently large n , the pair P_n, Q_n is Fredholm and

$$\text{index}(P_n, Q_n) = \text{index}(P, Q).$$

Proof. Since P, Q is a Fredholm pair, there exists $a > 0$ such that

$$\sigma(P - Q) \cap (-1, 1) \subset [-1 + 2a, 1 - 2a].$$

Then $-1 + a$ and $1 - a$ are not in the spectrum of $P - Q$ and so, by Proposition 4.1(ii),

$$(4.2) \quad \|E((1 - a, 2); P_n - Q_n) - E((1 - a, 2); P - Q)\| \rightarrow 0,$$

$$(4.3) \quad \|E((-2, -1 + a); P_n - Q_n) - E((-2, -1 + a); P - Q)\| \rightarrow 0,$$

as $n \rightarrow \infty$. In particular, $\text{rank } E((1 - a, 2); P_n - Q_n)$ and $\text{rank } E((-2, -1 + a); P_n - Q_n)$ are finite for all sufficiently large n and so the pair P_n, Q_n is Fredholm.

Finally, from the definition of index and (2.1) we get

$$\begin{aligned} \text{index}(P, Q) &= \text{rank } E((1 - a, 2); P - Q) - \text{rank } E((-2, -1 + a); P - Q), \\ \text{index}(P_n, Q_n) &= \text{rank } E((1 - a, 2); P_n - Q_n) - \text{rank } E((-2, -1 + a); P_n - Q_n) \end{aligned}$$

and so, applying (4.2), (4.3), we get the required statement. ■

In what follows, we will consider families of Fredholm pairs of projections P_s, Q_s such that the difference $P_s - Q_s$ depends continuously on s in the operator norm. Lemma 4.2 ensures that in this situation $\text{index}(P_s, Q_s)$ is independent of s .

4.2. Existence of Ξ . Assume that $H = H_0 + V$ where $V = G^* J G$ satisfies assumptions (2.11). First we need some notation. For $\lambda \in \mathbb{R}$, denote

$$\begin{aligned} F_0(\lambda) &= GE((-\infty, \lambda); H_0)(GE((-\infty, \lambda); H_0))^*, \\ F(\lambda) &= GE((-\infty, \lambda); H)(GE((-\infty, \lambda); H))^*. \end{aligned}$$

We note that by (2.11), (2.12), the operators $F_0(\lambda), F(\lambda)$ are compact. The existence of $\Xi(\lambda; H, H_0)$ will be derived from the following result of [21]:

Proposition 4.3. [21, Theorem 2.6] *Assume (2.11). Suppose that for some $\lambda \in \mathbb{R}$, the limits $T(\lambda + i0), T_0(\lambda + i0)$ and the derivatives $\frac{d}{d\lambda}F(\lambda), \frac{d}{d\lambda}F_0(\lambda)$ exist in the operator norm. Then the index $\Xi(\lambda; H, H_0)$ exists if and only if $J^{-1} + A_0(\lambda)$ is invertible.*

We need two simple lemmas.

Lemma 4.4. *Let \mathcal{M} be a bounded self-adjoint operator with a bounded inverse and let \mathcal{T} be a compact operator. Denote $\mathcal{A} = \text{Re } \mathcal{T}$, $\mathcal{B} = \text{Im } \mathcal{T}$ and assume that $\mathcal{B} \geq 0$ and $\text{Ker}(\mathcal{M} + \mathcal{A}) = \{0\}$. Then $\mathcal{M} + \mathcal{T}$ has a bounded inverse.*

Proof. Since \mathcal{M} has a bounded inverse and \mathcal{T} is compact, it suffices to prove that $\text{Ker}(\mathcal{M} + \mathcal{T}) = \{0\}$. Suppose that $(\mathcal{M} + \mathcal{T})f = 0$ for some vector f . Then

$$((\mathcal{M} + \mathcal{A})f, f) + i(\mathcal{B}f, f) = 0.$$

Taking imaginary parts yields $(\mathcal{B}f, f) = 0$. Since $\mathcal{B} \geq 0$, it follows that $\mathcal{B}f = 0$. Thus, $(\mathcal{M} + \mathcal{A})f = 0$ and so $f = 0$. ■

Lemma 4.5. *Assume (2.11) and (2.21). Then the derivative $\frac{d}{d\lambda}F_0(\lambda)$ exists in the operator norm for all $\lambda \in \Delta$.*

Proof. From the obvious inequality

$$0 \leq E(\{\lambda\}; H_0) \leq \frac{\varepsilon^2}{(H_0 - \lambda I)^2 + \varepsilon^2 I}, \quad \varepsilon > 0,$$

we get

$$(4.4) \quad 0 \leq GE(\{\lambda\}; H_0)(GE(\{\lambda\}; H_0))^* \leq \varepsilon \operatorname{Im} T_0(\lambda + i\varepsilon), \quad \varepsilon > 0.$$

By (5.2), this implies that $GE(\{\lambda\}; H_0) = 0$ for all $\lambda \in \Delta_0$. Using this, Stone's formula (see e.g. [22, Theorem VII.13]) yields

$$(4.5) \quad ((F_0(b) - F_0(a))f, f) = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \operatorname{Im} (T_0(\lambda + i\varepsilon)f, f) d\lambda = \frac{1}{\pi} \int_a^b (B_0(\lambda)f, f) d\lambda$$

for any interval $(a, b) \subset \Delta_0$ and any $f \in \mathcal{K}$. From here and the continuity of $B_0(\lambda)$ we get that $F_0(\lambda)$ is differentiable in λ in the operator norm. ■

Proof of Theorem 2.4(i) and (ii). (i) is a trivial consequence of the fact that the eigenvalues of $J^{-1} + A_0(\lambda)$ near zero depend continuously on $\lambda \in \Delta$.

(ii) Our aim is to use Proposition 4.3; we need to check that the limits and the derivatives mentioned in the hypothesis of this proposition exist in the operator norm.

1. The limit $T_0(\lambda + i0)$ exists in the operator norm for all $\lambda \in \Delta$; this trivially follows from (2.21). The derivative $\frac{d}{d\lambda} F_0(\lambda)$ exists in the operator norm for all $\lambda \in \Delta$ by Lemma 4.5.

2. Consider $T(\lambda + i0)$ and $\frac{d}{d\lambda} F(\lambda)$. Let us fix a closed interval $\Delta_0 \subset \Delta \setminus \mathcal{N}$. For any $\lambda \in \Delta_0$, we have $\operatorname{Ker}(J^{-1} + A_0(\lambda)) = \{0\}$ and therefore, by Lemma 4.4, the operator $J^{-1} + T_0(\lambda + i0)$ has a bounded inverse.

By the identity (2.16), we have

$$(4.6) \quad T(z) = J^{-1} - J^{-1}(J^{-1} + T_0(z))^{-1}J^{-1},$$

where the operator $J^{-1} + T_0(z)$ has a bounded inverse for all $\operatorname{Im} z \neq 0$. Since $J^{-1} + T_0(\lambda + i0)$ is invertible for all $\lambda \in \Delta_0$, we obtain that $T(z)$ is uniformly continuous in z in the rectangle $\operatorname{Re} z \in \Delta_0$, $\operatorname{Im} z \in (0, 1)$. In particular, the limit $T(\lambda + i0)$ exists in the operator norm for all $\lambda \in \Delta_0$.

Now we can apply Lemma 4.5 with Δ_0 instead of Δ and with $T(z)$ instead of $T_0(z)$. It follows that the derivative $\frac{d}{d\lambda} F(\lambda)$ exists in the operator norm for all $\lambda \in \Delta_0$.

3. Now we can apply Proposition 4.3 to any $\lambda \in \Delta_0$, and the required statement follows. ■

4.3. Proof of Theorem 2.4(iii) and (iv). In Sections 5 and 6 we prove

Theorem 4.6. *Assume (2.11) and suppose that $T_0(z)$ is uniformly continuous in the rectangle $|\operatorname{Re} z| < 1$, $\operatorname{Im} z \in (0, 1)$. Assume that $J^{-1} + A_0(0)$ is invertible. Then the identity*

$$(4.7) \quad \Xi(0; H, H_0) = -\Xi(0; J^{-1} + A_0(0), J^{-1})$$

holds true.

This theorem will be proved by using the Birman-Schwinger principle (Proposition 2.3) and a certain continuous deformation argument.

Now part (iii) of Theorem 2.4 follows directly from Theorem 4.6.

Let us prove Theorem 2.4(iv). Let us fix a closed interval $\Delta_0 \subset \Delta \setminus \mathcal{N}$. Since $A_0(\lambda)$ depends continuously on $\lambda \in \Delta$, by Proposition 4.1(ii) the projection $E((-\infty, 0); J^{-1} + A_0(\lambda))$ depends continuously on $\lambda \in \Delta_0$. Then by Lemma 4.2, the index $\Xi(0; J^{-1} + A_0(\lambda), J^{-1})$ is constant for $\lambda \in \Delta_0$. By the identity (2.24), the index $\Xi(\lambda; H, H_0)$ is constant for $\lambda \in \Delta_0$, as required.

5. PROOF OF THEOREM 4.6

5.1. Notation and preliminaries. Throughout the rest of the paper, we assume the hypothesis of Theorem 4.6. For a function $\omega \in L^\infty(\mathbb{R})$, $\omega \geq 0$, we denote $G(\omega) = G\omega(H_0)^{1/2}$. Since $\omega(H_0)$ is a bounded operator, we have by (2.11)

$$\text{Dom}(H_0 - aI)^{1/2} \subset \text{Dom } G(\omega) \quad \text{and} \quad G(\omega)(H_0 - aI)^{-1/2} \text{ is compact}$$

for any $a < \inf \sigma(H_0)$. Thus, we can define the selfadjoint operator

$$H(\omega) = H_0 + G(\omega)^* JG(\omega)$$

as a form sum and the compact operators

$$(5.1) \quad \begin{aligned} T_0(z; \omega) &= G(\omega)R_0(z)G(\omega)^* = G\omega(H_0)R_0(z)G^*, \\ T(z; \omega) &= G(\omega)(H(\omega) - zI)^{-1}G(\omega)^*. \end{aligned}$$

The definition of $T_0(z; \omega)$ and $T(z; \omega)$ can be made more rigorous similarly to (2.13), (2.14). If the limit $T_0(\lambda + i0; \omega)$ exists, we also denote $A_0(\lambda; \omega) = \text{Re } T_0(\lambda + i0; \omega)$.

Let χ_δ be the characteristic function of the interval $(-\delta, \delta)$ in \mathbb{R} , where $\delta \in (0, 1)$ will be chosen later. For $s \in [0, 1]$, we set $\omega_s(x) = 1 - s\chi_\delta(x)$. Let us discuss the existence of the limit $T_0(\lambda + i0; \omega_s)$. First note that $R_0(z)(1 - \chi_\delta(H_0))$ is analytic in z for $|\text{Re } z| < \delta$. It follows that $T_0(z; \omega_1)$ is analytic in z for $|\text{Re } z| < \delta$. Next, writing $\chi_\delta = 1 - \omega_1$, we get

$$(5.2) \quad T_0(z; \omega_s) = T_0(z) - sT_0(z; \chi_\delta) = (1 - s)T_0(z) + sT_0(z; \omega_1).$$

By the hypothesis of Theorem 4.6, it follows that for any $\delta' < \delta$, the operator $T_0(z; \omega_s)$ is uniformly continuous in the rectangle $|\text{Re } z| < \delta'$, $\text{Im } z \in (0, 1)$ in the operator norm. In particular, the limit $T_0(\lambda + i0; \omega_s)$ exists for all $\lambda \in (-\delta, \delta)$.

5.2. The strategy of the proof of Theorem 4.6. Our aim is to show that for all sufficiently small $\delta > 0$ and all $s \in [0, 1]$ one has

$$(5.3) \quad \Xi(0; H(\omega_s), H_0) = -\Xi(0; J^{-1} + A_0(0; \omega_s), J^{-1}).$$

Clearly, for $s = 0$ this is exactly the required identity (4.7). In order to prove (5.3), we first show that if δ is sufficiently small then the operator $J^{-1} + A_0(0; \omega_s)$ is invertible for all $s \in [0, 1]$. Using this fact, the stability of index and Proposition 4.3, we prove that both sides of (5.3) are independent of $s \in [0, 1]$. Thus it suffices to prove (5.3) for $s = 1$. Finally, for $s = 1$ we derive the identity (5.3) from the Birman-Schwinger principle (Proposition 2.3).

5.3. The limit $\delta \rightarrow 0$. Let us discuss the choice of δ .

Lemma 5.1. *Assume (2.11) and suppose that $T_0(z)$ is uniformly continuous in the rectangle $|\text{Re } z| < 1$, $\text{Im } z \in (0, 1)$. Then*

$$\|A_0(0; \chi_\delta)\| \rightarrow 0 \quad \text{as } \delta \rightarrow +0.$$

Using Lemma 5.1, we will choose δ such that

$$(5.4) \quad \|A_0(0; \chi_\delta)\| < \frac{1}{2} \|(J^{-1} + A_0(0))^{-1}\|^{-1}.$$

Then

$$(5.5) \quad J^{-1} + A_0(0; \omega_s) = J^{-1} + A_0(0) - sA_0(0; \chi_\delta) \text{ is invertible for all } s \in [0, 1].$$

This suffices for our construction.

Proof of Lemma 5.1. 1. From (4.5) we get that $\frac{d}{d\lambda}F_0(\lambda) = \frac{1}{\pi}B_0(\lambda)$ for any $\lambda \in \Delta$. By the spectral theorem, it follows that

$$(5.6) \quad T_0(z; \chi_\delta) = \int_{-\delta}^{\delta} (\lambda - z)^{-1} dF_0(\lambda) = \frac{1}{\pi} \int_{-\delta}^{\delta} (\lambda - z)^{-1} B_0(\lambda) d\lambda$$

for all $\text{Im } z > 0$.

2. By (5.6), we have

$$A_0(0; \chi_\delta) = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{\delta} \frac{B_0(\lambda)\lambda}{\lambda^2 + \varepsilon^2} d\lambda,$$

where, by our assumptions, the limit exists in the operator norm. Next, denote

$$\mathcal{A}(\delta_1, \delta_2) = \frac{1}{\pi} \int_{\delta_1}^{\delta_2} \frac{B_0(\lambda) - B_0(-\lambda)}{\lambda} d\lambda, \quad 0 < \delta_1 < \delta_2 < 1.$$

Let us prove that

$$(5.7) \quad \lim_{\varepsilon \rightarrow +0} \left\| \frac{1}{\pi} \int_{-\delta}^{\delta} \frac{\lambda B_0(\lambda)}{\lambda^2 + \varepsilon^2} d\lambda - \mathcal{A}(\varepsilon, \delta) \right\| = 0$$

for any $\delta > 0$. This is a well known argument, see e.g. [28, Lemma VI.1.2]. Let

$$\varphi(\lambda) = \begin{cases} \frac{\lambda}{\lambda^2+1} & \text{if } |\lambda| < 1, \\ \frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda} & \text{if } 1 \leq |\lambda|, \end{cases}$$

and $\varphi_\varepsilon(\lambda) = \varepsilon^{-1}\varphi(\lambda/\varepsilon)$, $\varepsilon > 0$. Note that φ is odd and $\varphi \in L^1(\mathbb{R})$. We have

$$(5.8) \quad \begin{aligned} \int_{-\delta}^{\delta} \frac{\lambda B_0(\lambda)}{\lambda^2 + \varepsilon^2} d\lambda - \pi \mathcal{A}(\varepsilon, \delta) &= \int_{\mathbb{R}} \frac{\lambda B_0(\lambda) \chi_\delta(\lambda)}{\lambda^2 + \varepsilon^2} d\lambda - \int_{\varepsilon < |\lambda|} \frac{B_0(\lambda) \chi_\delta(\lambda)}{\lambda} d\lambda \\ &= \int_{\mathbb{R}} B_0(\lambda) \chi_\delta(\lambda) \varphi_\varepsilon(\lambda) d\lambda = \int_{\mathbb{R}} B_0(\lambda) \chi_\delta(\lambda) \varphi_\varepsilon(\lambda) d\lambda - B_0(0) \int_{\mathbb{R}} \chi_\delta(\lambda) \varphi_\varepsilon(\lambda) d\lambda \\ &= \int_{\mathbb{R}} (B_0(\lambda) - B_0(0)) \chi_\delta(\lambda) \varphi_\varepsilon(\lambda) d\lambda. \end{aligned}$$

Using the fact that $B_0(\lambda)$ is continuous at $\lambda = 0$ in the operator norm, by a standard argument one checks that the integral in the r.h.s. of (5.8) tends to zero in the operator norm as $\varepsilon \rightarrow +0$. This proves (5.7).

3. By (5.7), the limit $\lim_{\varepsilon \rightarrow +0} \mathcal{A}(\varepsilon, \delta)$ exists in the operator norm and equals $A_0(0; \chi_\delta)$. We can rewrite the last statement as

$$\lim_{\varepsilon \rightarrow +0} (\mathcal{A}(\varepsilon, 1/2) - \mathcal{A}(\delta, 1/2)) = A_0(0; \chi_\delta), \quad \delta < 1/2.$$

Now it is clear that

$$\lim_{\delta \rightarrow +0} A_0(0; \chi_\delta) = \lim_{\delta \rightarrow +0} \lim_{\varepsilon \rightarrow +0} (\mathcal{A}(\varepsilon, 1/2) - \mathcal{A}(\delta, 1/2)) = 0$$

in the operator norm, as required. ■

5.4. The case $s = 1$.

Lemma 5.2. *Assume (2.11) and suppose that $T_0(z)$ is uniformly continuous in the rectangle $|\operatorname{Re} z| < 1$, $\operatorname{Im} z \in (0, 1)$. Assume that $J^{-1} + A_0(0)$ is invertible and let $\delta > 0$ be chosen as in (5.4). Then the index $\Xi(0; H(\omega_1), H_0)$ exists and*

$$(5.9) \quad \Xi(0; H(\omega_1), H_0) = -\Xi(0; J^{-1} + A_0(0; \omega_1), J^{-1}).$$

Proof. 1. Let $\mathcal{H}_0 = \operatorname{Ran} E(\mathbb{R} \setminus (-\delta, \delta); H_0)$. It is easy to see that the subspace \mathcal{H}_0 reduces both H_0 and $H(\omega_1)$ (i.e. both H_0 and $H(\omega_1)$ commute with $E(\mathbb{R} \setminus (-\delta, \delta); H_0) = \omega_1(H_0)$). Along with H_0 , $H(\omega_1)$, $G(\omega_1)$, consider the operators $h_0 = H_0|_{\mathcal{H}_0}$, $h = H(\omega_1)|_{\mathcal{H}_0}$, $g = G(\omega_1)|_{\mathcal{H}_0}$. We have $(-\delta, \delta) \cap \sigma(h_0) = \emptyset$. Since $h = h_0 + g^*Jg$ and g^*Jg is h_0 -form compact, we also have $(-\delta, \delta) \cap \sigma_{\text{ess}}(h) = \emptyset$. Next, let $t_0(z) = g(h_0 - zI)^{-1}g^*$. Note that $t_0(z) = T_0(z; \omega_1)$, $\operatorname{Im} z \neq 0$, and so

$$(5.10) \quad t_0(0) = \operatorname{Re} t_0(0) = A_0(0; \omega_1).$$

By our choice (5.4) of δ , it follows (cf. (5.5)) that the operator $J^{-1} + t_0(0)$ is invertible. Thus, we can apply Proposition 2.3 to the pair of operators h_0, h . This yields that $0 \notin \sigma(h)$ and

$$(5.11) \quad \Xi(0; h, h_0) = -\Xi(0; J^{-1} + t_0(0), J^{-1}),$$

where the indices Ξ on both sides exist.

2. Let us show that (5.11) is equivalent to (5.9). By (5.10), the r.h.s. of (5.11) coincides with the r.h.s. of (5.9). Consider the l.h.s. With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ we have (here and in what follows $\mathbb{R}_- = (-\infty, 0)$):

$$\begin{aligned} E(\mathbb{R}_-; H_0) &= E(\mathbb{R}_-; h_0) \oplus E((-\delta, 0); H_0), \\ E(\mathbb{R}_-; H(\omega_1)) &= E(\mathbb{R}_-; h) \oplus E((-\delta, 0); H_0), \end{aligned}$$

and therefore

$$E(\mathbb{R}_-; H(\omega_1)) - E(\mathbb{R}_-; H_0) = (E(\mathbb{R}_-; h) - E(\mathbb{R}_-; h_0)) \oplus 0.$$

It follows that the index $\Xi(0; H(\omega_1), H_0)$ exists if and only if $\Xi(0; h, h_0)$ exists and if these indices exist, they coincide. Thus, from (5.11) we get that $\Xi(0; H(\omega_1), H_0)$ exists and (5.9) holds true. ■

5.5. The proof of Theorem 4.6.

The key element in our proof is

Theorem 5.3. *Assume (2.11) and suppose that $T_0(z)$ is uniformly continuous in the rectangle $|\operatorname{Re} z| < 1$, $\operatorname{Im} z \in (0, 1)$. Assume that $J^{-1} + A_0(0)$ is invertible and let $\delta > 0$ be chosen as in (5.4). Then the spectral projections*

$$E(\mathbb{R}_-; H(\omega_s)) \quad \text{and} \quad E(\mathbb{R}_-; J^{-1} + A_0(0; \omega_s))$$

are continuous in $s \in [0, 1]$ in the operator norm.

Theorem 5.3 is proven in Section 6.

Proof of Theorem 4.6. Let δ be chosen as in (5.4).

1. By Proposition 2.1, the index $\Xi(0; J^{-1} + A_0(0; \omega_s), J^{-1})$ exists for all s . Thus, by Lemma 4.2 and Theorem 5.3, the index $\Xi(0; J^{-1} + A_0(0; \omega_s), J^{-1})$ is independent of $s \in [0, 1]$.

2. Let us prove that the index $\Xi(0; H(\omega_s), H_0)$ exists for any $s \in [0, 1]$. We will use part (ii) of Theorem 2.4 (this is not a circular argument: part (ii) has already been proven in Section 4.2). Let us apply Theorem 2.4(ii) with the operators $H_0, H(\omega_s), G(\omega_s)$ instead of H_0, H, G . As discussed in Section 5.1, for any $\delta' < \delta$ the operator $T_0(z; \omega_s)$ is uniformly continuous in z for $|\operatorname{Re} z| < \delta', \operatorname{Im} z \in (0, 1)$. Thus, the hypothesis of Theorem 2.4 is satisfied with $\Delta = (-\delta', \delta')$. By (5.5), we have $0 \notin \mathcal{N}$ and so the index $\Xi(0; H(\omega_s), H_0)$ exists for any $s \in [0, 1]$.

3. From the previous step of the proof, using Lemma 4.2 and Theorem 5.3 we obtain that $\Xi(0; H(\omega_s), H_0)$ is independent of $s \in [0, 1]$.

4. Using Lemma 5.2, we obtain

$$\begin{aligned} \Xi(0; H, H_0) &= \Xi(0; H(\omega_0), H_0) = \Xi(0; H(\omega_1), H_0) = -\Xi(0; J^{-1} + A_0(0; \omega_1), J^{-1}) \\ &= -\Xi(0; J^{-1} + A_0(0; \omega_0), J^{-1}) = -\Xi(0; J^{-1} + A_0(0), J^{-1}), \end{aligned}$$

which proves (4.7). Of course, this argument also shows that (5.3) holds true for any $s \in [0, 1]$. ■

6. PROOF OF THEOREM 5.3

6.1. **Estimates for $T(z; \omega_s)$.** We use the notation (5.1).

Lemma 6.1. *Assume (2.11) and suppose that $T_0(z)$ is uniformly continuous in the rectangle $|\operatorname{Re} z| < 1, \operatorname{Im} z \in (0, 1)$. Assume that $J^{-1} + A_0(0)$ is invertible and let $\delta > 0$ be chosen as in (5.4). Then for some $C > 0$ the estimates*

$$(6.1) \quad \|T(it; \omega_s)\| \leq C, \quad t \in (0, 1), \quad s \in [0, 1],$$

$$(6.2) \quad \|T(it; \omega_s) - T(it; \omega_r)\| \leq C|s - r|, \quad t \in (0, 1), \quad s, r \in [0, 1],$$

hold true.

Proof. 1. Similarly to (2.16), we have

$$(J^{-1} + T_0(z; \omega_s))(J - JT(z; \omega_s)J) = (J - JT(z; \omega_s)J)(J^{-1} + T_0(z; \omega_s)) = I$$

and therefore

$$(6.3) \quad T(z; \omega_s) = J^{-1} - J^{-1}(J^{-1} + T_0(z; \omega_s))^{-1}J^{-1}$$

for all $s \in [0, 1]$ and all $\operatorname{Im} z \neq 0$.

2. By (5.5), the operator $J^{-1} + A_0(0; \omega_s)$ is invertible for all $s \in [0, 1]$. By Lemma 4.4, it follows that $J^{-1} + T_0(+i0; \omega_s)$ is also invertible for all $s \in [0, 1]$. Since the operator $J^{-1} + T_0(it; \omega_s)$ is uniformly continuous in $s \in [0, 1], t \in (0, 1)$ in the operator norm, it follows that the norm of the inverse $(J^{-1} + T_0(it; \omega_s))^{-1}$ is uniformly bounded for $s \in [0, 1], t \in (0, 1)$. By (6.3), we obtain the bound (6.1).

3. Using (6.3), for any $z \in \mathbb{C} \setminus \mathbb{R}$ we obtain

$$\begin{aligned} (6.4) \quad T(z; \omega_s) - T(z; \omega_r) &= J^{-1}(J^{-1} + T_0(z; \omega_r))^{-1}J^{-1} - J^{-1}(J^{-1} + T_0(z; \omega_s))^{-1}J^{-1} \\ &= J^{-1}(J^{-1} + T_0(z; \omega_s))^{-1}(T_0(z; \omega_s) - T_0(z; \omega_r))(J^{-1} + T_0(z; \omega_r))^{-1}J^{-1} \\ &= (r - s)(I - T(z; \omega_s)J)T_0(z; \chi_\delta)(I - JT(z; \omega_r)). \end{aligned}$$

Since $T_0(z; \chi_\delta) = T_0(z) - T_0(z; \omega_1)$, the limit $T_0(+i0; \chi_\delta)$ exists in the operator norm and therefore $\|T_0(it; \chi_\delta)\|$ is uniformly bounded for $t \in (0, 1)$. Combining this with (6.4) and the estimate (6.1), we obtain (6.2). ■

6.2. Proof of Theorem 5.3.

Lemma 6.2. *Under the assumptions of Theorem 5.3, for all $s \in [0, 1]$ one has*

$$(6.5) \quad \text{Ker } H(\omega_s) = \text{Ker } H_0.$$

Proof. Since $T_0(it; \omega_s)$ is bounded uniformly in $t \in (0, 1)$, we obtain, as in (4.4):

$$G(\omega_s)E(\{0\}; H_0) = 0.$$

Thus, for any $f \in \text{Ker } H_0$ we get $H(\omega_s)f = H_0f + G(\omega_s)^*JG(\omega_s)f = 0$. We see that $\text{Ker } H_0 \subset \text{Ker } H(\omega_s)$. Conversely, using the bound (6.1) in the same way we obtain $G(\omega_s)E(\{0\}; H(\omega_s)) = 0$. It follows that for any $f \in \text{Ker } H(\omega_s)$ we have $H_0f = H(\omega_s)f - G(\omega_s)^*JG(\omega_s)f = 0$ and so $\text{Ker } H(\omega_s) \subset \text{Ker } H_0$. ■

Let us define the functions χ_-, ζ, ψ as follows:

$$\chi_-(x) = \begin{cases} 1, & x < 0, \\ 1/2, & x = 0, \\ 0, & x > 0, \end{cases} \quad \zeta(x) = \begin{cases} \frac{1}{\pi} \tan^{-1}(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and $\psi(x) = \chi_-(x) + \zeta(x)$. By definition, $\psi \in C(\mathbb{R})$, $\psi(x) \rightarrow 0$ as $x \rightarrow \infty$ and $\psi(x) \rightarrow 1$ as $x \rightarrow -\infty$. The key statement in the proof of Theorem 5.3 is

Lemma 6.3. *Under the assumptions of Theorem 5.3, the operator $\zeta(H(\omega_s))$ depends continuously on $s \in [0, 1]$ in the operator norm.*

The proof of Lemma 6.3 is given in Sections 6.3, 6.4. Now we are ready to provide

Proof of Theorem 5.3. 1. Clearly, $A_0(0; \omega_s)$ is continuous in s in the operator norm. By our choice of δ the operator $J^{-1} + A_0(0; \omega_s)$ is invertible for all $s \in [0, 1]$. Thus, the continuity of the projection $E(\mathbb{R}_-; J^{-1} + A_0(0; \omega_s))$ follows directly from Proposition 4.1(ii).

2. Consider the projection $E(\mathbb{R}_-; H(\omega_s))$. Using (6.5), we obtain

$$E(\mathbb{R}_-; H(\omega_s)) = \chi_-(H(\omega_s)) + \frac{1}{2}E(\{0\}; H(\omega_s)) = \psi(H(\omega_s)) - \zeta(H(\omega_s)) + \frac{1}{2}E(\{0\}; H_0).$$

By Lemma 6.3, it remains to prove that $\psi(H(\omega_s))$ depends continuously on $s \in [0, 1]$ in the operator norm.

3. Let us prove that $H(\omega_s)$ is continuous in s in the norm resolvent sense. For any $z \in \mathbb{C} \setminus \mathbb{R}$, similarly to (2.15), we have the iterated resolvent identity

$$(6.6) \quad (H(\omega_s) - zI)^{-1} - R_0(z) = -\omega_s(H_0)^{1/2}(GR_0(\bar{z}))^*(J - JT(z; \omega_s)J)GR_0(z)\omega_s(H_0)^{1/2}.$$

Clearly, $\omega_s(H_0)^{1/2}$ depends continuously on s in the operator norm. By (6.4), the operator $T(z; \omega_s)$ depends continuously on s in the operator norm. It follows that $(H(\omega_s) - zI)^{-1}$ depends continuously on s in the operator norm.

4. It is easy to see that there exists $a \in \mathbb{R}$ such that $a < \inf(\sigma(H(\omega_s)))$ for all $s \in [0, 1]$. Let $\tilde{\psi} \in C(\mathbb{R})$ be such that $\tilde{\psi}(x) = \psi(x)$ for all $x \geq a$ and $\tilde{\psi}(x) = 0$ for $x \leq a - 1$. Then

$\psi(H(\omega_s)) = \tilde{\psi}(H(\omega_s))$ for all s . By Proposition 4.1(i), the operator $\tilde{\psi}(H(\omega_s))$ is continuous in s in the operator norm. This proves the required statement. ■

6.3. Proof of Lemma 6.3. We will use the following elementary representation for the function ζ :

$$\zeta(x) = \frac{1}{\pi} \tan^{-1}(1/x) = \frac{1}{2\pi} \int_{-1}^1 \frac{dt}{x - it}, \quad x \neq 0.$$

Using the resolvent identity (6.6), from this representation we formally obtain:

$$\begin{aligned} (6.7) \quad 2\pi(\zeta(H_0) - \zeta(H(\omega_s))) &= \int_{-1}^1 ((H(\omega_s) - it)^{-1} - R_0(it)) dt \\ &= \omega_s(H_0)^{1/2} \int_{-1}^1 (GR_0(-it))^*(J - JT(it; \omega_s)J)GR_0(it)\omega_s(H_0)^{1/2} dt. \end{aligned}$$

Of course, the validity of this formula and the convergence of the integral in the r.h.s. have to be rigorously justified; this will be done below. We note that, by (6.5), the value $\zeta(0)$ is unimportant; the contribution from this value cancels out in the l.h.s. of (6.7).

Let us denote by X_+ and X_- the operators from $L^2((-1, 1); \mathcal{K})$ to \mathcal{H} defined by

$$(6.8) \quad X_{\pm}f = \int_{-1}^1 (GR_0(\mp it))^* f(t) dt,$$

where f belongs to the dense set of functions vanishing in a neighbourhood of $t = 0$. In what follows we prove that X_{\pm} extend to bounded operators from $L^2((-1, 1); \mathcal{K})$ to \mathcal{H} .

Next, denote by $Y(\omega_s)$ the operator in $L^2((-1, 1); \mathcal{K})$ defined by

$$(6.9) \quad (Y(\omega_s)f)(t) = (J - JT(it; \omega_s)J)f(t), \quad t \neq 0.$$

Note that $T(-it; \omega_s) = T(it; \omega_s)^*$. By Lemma 6.1, the operators $Y(\omega_s)$, are bounded for all s and

$$(6.10) \quad \|Y(\omega_s) - Y(\omega_r)\| \leq C|s - r|.$$

In what follows we prove

Lemma 6.4. (i) *The operators X_{\pm} defined by (6.8) extend to bounded operators from $L^2((-1, 1); \mathcal{K})$ to \mathcal{H} .*

(ii) *The identity*

$$(6.11) \quad 2\pi(\zeta(H_0) - \zeta(H(\omega_s))) = \omega_s(H_0)^{1/2} X_+ Y(\omega_s) X_-^* \omega_s(H_0)^{1/2}$$

holds true.

Now we can provide

Proof of Lemma 6.3. Since $\omega_s(H_0)^{1/2}$ depend continuously on s in the operator norm, from (6.10) and (6.11) we immediately obtain the required statement. ■

6.4. Proof of Lemma 6.4. (i) We will prove the boundedness of X_+ ; the operator X_- can be considered in the same way. Let

$$\mathcal{D} = C_0^\infty((-1, 1) \setminus \{0\}; \mathcal{K});$$

clearly, \mathcal{D} is dense in $L^2((-1, 1); \mathcal{K})$. For $f \in \mathcal{D}$, using the resolvent identity

$$(z_1 - z_2)R_0(z_1)R_0(z_2) = (R_0(z_1) - R_0(z_2)),$$

we obtain

$$\begin{aligned} (6.12) \quad \|X_+f\|^2 &= \int_{-1}^1 dt_1 \int_{-1}^1 dt_2 ((GR_0(-it_2))^* f(t_2), (GR_0(-it_1))^* f(t_1)) \\ &= \int_{-1}^1 dt_1 \int_{-1}^1 dt_2 \frac{i}{t_1 + t_2} ((T_0(-it_1) - T_0(it_2))f(t_2), f(t_1)). \end{aligned}$$

Thus, we are led to the consideration of the operator in $L^2((-1, 1); \mathcal{K})$ with the integral kernel $(T_0(-it_1) - T_0(it_2))/(t_1 + t_2)$. For $f \in \mathcal{D}$, let us define

$$(Mf)(t_1) = \text{v.p.} \int_{-1}^1 \frac{f(t_2)}{t_1 + t_2} dt_2.$$

Up to the change of variables $t \mapsto (-t)$, this is the operator of the Hilbert transform restricted onto the interval $(-1, 1)$. Since the Hilbert transform is bounded in L^2 , the operator M is bounded in $L^2((-1, 1); \mathcal{K})$.

Next, let \mathbb{T} be the operator in $L^2((-1, 1); \mathcal{K})$ given by

$$(\mathbb{T}f)(t) = T_0(it)f(t), \quad t \neq 0.$$

Since the norm of $T_0(it)$ is uniformly bounded, the operator \mathbb{T} is bounded. The r.h.s. of (6.12) can be rewritten as

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \left(\iint_{\substack{|t_1| \leq 1, |t_2| \leq 1 \\ |t_1 + t_2| > \varepsilon}} \frac{i(T_0(-it_1)f(t_2), f(t_1))}{t_1 + t_2} dt_1 dt_2 - \iint_{\substack{|t_1| \leq 1, |t_2| \leq 1 \\ |t_1 + t_2| > \varepsilon}} \frac{i(T_0(it_2)f(t_2), f(t_1))}{t_1 + t_2} dt_1 dt_2 \right) \\ = i(\mathbb{T}^* Mf, f) - i(M\mathbb{T}f, f), \quad f \in \mathcal{D}, \end{aligned}$$

and therefore X_+ extends to a bounded operator.

(ii) For any $\varepsilon > 0$, let

$$\zeta_\varepsilon(x) = \frac{1}{2\pi} \int_{-1}^{-\varepsilon} \frac{dt}{x - it} + \frac{1}{2\pi} \int_{\varepsilon}^1 \frac{dt}{x - it},$$

and let $X_\pm(\varepsilon) : L^2((-1, 1); \mathcal{K}) \rightarrow \mathcal{H}$ be the operators

$$X_\pm(\varepsilon)f = \int_{-1}^{-\varepsilon} (GR_0(\mp it))^* f(t) dt + \int_{\varepsilon}^1 (GR_0(\mp it))^* f(t) dt.$$

Since the norm $\|GR_0(it)\|$ is uniformly bounded for $|t| > \varepsilon$, it is clear directly from the definition of $X_\pm(\varepsilon)$ that these operators are bounded for each $\varepsilon > 0$. Applying the resolvent identity (6.6), by a calculation similar to (6.7) we see that

$$(6.13) \quad 2\pi(\zeta_\varepsilon(H_0) - \zeta_\varepsilon(H(\omega_s))) = \omega_s(H_0)^{1/2} X_+(\varepsilon) Y(\omega_s) X_-(\varepsilon)^* \omega_s(H_0)^{1/2}$$

holds true. Let us prove that both sides of (6.13) converge weakly to the corresponding sides of (6.11) as $\varepsilon \rightarrow +0$.

Since ζ_ε is uniformly bounded and $\zeta_\varepsilon(x) \rightarrow \zeta(x)$ as $\varepsilon \rightarrow +0$ for all $x \in \mathbb{R}$ (it is here that the choice of the value $\zeta(0)$ is important) we get that the l.h.s. of (6.13) converges weakly to the l.h.s. of (6.11).

Next, since X_+^* and X_-^* are bounded by part (i) of the Lemma, for any $g \in \mathcal{H}$ we have

$$(X_\pm^*g)(t) = GR_0(\mp it)g, \quad t \neq 0,$$

and

$$\int_{-1}^1 \|GR_0(it)g\|_{\mathcal{K}}^2 dt < \infty.$$

It follows that for any $g \in \mathcal{H}$

$$\|(X_\pm^*(\varepsilon) - X_\pm^*)g\|^2 = \int_{-\varepsilon}^{\varepsilon} \|GR_0(it)g\|_{\mathcal{K}}^2 dt \rightarrow 0$$

as $\varepsilon \rightarrow +0$. Thus, $X_\pm^*(\varepsilon)$ converges strongly to X_\pm^* as $\varepsilon \rightarrow +0$. It follows that the r.h.s. of (6.13) converges weakly to the r.h.s. of (6.11). This completes the proof. ■

ACKNOWLEDGEMENTS

The author is grateful to Serge Richard, Dmitri Yafaev and Nikolai Filonov for careful critical reading of the manuscript and for offering a number of useful remarks.

REFERENCES

- [1] S. ALAMA, P. A. DEIFT, R. HEMPEL, *Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$* . Comm. Math. Phys. **121** (1989), no. 2, 291–321.
- [2] J. AVRON, R. SEILER, B. SIMON, *The index of a pair of projections* J. Funct. Anal. **120** (1994), no. 1, 220–237.
- [3] N. A. AZAMOV, A. L. CAREY, F. A. SUKOCHEV, *The spectral shift function and spectral flow*, Comm. Math. Phys. **276** (2007), no. 1, 51–91.
- [4] N. A. AZAMOV, A. L. CAREY, P. G. DODDS, F. A. SUKOCHEV, *Operator integrals, spectral shift and spectral flow*, Canadian J. of Mathematics, **61** (2009), no. 2, 241–263.
- [5] M. SH. BIRMAN, *The spectrum of singular boundary problems*, Amer. Math. Soc. Transl. (2) **53** (1966), 23–80.
- [6] M. SH. BIRMAN, M. G. KREIN, *On the theory of wave operators and scattering operators*, Soviet Math. Dokl. **3** (1962), 740–744.
- [7] V. BRUNEAU, A. PUSHNITSKI, G. RAIKOV, *Spectral Shift Function in Strong Magnetic Fields*, St. Petersburg Math. J. **16** (2004), no. 1, 207–238.
- [8] P. A. DEIFT, R. HEMPEL, *On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$* , Commun. Math. Phys. **103** (1986), 461–490.
- [9] F. GESZTESY, K. MAKAROV, *The Ξ operator and its relation to Krein's spectral shift function*, J. Anal. Math. **81** (2000), 139–183.
- [10] F. GESZTESY, K. MAKAROV, S. NABOKO, *The spectral shift operator*, in Mathematical Results in Quantum Mechanics, J. Dittrich, P. Exner, and M. Tater (eds.), Operator Theory: Advances and Applications, Vol. 108, Birkhäuser, Basel, 1999, 59–90.
- [11] R. HEMPEL, *A left-indefinite generalized eigenvalue problem for Schrödinger operators*, Habilitation thesis, Munich University, 1987.
- [12] R. HEMPEL, *On the asymptotic distribution of the eigenvalue branches of a Schrödinger operator $H - \lambda W$ in a spectral gap of H* , J. Reine Angew. Math. **399** (1989), 38–59.

- [13] R. HEMPEL, *Eigenvalues of Schrödinger operators in gaps of the essential spectrum — an overview*, Contemporary Mathematics, **458** (2008), Amer. Math. Soc.
- [14] M. KLAUS, *Some applications of the Birman-Schwinger principle*, Helv. Phys. Acta. **55** (1982), 49–68.
- [15] D. HUNDERTMARK, B. SIMON, *Eigenvalue bounds in the gaps of Schrödinger operators and Jacobi matrices*, J. Math. Anal. Appl. **340** (2008), 892–900.
- [16] V. KOSTRYKIN, K. MAKAROV, *On Krein’s example*, Proc. Amer. Math. Soc. **136** (2008), no. 6, 2067–2071.
- [17] V. KOSTRYKIN, K. A. MAKAROV, A. SKRIPKA, *The Birman-Schwinger principle in von Neumann algebras of finite type*, J. Funct. Anal. **247** (2007), no. 2, 492–508.
- [18] M. G. KREĪN, *On the trace formula in perturbation theory (Russian)*, Mat. Sb. **33 (75)** (1953), no. 3, 597–626.
- [19] A. PUSHNITSKI, *The spectral shift function and the invariance principle*, J. Functional Analysis, **183** (2001), no. 2, 269–320.
- [20] A. PUSHNITSKI, *Operator theoretic methods for the eigenvalue counting function in spectral gaps*, Ann. Henri Poincaré, **10** (2009), no. 4, 793–822.
- [21] A. PUSHNITSKI, *Spectral theory of discontinuous functions of self-adjoint operators: essential spectrum*, to appear in Integral Eq. Oper. Theory.
- [22] M. REED, B. SIMON, *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, 1972.
- [23] M. REED, B. SIMON, *Methods of modern mathematical physics. II. Fourier Analysis, Self-adjointness*. Academic Press, 1975.
- [24] M. REED, B. SIMON, *Methods of modern mathematical physics. III. Scattering theory*. Academic Press, 1979.
- [25] M. REED, B. SIMON, *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, 1978.
- [26] J. SCHWINGER, *On the bound states of a given potential*, Proc. Natl. Acad. Sci. USA **47** (1961), 122–129.
- [27] A. V. SOBOLEV, *Efficient bounds for the spectral shift function*, Ann. Inst. H. Poincaré Phys. Théor. **58** (1993), no. 1, 55–83.
- [28] E. M. STEIN, G. WEISS, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.
- [29] D. R. YAFAEV, *Mathematical scattering theory. General theory*. American Mathematical Society, Providence, RI, 1992.

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