

SPECTRAL THEORY OF DISCONTINUOUS FUNCTIONS OF SELF-ADJOINT OPERATORS: ESSENTIAL SPECTRUM

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ABSTRACT. Let H_0 and H be self-adjoint operators in a Hilbert space. In the scattering theory framework, we describe the essential spectrum of the difference $\varphi(H) - \varphi(H_0)$ for piecewise continuous functions φ . This description involves the scattering matrix for the pair H, H_0 .

1. INTRODUCTION

Let H_0 and H be self-adjoint operators in a Hilbert space \mathcal{H} and suppose that the difference $V = H - H_0$ is a compact operator. If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which tends to zero at infinity then a well known simple argument shows that the difference

$$\varphi(H) - \varphi(H_0)$$

is a compact operator. Moreover, there is a large family of results that assert that if the function φ is sufficiently “nice” and V belongs to some Schatten–von Neumann class of compact operators, then $\varphi(H) - \varphi(H_0)$ also belongs to this class. See [10, 3] or the survey [4] for early results of this type; they were later made much more precise by V. V. Peller, see [12, 13]. See also [1, 11] for some recent progress in this area.

In all of the above mentioned results, the function φ is assumed to be continuous. If φ has discontinuities on the essential spectrum of H_0 , then the difference $\varphi(H) - \varphi(H_0)$ in general fails to be compact even if V is a rank one operator; see [10, 9]. In this paper we study the essential spectrum of $\varphi(H) - \varphi(H_0)$ for piecewise continuous functions φ . Some initial results in this direction have been obtained in [17]; we begin by describing these results.

For a Borel set $\Lambda \subset \mathbb{R}$, we denote by $E(\Lambda)$ (resp. $E_0(\Lambda)$) the spectral projection of H (resp. H_0) corresponding to the set Λ . If Λ is an interval, say $\Lambda = [a, b)$, we write $E[a, b)$ instead of $E([a, b))$ in order to make our formulas more readable. In [17], under some assumptions typical for smooth scattering theory, it was proven that for compact V one has

$$(1.1) \quad \sigma_{\text{ess}}(E(-\infty, \lambda) - E_0(-\infty, \lambda)) = \left[-\frac{1}{2}\|S(\lambda) - I\|, \frac{1}{2}\|S(\lambda) - I\|\right],$$

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where $S(\lambda)$ is the scattering matrix for the pair H_0, H .

In this paper, we prove the following generalisation of (1.1). Assume that for some $\lambda \in \mathbb{R}$ the derivatives

$$(1.2) \quad \frac{d}{d\lambda}|V|^{1/2}E_0(-\infty, \lambda)|V|^{1/2}, \quad \frac{d}{d\lambda}|V|^{1/2}E(-\infty, \lambda)|V|^{1/2}$$

exist in the operator norm. Then we prove (see Theorem 2.1) that the limit

$$(1.3) \quad \alpha(\lambda) = \lim_{\varepsilon \rightarrow +0} \frac{\pi}{2\varepsilon} \|E_0(\lambda - \varepsilon, \lambda + \varepsilon)VE(\lambda - \varepsilon, \lambda + \varepsilon)\|$$

exists and the identity

$$(1.4) \quad \sigma_{\text{ess}}(E(-\infty, \lambda) - E_0(-\infty, \lambda)) = [-\alpha(\lambda), \alpha(\lambda)]$$

holds true. If the standard assumptions of either trace class or smooth variant of scattering theory are fulfilled, we prove (see Section 2.4) that $\alpha(\lambda) = \frac{1}{2}\|S(\lambda) - I\|$. Thus, (1.1) becomes a corollary of (1.4). Using (1.4), we obtain the following results:

- (i) Applying (1.4) in the trace class framework, we prove (see Section 2.3 for the definition of the core of the absolutely continuous spectrum):

Theorem. *Let V be a trace class operator. Then for a.e. $\lambda \in \mathbb{R}$ the relation (1.4) holds true and for a.e. λ in the core of the absolutely continuous spectrum of H_0 , the relation (1.1) holds true.*

This is stated as Theorem 2.3 below.

- (ii) In Section 2.5 we describe the essential spectrum of the difference $\varphi(H) - \varphi(H_0)$ for piecewise continuous functions φ .
- (iii) In Section 2.6 we give a convenient criterion for $E_0(-\infty, \lambda), E(-\infty, \lambda)$ to be a Fredholm pair of projections.
- (iv) In Sections 2.7, 2.8, we give some applications to the Schrödinger operator.

In the proof of (1.4) we use the technique of [17] with some minor improvements. In (ii) above, we follow the method of proof used by S. Power in his description [15] of the essential spectrum of Hankel operators with piecewise continuous symbols.

Finally, we note that a description of the absolutely continuous spectrum of the difference

$$(1.5) \quad E(-\infty, \lambda) - E_0(-\infty, \lambda)$$

is also available in terms of the spectrum of the scattering matrix; see [17, 19].

2. MAIN RESULTS

2.1. The definition of the operator H . Let H_0 be a self-adjoint operator in a Hilbert space \mathcal{H} . We would like to introduce a self-adjoint perturbation V and define the sum $H = H_0 + V$. Informally speaking, we would like to define $H_0 + V$ as a quadratic form sum; however, since we do not assume H_0 or V to be semi-bounded, the language of quadratic forms is not applicable here. The definition of $H_0 + V$

requires some care; we follow the approach which goes back at least to [8] and was developed in more detail in [27, Sections 1.9, 1.10]. We assume that V is factorised as $V = G^* JG$, where G is an operator from \mathcal{H} to an auxiliary Hilbert space \mathcal{K} and J is an operator in \mathcal{K} . We assume that

$$(2.1) \quad \begin{aligned} & J = J^* \text{ is bounded in } \mathcal{K}, \\ & \text{Dom}|H_0|^{1/2} \subset \text{Dom } G \text{ and } G(|H_0| + I)^{-1/2} \text{ is compact.} \end{aligned}$$

We denote by (\cdot, \cdot) and $\|\cdot\|$ the inner product and the norm in \mathcal{H} and by $(\cdot, \cdot)_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$ the inner product and the norm in \mathcal{K} . In applications a factorisation $V = G^* JG$ with these properties often arises naturally from the structure of the problem. In any case, one can always take $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$ and $J = \text{sign}(V)$.

For $z \in \mathbb{C} \setminus \sigma(H_0)$, we denote $R_0(z) = (H_0 - zI)^{-1}$. Formally, we define the operator $T_0(z)$ (sandwiched resolvent) by setting

$$(2.2) \quad T_0(z) = GR_0(z)G^*;$$

more precisely, this means

$$T_0(z) = G(|H_0| + I)^{-1/2}(|H_0| + I)R_0(z)(G(|H_0| + I)^{-1/2})^*.$$

By (2.1), the operator $T_0(z)$ is compact. It can be shown (see [27, Sections 1.9, 1.10]) that under the assumption (2.1) the operator $I + T_0(z)J$ has a bounded inverse for all $z \in \mathbb{C} \setminus \mathbb{R}$ and that the operator valued function

$$(2.3) \quad R(z) = R_0(z) - (GR_0(\bar{z}))^* J(I + T_0(z)J)^{-1} GR_0(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

is a resolvent of a self-adjoint operator; we denote this self-adjoint operator by H . Thus, formula (2.3), which is usually treated as a resolvent identity for H_0 and $H = H_0 + V$, is now accepted as the definition of H . If V is bounded, then the above defined operator H coincides with the operator sum $H_0 + V$. If H_0 is semi-bounded from below, then (2.1) means that V is H_0 -form compact and then H coincides with the quadratic form sum $H_0 + V$ (in the sense of the KLMN Theorem, see [21]). In general, we have

$$(2.4) \quad (f_0, Hf) = (H_0 f_0, f) + (JGf_0, Gf)_{\mathcal{K}}, \quad \forall f_0 \in \text{Dom } H_0, \quad \forall f \in \text{Dom } H.$$

Finally, it is not difficult to check that by (2.1) and (2.3), the resolvent $R(z)$ can be written as

$$(2.5) \quad R(z) = (|H_0| + I)^{-1/2} \mathcal{B}(z) (|H_0| + I)^{-1/2}$$

with a bounded operator $\mathcal{B}(z)$. In particular, this implies that

$$(2.6) \quad \text{the operator } GR(z) \text{ is well defined and compact for any } z \in \mathbb{C} \setminus \mathbb{R}.$$

2.2. Main result. Let us fix a “reference point” $\nu \in \mathbb{R}$ and for $\lambda > \nu$ denote

$$(2.7) \quad \begin{aligned} F_0(\lambda) &= GE_0[\nu, \lambda](GE_0[\nu, \lambda])^*, \\ F(\lambda) &= GE[\nu, \lambda](GE[\nu, \lambda])^*. \end{aligned}$$

Note that by (2.1) and (2.6), the operators $GE_0[\nu, \lambda]$ and $GE[\nu, \lambda]$ are well defined and compact. For $\nu < \lambda_1 < \lambda_2$, we have

$$(2.8) \quad F_0(\lambda_2) - F_0(\lambda_1) = GE_0[\lambda_1, \lambda_2](GE_0[\lambda_1, \lambda_2])^*$$

and a similar identity holds true for $F(\lambda)$. In what follows, we discuss the derivatives

$$(2.9) \quad F'_0(\lambda) = \frac{d}{d\lambda}F_0(\lambda), \quad F'(\lambda) = \frac{d}{d\lambda}F(\lambda)$$

understood in the operator norm sense. By (2.8), it is clear that neither the existence nor the values of these derivatives depend on the choice of the reference point ν . In fact, if H_0 is semi-bounded from below, then we can take $\nu = -\infty$. It is also clear that if these derivatives exist in the operator norm, then $F'_0(\lambda) \geq 0$ and $F'(\lambda) \geq 0$ in the quadratic form sense.

Theorem 2.1. *Assume (2.1) and suppose that for some $\lambda > \nu$, the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist in the operator norm. Then the limit*

$$(2.10) \quad \alpha(\lambda) \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow +0} \frac{\pi}{2\varepsilon} \|(GE_0(\lambda - \varepsilon, \lambda + \varepsilon))^* JGE(\lambda - \varepsilon, \lambda + \varepsilon)\|$$

exists and the identity

$$(2.11) \quad \sigma_{\text{ess}}(E(-\infty, \lambda) - E_0(-\infty, \lambda)) = [-\alpha(\lambda), \alpha(\lambda)]$$

holds true. One also has

$$(2.12) \quad \alpha(\lambda) = \pi \|F'_0(\lambda)^{1/2} JF'(\lambda)^{1/2}\|.$$

The proof is given in Section 3.

Remark. 1. It is straightforward to see that $\sigma(E(-\infty, \lambda) - E_0(-\infty, \lambda)) \subset [-1, 1]$.

Thus, Theorem 2.1 implies, in particular, that $\alpha(\lambda) \leq 1$.

2. If $\lambda \notin \sigma(H_0)$, then $F'_0(\lambda) = 0$, and we obtain that the difference of the spectral projections in (2.11) is compact. This is not difficult to prove directly (see Remark 3.5).

3. If the operator $VR_0(i)$ is bounded, then it is obvious that (2.10) can be rewritten as (1.3).

In what follows we prove that under the standard assumptions of either trace class or smooth version of scattering theory, one has

$$(2.13) \quad \alpha(\lambda) = \|S(\lambda) - I\|/2,$$

where $S(\lambda)$ is the scattering matrix for the pair H_0, H . Thus, the verification of (1.1) splits into two parts: (2.11) and (2.13). The statement (2.11) is more general

than (2.13). Indeed, in order to prove (2.13), one has to ensure that the scattering matrix $S(\lambda)$ is well defined; this requires some assumptions stronger than those of Theorem 2.1, see Sections 2.3 and 2.4.

2.3. The scattering matrix. Here, following [27], we recall the definition of the scattering matrix in abstract scattering theory. This requires some rather lengthy preliminaries. First we need to recall the definition of the core of the absolutely continuous (a.c.) spectrum of H_0 . Let $E_0^{(\text{ac})}(\cdot)$ (resp. $E^{(\text{ac})}(\cdot)$) be the a.c. part of the spectral measure of H_0 (resp. H) and let $\sigma_{\text{ac}}(H_0)$ be the a.c. spectrum of H_0 defined as usual as the minimal *closed* set such that $E_0^{(\text{ac})}(\mathbb{R} \setminus \sigma_{\text{ac}}(H_0)) = 0$.

The set $\sigma_{\text{ac}}(H_0)$ is “too large” for general scattering theory considerations. Indeed, it is not difficult to construct examples when $\sigma_{\text{ac}}(H_0)$ contains a closed set A of a *positive Lebesgue measure* such that $E_0^{(\text{ac})}(A) = 0$ (consider $E_0^{(\text{ac})}$ being supported on the intervals $(a_n - 2^{-n}, a_n + 2^{-n})$, where a_1, a_2, \dots is a dense sequence in \mathbb{R}). Thus, it is convenient to use the notion of the *core of the a.c. spectrum of H_0* , denoted by $\hat{\sigma}_{\text{ac}}(H_0)$ and defined as a Borel set such that:

- (i) $\hat{\sigma}_{\text{ac}}(H_0)$ is a Borel support of $E_0^{(\text{ac})}$, i.e. $E_0^{(\text{ac})}(\mathbb{R} \setminus \hat{\sigma}_{\text{ac}}(H_0)) = 0$;
- (ii) if A is any other Borel support of $E_0^{(\text{ac})}$, then the set $\hat{\sigma}_{\text{ac}}(H_0) \setminus A$ has a zero Lebesgue measure.

The set $\hat{\sigma}_{\text{ac}}(H_0)$ is not unique but is defined up to a set of a zero Lebesgue measure.

Suppose that for some interval $\Delta \subset \mathbb{R}$, the (local) wave operators

$$W_{\pm} = W_{\pm}(H_0, H; \Delta) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} E_0^{(\text{ac})}(\Delta)$$

exist and $\text{Ran } W_+(H_0, H; \Delta) = \text{Ran } W_-(H_0, H; \Delta)$. Then the (local) scattering operator $\mathbf{S} = W_+^* W_-$ is unitary in $\text{Ran } E_0^{(\text{ac})}(\Delta)$ and commutes with H_0 . Consider the direct integral decomposition

$$(2.14) \quad \text{Ran } E_0^{(\text{ac})}(\Delta) = \int_{\hat{\sigma}_{\text{ac}}(H_0) \cap \Delta}^{\oplus} \mathfrak{h}(\lambda) d\lambda$$

which diagonalises H_0 . Since \mathbf{S} commutes with H_0 , the decomposition (2.14) represents \mathbf{S} as the operator of multiplication by the operator valued function $S(\lambda) : \mathfrak{h}(\lambda) \rightarrow \mathfrak{h}(\lambda)$. The unitary operator $S(\lambda)$ is called the scattering matrix. With this definition, $S(\lambda)$ is defined for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0)$. In abstract scattering theory, it does not make sense to speak of $S(\lambda)$ at an individual point $\lambda \in \hat{\sigma}_{\text{ac}}(H_0)$, since even the set $\hat{\sigma}_{\text{ac}}(H_0)$ is defined only up to addition or subtraction of sets of zero Lebesgue measure. Also, in general there is no distinguished choice of the direct integral decomposition (2.14); any unitary transformation in the fiber spaces $\mathfrak{h}(\lambda)$ yields another suitable decomposition. Thus, the scattering matrix is, in general, defined only up to a unitary equivalence.

The above discussion refers only to the “abstract” version of the mathematical scattering theory. In concrete problems, there is often a natural distinguished choice

of the core $\hat{\sigma}_{\text{ac}}(H_0)$ and of the direct integral decomposition (2.14). This usually allows one to consider $S(\lambda)$ as an operator defined for *all* (rather than for a.e.) $\lambda \in \hat{\sigma}_{\text{ac}}(H_0)$.

In what follows we set

$$(2.15) \quad S(\lambda) = I \quad \text{for } \lambda \in \mathbb{R} \setminus \hat{\sigma}_{\text{ac}}(H_0);$$

thus, $S(\lambda)$ is now defined for a.e. $\lambda \in \mathbb{R}$. This will make the statements below more succinct.

2.4. The scattering matrix and $\alpha(\lambda)$. Similarly to the definition (2.2) of $T_0(z)$, let us formally define $T(z) = GR(z)G^*$. More precisely, using (2.5), we set

$$T(z) = G(|H_0| + I)^{-1/2} \mathcal{B}(z) (G(|H_0| + I)^{-1/2})^*.$$

By (2.1), the operator $T(z)$ is compact. From the resolvent identity (2.3) it follows that

$$(2.16) \quad T(z) = T_0(z) - T_0(z)J(I + T_0(z)J)^{-1}T_0(z) = (I + T_0(z)J)^{-1}T_0(z).$$

First let us consider the framework of smooth perturbations. Suppose that for some bounded open interval $\Delta \subset \mathbb{R}$,

$$(2.17) \quad \begin{aligned} &T_0(z) \text{ and } T(z) \text{ are uniformly continuous in the operator norm} \\ &\text{in the rectangle } \text{Re } z \in \Delta, \text{ Im } z \in (0, 1). \end{aligned}$$

Of course, from here it trivially follows that the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist in the operator norm for all $\lambda \in \Delta$. Under the assumption (2.17) the operator G is locally H_0 -smooth and H -smooth on Δ , and therefore the local wave operators $W_{\pm}(H_0, H; \Delta)$ exist and are complete (see e.g. [27] for the details). The scattering matrix $S(\lambda)$ is defined for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0) \cap \Delta$.

Theorem 2.2. *Assume (2.1) and (2.17). Then for all $\lambda \in \Delta$, the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ exist in the operator norm and so (2.11) holds true. For a.e. $\lambda \in \Delta$, the identities (2.13) and (1.1) hold true.*

The proof is given in Section 4.2. In [17], formula (1.1) was proven under the additional assumptions of the compactness of G (which is a stronger assumption than (2.1)) and the Hölder continuity of $F'_0(\lambda)$ and $F'(\lambda)$.

Next, consider the trace class scheme. Let \mathbf{S}_2 be the Hilbert-Schmidt class. Suppose that $H = H_0 + V$, where $V = V^*$ is a trace class operator. Then we can factorise $V = GJG^*$ with $G = |V|^{1/2} \in \mathbf{S}_2$ and $J = \text{sign}(V)$. It is well known that under these assumptions, the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ exist in the operator norm for a.e. $\lambda \in \mathbb{R}$ (see e.g. [27, Section 6.1]). We have

Theorem 2.3. *Let $H = H_0 + V$, where V is a trace class operator. Set $G = |V|^{1/2}$. Then for a.e. $\lambda \in \mathbb{R}$, the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist and (2.11), (2.13) and (1.1) hold true.*

Alternatively, we have the following statement more suitable for applications to differential operators:

Theorem 2.4. *Let H_0 be semi-bounded from below; assume that (2.1) holds true and also, for some $m > 0$,*

$$(2.18) \quad G(|H_0| + I)^{-m} \in \mathbf{S}_2.$$

Then the conclusion of Theorem 2.3 holds true.

The proofs of Theorems 2.3 and 2.4 are given in Section 4.2.

Remark. 1. The existence and completeness of the wave operators under the assumptions of Theorems 2.3 and 2.4 is well known; see e.g. [27, Section 4.5 and Section 6.4].

2. According to our convention (2.15), we have

$$\|S(\lambda) - I\| = 0 \quad \text{for } \lambda \in \mathbb{R} \setminus \hat{\sigma}_{\text{ac}}(H_0).$$

Thus, Theorems 2.2–2.4 in particular, include the statement that for a.e. $\lambda \in \mathbb{R} \setminus \hat{\sigma}_{\text{ac}}(H_0)$, the difference of the spectral projections (1.5) is compact.

2.5. Piecewise continuous functions φ . Let us consider the spectrum of $\varphi(H) - \varphi(H_0)$ for piecewise continuous functions φ . It is natural to consider complex-valued functions φ ; in this case $\varphi(H) - \varphi(H_0)$ is non-selfadjoint. For a bounded operator M , we denote by $\sigma_{\text{ess}}(M)$ the compact set of all $z \in \mathbb{C}$ such that the operator $M - zI$ is not Fredholm. The reader should be warned that there are several non-equivalent definitions of the essential spectrum of a non-selfadjoint operator in the literature; see e.g. [5, Sections 1.4 and 9.1] for a comprehensive discussion. However, as we shall see, the essential spectrum of $\varphi(H) - \varphi(H_0)$ has an empty interior and a connected complement in \mathbb{C} , and so in our case most of these definitions coincide.

A function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is called piecewise continuous if for any $\lambda \in \mathbb{R}$ the limits $\varphi(\lambda \pm 0) = \lim_{\varepsilon \rightarrow +0} \varphi(\lambda \pm \varepsilon)$ exist. We denote by $PC_0(\mathbb{R})$ (resp. $C_0(\mathbb{R})$) the set of all piecewise continuous (resp. continuous) functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ such that $\lim_{|x| \rightarrow \infty} \varphi(x) = 0$. For $\varphi \in PC_0(\mathbb{R})$ we denote

$$\varkappa_\lambda(\varphi) = \varphi(\lambda + 0) - \varphi(\lambda - 0), \quad \text{sing supp } \varphi = \overline{\{\lambda \in \mathbb{R} \mid \varkappa_\lambda(\varphi) \neq 0\}}.$$

It is easy to see that for any $\varepsilon > 0$, the set $\{\lambda \in \mathbb{R} \mid |\varkappa_\lambda(\varphi)| > \varepsilon\}$ is finite. For $z_1, z_2 \in \mathbb{C}$, we denote by $[z_1, z_2]$ the closed interval of the straight line in \mathbb{C} that joins z_1 and z_2 .

Theorem 2.5. *Assume (2.1) and let (2.17) hold true for some open bounded interval $\Delta \subset \mathbb{R}$. Let $\varphi \in PC_0(\mathbb{R})$ be a function with $\text{sing supp } \varphi \subset \Delta$. Then we have*

$$(2.19) \quad \sigma_{\text{ess}}(\varphi(H) - \varphi(H_0)) = \cup_{\lambda \in \Delta} [-\alpha(\lambda)\varkappa_\lambda(\varphi), \alpha(\lambda)\varkappa_\lambda(\varphi)],$$

where $\alpha(\lambda)$ is defined by (2.10). In particular, if φ is real valued, then

$$\sigma_{\text{ess}}(\varphi(H) - \varphi(H_0)) = [-a, a], \quad a = \sup_{\lambda \in \Delta} |\alpha(\lambda)\varkappa_\lambda(\varphi)|.$$

The proof is given in Section 5.

2.6. The Fredholm property. A pair of orthogonal projections P, Q in a Hilbert space is called Fredholm, if (see e.g. [2])

$$(2.20) \quad \pm 1 \notin \sigma_{\text{ess}}(P - Q).$$

If P, Q is a Fredholm pair, one defines the index of P, Q by

$$\text{index}(P, Q) = \dim \text{Ker}(P - Q - I) - \dim \text{Ker}(P - Q + I).$$

In a forthcoming publication [18], we study the index of the pair

$$(2.21) \quad E(-\infty, \lambda), \quad E_0(-\infty, \lambda).$$

In connection with this (and perhaps otherwise) it is interesting to know whether the pair (2.21) is Fredholm. Under the assumptions of Theorem 2.1, the question reduces to deciding whether $\alpha(\lambda) < 1$ or $\alpha(\lambda) = 1$. If (2.13) holds true, then, clearly, the pair (2.21) is Fredholm if and only if -1 is not an eigenvalue of the scattering matrix $S(\lambda)$. Below we give a convenient criterion for this in terms of the operators T_0, T .

For a bounded operator M , we denote $\text{Re } M = (M + M^*)/2$, $\text{Im } M = (M - M^*)/2i$. If the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist, we denote

$$(2.22) \quad A_0(\lambda) = \text{Re } T_0(\lambda + i0), \quad A(\lambda) = \text{Re } T(\lambda + i0).$$

Theorem 2.6. *Assume (2.1). Suppose that for some $\lambda \in \mathbb{R}$, the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ and the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist in the operator norm. Then the following statements are equivalent:*

- (i) the pair (2.21) is Fredholm;
- (ii) $\text{Ker}(I + A_0(\lambda)J) = \{0\}$;
- (iii) $\text{Ker}(I - A(\lambda)J) = \{0\}$.

The proof is given in Section 4. Theorem 2.6 can be applied to either smooth or trace class framework. In applications, one can often obtain some information about the spectrum of $A_0(\lambda)$ or $A(\lambda)$; for example, one can sometimes ensure that the norm of $A_0(\lambda)$ is small. By Theorem 2.6, this can be used to ensure that the pair (2.21) is Fredholm.

Remark. 1. Since $\dim \text{Ker}(I + XY) = \dim \text{Ker}(I + YX)$ for any bounded operators X, Y , we can equivalently restate (ii), (iii) as

- (iia) $\text{Ker}(I + JA_0(\lambda)) = \{0\}$;
- (iiia) $\text{Ker}(I - JA(\lambda)) = \{0\}$.

2. If the operator J has a bounded inverse, we can equivalently restate (ii), (iii) in a more symmetric form as

- (iib) $\text{Ker}(J^{-1} + A_0(\lambda)) = \{0\}$;
- (iiib) $\text{Ker}(J^{-1} - A(\lambda)) = \{0\}$.

2.7. Schrödinger operator: smooth framework. Let $H_0 = -\Delta$ in $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$, and let $H = H_0 + V$, where V is the operator of multiplication by a function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ which is assumed to satisfy

$$(2.23) \quad |V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1.$$

Let $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$, $J = \text{sign } V$. Under the assumption (2.23), the hypotheses (2.1) and (2.17) hold true with any $\Delta = (c_1, c_2)$, $0 < c_1 < c_2 < \infty$, see e.g. [23, Theorem XIII.33]. It is also easy to see that the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist in the operator norm for all $\lambda > 0$. Thus, for any $\lambda > 0$, the conclusions of Theorems 2.2 and 2.6 hold true. In [17], formula (1.1) was proven for H_0 and H as above only for $d \leq 3$. We also see that the conclusion of Theorem 2.5 holds true for any $\varphi \in PC_0(\mathbb{R})$ which is continuous in an open neighbourhood of zero.

In this example there is a well known natural choice of the core $\hat{\sigma}_{\text{ac}}(H_0) = (0, \infty)$ and of the direct integral decomposition (2.14) with $\mathfrak{h}(\lambda) = L^2(\mathbb{S}^{d-1})$. Moreover, in this case the scattering matrix $S(\lambda) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ is continuous in $\lambda > 0$. Thus, in this case the statement (1.1) holds true for all $\lambda > 0$.

2.8. Schrödinger operator: trace class framework. Let $H_0 = -\Delta + U$ in $L^2(\mathbb{R}^d)$, $d \geq 1$, where U is the operator of multiplication by a real valued bounded function. Next, let $H = H_0 + V$, where V is the operator of multiplication by a real valued function $V \in L^1(\mathbb{R}^d)$ such that V is $(-\Delta)$ -form compact. Then V is also H_0 -form compact and $H = H_0 + V$ is well defined as a form sum. It is well known (see e.g. [25, Theorem B.9.1]) that under the above assumptions, (2.18) holds true with $G = |V|^{1/2}$ for $m > d/4$. Thus, the conclusions of Theorem 2.4 hold true.

The assumptions on H_0 in this example can be considerably relaxed by allowing U to have local singularities, by including a background magnetic field, etc. Note that in this example we have no information on the a.c. spectrum of H_0 .

3. PROOF OF THEOREM 2.1

We follow the method of [17] with some minor technical improvements. In order to simplify our notation, we assume $\lambda = 0$ and denote $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$.

3.1. The proof of (2.12). Let us prove that if the derivatives $F'_0(0)$ and $F'(0)$ exist in the operator norm, then the limit (2.10) also exists and the identity (2.12) holds true. Let us start from the r.h.s. of (2.12). Denoting $\delta_\varepsilon = (-\varepsilon, \varepsilon)$ and using the

identities $\|X\|^2 = \|XX^*\| = \|X^*X\|$, we get

$$\begin{aligned}
\|F'_0(0)^{1/2} JF'(0)^{1/2}\|^2 &= \|F'_0(0)^{1/2} JF'(0) JF'_0(0)^{1/2}\| \\
&= \lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon} \|F'_0(0)^{1/2} JGE(\delta_\varepsilon)(GE(\delta_\varepsilon))^* JF'_0(0)^{1/2}\| \\
&= \lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon} \|(GE(\delta_\varepsilon))^* JF'_0(0) JGE(\delta_\varepsilon)\| \\
&= \lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon^2} \|(GE(\delta_\varepsilon))^* JGE_0(\delta_\varepsilon)(GE_0(\delta_\varepsilon))^* JGE(\delta_\varepsilon)\| \\
&= \lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon^2} \|(GE(\delta_\varepsilon))^* JGE_0(\delta_\varepsilon)\|^2,
\end{aligned}$$

as required.

In the rest of this section, we prove that if the derivatives $F'_0(0)$ and $F'(0)$ exist in the operator norm, then the identity

$$(3.1) \quad \sigma_{\text{ess}}(E(\mathbb{R}_-) - E_0(\mathbb{R}_-)) = [-\alpha(0), \alpha(0)]$$

holds true with $\alpha(0) = \pi \|F'_0(0)^{1/2} JF'(0)^{1/2}\|$.

3.2. The kernels of H_0 and H .

Lemma 3.1. *Assume (2.1) and suppose that the derivatives $F'_0(0)$ and $F'(0)$ exist in the operator norm. Then $\text{Ker } H_0 = \text{Ker } H$.*

Proof. 1. By our assumptions, $GE_0(\{0\}) = 0$ (otherwise $F'_0(0)$ cannot exist). Suppose $\psi \in \text{Ker } H_0$; then $G\psi = 0$ and the resolvent identity (2.3) yields

$$R(z)\psi = R_0(z)\psi = -\frac{1}{z}\psi.$$

Thus, $H\psi = 0$. This proves that $\text{Ker } H_0 \subset \text{Ker } H$.

2. From (2.3) it is not difficult to obtain the ‘‘usual’’ resolvent identity (see e.g. [27, Section 1.10]):

$$(3.2) \quad R(z) = R_0(z) - (GR_0(\bar{z}))^* JGR(z).$$

Now let $\psi \in \text{Ker } H$. As above, $GE(\{0\}) = 0$, and so from (3.2) one obtains

$$R_0(z)\psi = R(z)\psi = -\frac{1}{z}\psi.$$

Thus, $H_0\psi = 0$ and so $\text{Ker } H \subset \text{Ker } H_0$. ■

3.3. Reduction to the products of spectral projections.

Let us denote $D = E(\mathbb{R}_-) - E_0(\mathbb{R}_-)$, $\mathcal{H}_+ = \text{Ker}(D - I)$, $\mathcal{H}_- = \text{Ker}(D + I)$, $\mathcal{H}_0 = (\mathcal{H}_+ \oplus \mathcal{H}_-)^{\perp}$.

It is well known (see e.g. [6] or [2]) that

$$(3.3) \quad D|_{\mathcal{H}_0} \text{ is unitarily equivalent to } (-D)|_{\mathcal{H}_0}.$$

Therefore, the spectral analysis of D reduces to the spectral analysis of D^2 and to the evaluation of the dimensions of \mathcal{H}_+ and \mathcal{H}_- .

Next, using Lemma 3.1, by a simple algebra we obtain the identity

$$(3.4) \quad D^2 = E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-) + E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+),$$

where the r.h.s. provides a block-diagonal decomposition of D^2 with respect to the direct sum

$$\mathcal{H} = \text{Ran } E_0(\mathbb{R}_-) \oplus \text{Ran } E_0(\{0\}) \oplus \text{Ran } E_0(\mathbb{R}_+).$$

Thus, the spectral analysis of D^2 reduces to the spectral analysis of the two terms in the r.h.s. of (3.4). In Sections 3.4–3.7 we prove

Lemma 3.2. *Assume (2.1). Then the differences*

$$(3.5) \quad E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+) - E_0(0, 1)E(-1, 0)E_0(0, 1)$$

$$(3.6) \quad E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-) - E_0(-1, 0)E(0, 1)E_0(-1, 0)$$

are compact operators.

Theorem 3.3. *Assume (2.1) and suppose that the derivatives $F'_0(0)$ and $F'(0)$ exist in the operator norm. Then*

$$(3.7) \quad \sigma_{\text{ess}}(E_0(0, 1)E(-1, 0)E_0(0, 1)) = [0, \alpha(0)^2],$$

$$(3.8) \quad \sigma_{\text{ess}}(E_0(-1, 0)E(0, 1)E_0(-1, 0)) = [0, \alpha(0)^2],$$

where $\alpha(0)$ is given by $\alpha(0) = \pi \|F'_0(0)^{1/2} J F'(0)^{1/2}\|$. In particular, $\alpha(0) \leq 1$.

With these two statements, it is easy to provide

Proof of Theorem 2.1. Combining Lemma 3.2, Theorem 3.3 and Weyl's theorem on the invariance of the essential spectrum under compact perturbations, we obtain

$$(3.9) \quad \sigma_{\text{ess}}(E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-)) = \sigma_{\text{ess}}(E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+)) = [0, \alpha(0)^2].$$

By (3.4), it follows that

$$(3.10) \quad \sigma_{\text{ess}}(D^2) = [0, \alpha(0)^2].$$

Suppose first that $\alpha(0) = 1$. Then from (3.10) and (3.3) we obtain $\sigma_{\text{ess}}(D) = [-1, 1]$, as required. Next, suppose $\alpha(0) < 1$. Then from (3.10) it follows that the dimensions of \mathcal{H}_- and \mathcal{H}_+ are finite, and therefore

$$\sigma_{\text{ess}}(D) = \sigma_{\text{ess}}(D|_{\mathcal{H}_0}) \text{ and } \sigma_{\text{ess}}(D^2) = \sigma_{\text{ess}}((D|_{\mathcal{H}_0})^2).$$

Recalling (3.3), we obtain

$$\sigma_{\text{ess}}(D|_{\mathcal{H}_0}) = [-\alpha(0), \alpha(0)],$$

and (3.1) follows. ■

3.4. Proof of Lemma 3.2.

Lemma 3.4. *Assume (2.1). Let $\varphi \in C(\mathbb{R})$ be a function such that the limits $\lim_{x \rightarrow \pm\infty} \varphi(x)$ exist. Then the difference $\varphi(H) - \varphi(H_0)$ is compact.*

Proof. As is well known (and can easily be deduced from the compactness of $R(z) - R_0(z)$ for $\text{Im } z \neq 0$), the operator $\varphi(H) - \varphi(H_0)$ is compact for any function $\varphi \in C_0(\mathbb{R})$. Therefore, it suffices to prove that $\varphi(H) - \varphi(H_0)$ is compact for at least one function $\varphi \in C(\mathbb{R})$ such that $\lim_{x \rightarrow \infty} \varphi(x) \neq \lim_{x \rightarrow -\infty} \varphi(x)$ and both limits exist. The latter fact is provided by [16, Theorem 7.3] where it is proven that if (2.1) holds true then the difference $\tan^{-1}(H) - \tan^{-1}(H_0)$ is compact. ■

Remark 3.5. Let $\mu \in \mathbb{R} \setminus (\sigma(H_0) \cup \sigma(H))$. Then

$$(3.11) \quad E(-\infty, \mu) - E_0(-\infty, \mu) = \varphi(H) - \varphi(H_0)$$

for an appropriately chosen continuous function φ with $\varphi(x) = 1$ for $x \in \sigma(H) \cup \sigma(H_0)$, $x < \mu$ and $\varphi(x) = 0$ for $x \in \sigma(H) \cup \sigma(H_0)$, $x > \mu$. It follows that the difference (3.11) is compact.

Proof of Lemma 3.2. 1. Let $\varphi_1 \in C(\mathbb{R})$ be such that $\varphi_1(x) = 1$ for $x \leq -1$ and $\varphi_1(x) = 0$ for $x \geq 0$. Then

$$(3.12) \quad E(-\infty, -1)E_0(\mathbb{R}_+) = E(-\infty, -1)(\varphi_1(H) - \varphi_1(H_0))E_0(\mathbb{R}_+)$$

and so by Lemma 3.4 the r.h.s. is compact.

2. Let $\varphi_2 \in C(\mathbb{R})$ be such that $\varphi_2(x) = 1$ for $x \geq 1$ and $\varphi_2(x) = 0$ for $x \leq 0$. Then

$$(3.13) \quad E_0(1, \infty)E(\mathbb{R}_-) = E_0(1, \infty)(\varphi_2(H_0) - \varphi_2(H))E(\mathbb{R}_-),$$

and so by Lemma 3.4 the r.h.s. is compact.

3. From the compactness of the l.h.s. of (3.12) and (3.13), the compactness of the difference (3.5) follows by some simple algebra. Compactness of (3.6) is proven in the same way. ■

3.5. Hankel operators. In order to prove Theorem 3.3, we need some basic facts concerning operator valued Hankel integral operators. Suppose that for each $t > 0$, a bounded self-adjoint operator $K(t)$ in \mathcal{K} is given. Suppose that $K(t)$ is continuous in $t > 0$ in the operator norm. Define a Hankel integral operator K in $L^2(\mathbb{R}_+, \mathcal{K})$ by

$$(3.14) \quad (Kf, g)_{L^2(\mathbb{R}_+, \mathcal{K})} = \int_0^\infty \int_0^\infty (K(t+s)f(t), g(s))_{\mathcal{K}} dt ds,$$

when $f, g \in L^2(\mathbb{R}_+, \mathcal{K})$ are functions with compact support in \mathbb{R}_+ . The statement below is a straightforward generalisation of [7, Proposition 1.1] to the operator valued case.

Proposition 3.6. (i) Suppose $\|K(t)\| \leq C/t$ for all $t > 0$. Then the operator K is bounded and $\|K\| \leq \pi C$.

(ii) Suppose $K(t)$ is compact for all t and $\|K(t)\| = o(1/t)$ as $t \rightarrow +0$ and as $t \rightarrow +\infty$. Then K is compact.

Proof. Since the Carleman operator on $L^2(\mathbb{R}_+)$ with the kernel $(t+s)^{-1}$ is bounded with the norm π , we have

$$|(Kf, g)_{L^2(\mathbb{R}_+, \mathcal{K})}| \leq C \int_0^\infty \int_0^\infty \frac{\|f(t)\|_{\mathcal{K}} \|g(s)\|_{\mathcal{K}}}{t+s} dt ds \leq \pi C \|f\|_{L^2(\mathbb{R}_+, \mathcal{K})} \|g\|_{L^2(\mathbb{R}_+, \mathcal{K})},$$

which proves (i). To prove (ii), we need to approximate K by compact operators. Let $K_n(t) = K(t)\chi_n(t)$, where χ_n is the characteristic function of the interval $(1/n, n)$ and let K_n be the corresponding operator in $L^2(\mathbb{R}_+, \mathcal{K})$. By (i), $\|K - K_n\|_{L^2(\mathbb{R}_+, \mathcal{K})} \rightarrow 0$ as $n \rightarrow \infty$. Thus, it remains to show that each K_n is compact.

For each n , the Hankel type integral operator with the kernel $\chi_n(t+s)/(t+s)$ in $L^2(\mathbb{R}_+)$ is compact (in fact, Hilbert-Schmidt). It follows that K_n is compact if $K(t)$ is independent of t . Now the result follows from the fact that $K(t)$ can be uniformly approximated by piecewise constant functions on the interval $(1/n, n)$. ■

Important model operators in our construction below are the Hankel integral operators in $L^2(\mathbb{R}_+, \mathcal{K})$ of the type (3.14) with $K(t)$ given by

$$(3.15) \quad \frac{1 - e^{-t}}{t} F'_0(0) \quad \text{and} \quad \frac{1 - e^{-t}}{t} F'(0).$$

For this reason, we need to discuss the integral Hankel operator Γ in $L^2(\mathbb{R}_+)$ with the integral kernel $\Gamma(t, s) = \frac{1 - e^{-t-s}}{t+s}$. One can show (see e.g. [17, Lemma 7]) that

$$(3.16) \quad \sigma(\Gamma) = [0, \pi].$$

In fact, the spectrum of Γ is purely absolutely continuous, but we will not need this fact. Identifying $L^2(\mathbb{R}_+, \mathcal{K})$ with $L^2(\mathbb{R}_+) \otimes \mathcal{K}$, we denote the operators (3.15) by $\Gamma \otimes F'_0(0)$ and $\Gamma \otimes F'(0)$.

3.6. The operators L and L_0 . The crucial point of our proof of Theorem 3.3 is the representation

$$(3.17) \quad E(-1, 0)E_0(0, 1) = -LJL_0^*$$

in terms of some auxiliary operators L_0 and L which we proceed to define. These operators act from $L^2(\mathbb{R}_+, \mathcal{K})$ to \mathcal{H} . On the dense set $L^2(\mathbb{R}_+, \mathcal{K}) \cap L^1(\mathbb{R}_+, \mathcal{K})$ we define L_0, L by

$$(3.18) \quad L_0 f = \int_0^\infty e^{-tH_0} (GE_0(0, 1))^* f(t) dt,$$

$$(3.19) \quad Lf = \int_0^\infty e^{tH} (GE(-1, 0))^* f(t) dt.$$

Lemma 3.7. *Assume (2.1) and suppose that the derivatives $F'_0(0)$, $F'(0)$ exist in the operator norm. Then:*

(i) *The operators L_0 and L defined by (3.18) and (3.19) extend to bounded operators from $L^2(\mathbb{R}_+, \mathcal{K})$ to \mathcal{H} .*

(ii) *The differences*

$$(3.20) \quad L_0^* L_0 - \Gamma \otimes F'_0(0), \quad L^* L - \Gamma \otimes F'(0)$$

are compact operators.

(iii) *The identity (3.17) holds true.*

Proof. (i) Let us prove that L_0 is bounded; the boundedness of L is proven in the same way. For $f \in L^2(\mathbb{R}_+, \mathcal{K}) \cap L^1(\mathbb{R}_+; \mathcal{K})$ we have

$$\|L_0 f\|^2 = \int_0^\infty \int_0^\infty (GE_0(0, 1)e^{-(t+s)H_0}(GE_0(0, 1))^* f(t), f(s))_{\mathcal{K}} dt ds,$$

and so the above expression is a quadratic form of the operator of the type (3.14) with the kernel $K(t) = GE_0(0, 1)e^{-tH_0}(GE_0(0, 1))^*$. By Proposition 3.6, it suffices to prove the bound $\|K(t)\|_{\mathcal{K}} \leq C/t$, $t > 0$. Let $f \in \mathcal{H}$ and $\rho(\lambda) = (E_0(-\infty, \lambda)f, f)$. Integrating by parts, one obtains

$$\int_0^1 e^{-t\lambda} d\rho(\lambda) = e^{-t} \int_0^1 d\rho(\lambda) + t \int_0^1 d\mu e^{-t\mu} \int_0^\mu d\rho(\lambda).$$

It follows that

$$e^{-tH_0} E_0(0, 1) = e^{-t} E_0(0, 1) + t \int_0^1 e^{-t\mu} E_0(0, \mu) d\mu.$$

Using this expression, the relation (2.8) and the fact that $GE_0(\{0\}) = 0$, we get

$$(3.21) \quad K(t) = e^{-t}(F_0(1) - F_0(0)) + t \int_0^1 e^{-t\mu}(F_0(\mu) - F_0(0)) d\mu.$$

By our assumption on the differentiability of F_0 , we have

$$\|F_0(\mu) - F_0(0)\| \leq C|\mu| \text{ for } |\mu| \leq 1.$$

Using this, we obtain:

$$\begin{aligned} \|K(t)\| &\leq e^{-t}\|F_0(1) - F_0(0)\| + t \int_0^1 e^{-t\mu}\|F_0(\mu) - F_0(0)\| d\mu \\ &\leq Ce^{-t} + Ct \int_0^1 e^{-t\mu} \mu d\mu = C(1 - e^{-t})/t \leq C/t, \quad t > 0, \end{aligned}$$

as required.

(ii) Let us consider the first of the differences (3.20); the second one is considered in the same way. By the same reasoning as above, $L_0^*L_0 - \Gamma \otimes F'_0(0)$ is the operator of the type (3.14) with

$$K(t) = GE_0(0, 1)e^{-tH_0}(GE_0(0, 1))^* - F'_0(0)(1 - e^{-t})/t.$$

By (2.1), $F_0(\lambda)$ is compact for all λ . Since the derivative $F'_0(0)$ exists in the operator norm, the operator $F'_0(0)$ is also compact. Thus, $K(t)$ is compact for all $t > 0$. By Proposition 3.6(ii), it suffices to prove that $\|K(t)\| = o(1/t)$ as $t \rightarrow 0$ and $t \rightarrow \infty$. For $t \rightarrow 0$, the statement is obvious. Consider the limit $t \rightarrow \infty$. By the same calculation as in part (i) of the proof (see (3.21)), we have

$$\begin{aligned} K(t) &= e^{-t}(F_0(1) - F_0(0)) + t \int_0^1 e^{-t\mu}(F_0(\mu) - F_0(0))d\mu \\ &\quad - F'_0(0)t \int_0^1 e^{-t\mu}\mu d\mu - F'_0(0)e^{-t}. \end{aligned}$$

It follows that

$$(3.22) \quad \|K(t)\| \leq e^{-t}\|F_0(1) - F_0(0) - F'_0(0)\| + t \int_0^1 e^{-t\mu}\|F_0(\mu) - F_0(0) - F'_0(0)\mu\|d\mu.$$

By our assumption,

$$(3.23) \quad \|F_0(\mu) - F_0(0) - F'_0(0)\mu\| = o(\mu) \text{ as } \mu \rightarrow 0.$$

Using (3.22) and (3.23), it is easy to see that $\|K(t)\| = o(1/t)$ as $t \rightarrow \infty$.

(iii) Let $f, f_0 \in \mathcal{H}$. Using (2.4), we obtain

$$\begin{aligned} &\frac{d}{dt}(E_0(0, 1)e^{-tH_0}f_0, E(-1, 0)e^{tH}f) \\ &= (E_0(0, 1)e^{-tH_0}f_0, HE(-1, 0)e^{tH}f) - (H_0E_0(0, 1)e^{-tH_0}f_0, E(-1, 0)e^{tH}f) \\ &= (JGE_0(0, 1)e^{-tH_0}f_0, GE(-1, 0)e^{tH}f)_{\mathcal{K}}. \end{aligned}$$

Using this and the easily verifiable relations

$$\|E_0(0, 1)e^{-tH_0}f_0\|_{\mathcal{H}} \rightarrow 0, \quad \|E_0(-1, 0)e^{tH}f\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

we get

$$\begin{aligned} (JL_0^*f_0, L^*f)_{L^2(\mathbb{R}_+, \mathcal{K})} &= \int_0^\infty (JGE_0(0, 1)e^{-tH_0}f_0, GE(-1, 0)e^{tH}f)_{\mathcal{K}}dt \\ &= \int_0^\infty \frac{d}{dt}(E_0(0, 1)e^{-tH_0}f_0, E(-1, 0)e^{tH}f)dt = -(E_0(0, 1)f_0, E(-1, 0)f), \end{aligned}$$

which proves (3.17). ■

3.7. Proof of Theorem 3.3. We will prove (3.7); the relation (3.8) is proven in the same manner.

1. First we introduce some notation. For bounded self-adjoint operators M and N we shall write

$$M \approx N \text{ if } M|_{(\text{Ker } M)^\perp} \text{ is unitarily equivalent to } N|_{(\text{Ker } N)^\perp}.$$

It is well known that $M^*M \approx MM^*$ for any bounded operator M ; below we use this fact.

2. Using Lemma 3.7 we get, for some compact operators X_0 and X :

$$E_0(0,1)E(-1,0)E_0(0,1) = L_0JL^*LJL_0^* = L_0(\Gamma \otimes JF'(0)J)L_0^* + X,$$

$$\begin{aligned} L_0(\Gamma \otimes JF'(0)J)L_0^* &= L_0(\Gamma^{1/2} \otimes JF'(0)^{1/2})(\Gamma^{1/2} \otimes F'(0)^{1/2}J)L_0^* \\ &\approx (\Gamma^{1/2} \otimes F'(0)^{1/2}J)L_0^*L_0(\Gamma^{1/2} \otimes JF'(0)^{1/2}) \\ &= (\Gamma^{1/2} \otimes F'(0)^{1/2}J)(\Gamma \otimes F'_0(0))(\Gamma^{1/2} \otimes JF'(0)^{1/2}) + X_0 \\ &= \Gamma^2 \otimes (F'(0)^{1/2}JF'_0(0)JF'(0)^{1/2}) + X_0. \end{aligned}$$

Thus, by Weyl's theorem, we obtain

$$\sigma_{\text{ess}}(E_0(0,1)E(-1,0)E_0(0,1)) = \sigma_{\text{ess}}(\Gamma^2 \otimes Q), \quad Q = F'(0)^{1/2}JF'_0(0)JF'(0)^{1/2}.$$

3. The operator Q above is compact, selfadjoint and $Q \geq 0$. Let $Q = \sum_{n=1}^{\infty} \lambda_n(\cdot, f_n)f_n$ be the spectral decomposition of Q , where $\lambda_1 \geq \lambda_2 \geq \dots$ are the eigenvalues of Q . Then

$$\Gamma^2 \otimes Q = \sum_{n=1}^{\infty} \lambda_n \Gamma^2 \otimes (\cdot, f_n)f_n$$

is an orthogonal sum decomposition of $\Gamma^2 \otimes Q$, and therefore

$$\sigma_{\text{ess}}(\Gamma^2 \otimes Q) = \cup_{n=1}^{\infty} \sigma_{\text{ess}}(\lambda_n \Gamma^2 \otimes (\cdot, f_n)f_n).$$

Taking into account (3.16) and recalling that $\lambda_1 = \|Q\|$, we obtain

$$\begin{aligned} \sigma_{\text{ess}}(\Gamma^2 \otimes Q) &= \cup_{n=1}^{\infty} [0, \lambda_n \pi^2] = [0, \pi^2 \|Q\|] = [0, \pi^2 \|F'(0)^{1/2}JF'_0(0)JF'(0)^{1/2}\|] \\ &= [0, \pi^2 \|F'_0(0)^{1/2}JF'(0)^{1/2}\|^2] = [0, \alpha(0)^2], \end{aligned}$$

as required. ■

4. PROOFS OF THEOREMS 2.2, 2.3, 2.4 AND 2.6

4.1. Existence of F'_0 , F' and T_0 , T . Here we recall various statements concerning the existence of the derivatives $F'_0(\lambda)$, $F'(\lambda)$ and the limits $T_0(\lambda+i0)$, $T(\lambda+i0)$ under the assumptions of Theorems 2.2, 2.3, 2.4. All of these statements are essentially well known. If the limits $T_0(\lambda+i0)$, $T(\lambda+i0)$ exist, we denote

$$B_0(\lambda) = \text{Im } T_0(\lambda+i0), \quad B(\lambda) = \text{Im } T(\lambda+i0).$$

We first note that if the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ and the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist at some point λ , then

$$(4.1) \quad \pi F'_0(\lambda) = B_0(\lambda), \quad \pi F'(\lambda) = B(\lambda).$$

Indeed, this follows from the spectral theorem and the following well known fact (see e.g. [24, Theorem 11.22]): if μ is a measure on \mathbb{R} and the derivative $\frac{d}{d\lambda}\mu(-\infty, \lambda)$ exists, then

$$\pi \frac{d}{d\lambda}\mu(-\infty, \lambda) = \lim_{\varepsilon \rightarrow +0} \operatorname{Im} \int_{\mathbb{R}} \frac{d\mu(t)}{t - \lambda - i\varepsilon}.$$

Lemma 4.1. *Assume (2.1) and suppose that (2.17) holds true for some bounded open interval $\Delta \subset \mathbb{R}$. Then for all $\lambda \in \Delta$ the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist in the operator norm.*

Proof. From the obvious operator inequality

$$0 \leq E_0(\{\lambda\}) \leq \frac{\varepsilon^2}{(H_0 - \lambda)^2 + \varepsilon^2}, \quad \varepsilon > 0,$$

we get

$$0 \leq GE_0(\{\lambda\})(GE_0(\{\lambda\}))^* \leq \varepsilon \operatorname{Im} T_0(\lambda + i\varepsilon), \quad \varepsilon > 0.$$

This implies that $GE_0(\{\lambda\}) = 0$ for all $\lambda \in \Delta$. Using this, Stone's formula (see e.g. [20, Theorem VII.13]) yields

$$((F_0(b) - F_0(a))f, f) = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \operatorname{Im} (T_0(\lambda + i\varepsilon)f, f) d\lambda = \frac{1}{\pi} \int_a^b (B_0(\lambda)f, f) d\lambda$$

for any interval $[a, b] \subset \Delta$. From here and the continuity of $B_0(\lambda)$ we get the statement concerning $F'_0(\lambda)$. The case of $F'(\lambda)$ is considered in the same way. ■

Lemma 4.2. (i) *Assume that G is a Hilbert-Schmidt operator. Then for a.e. $\lambda \in \mathbb{R}$, the derivatives $F'_0(\lambda)$, $F'(\lambda)$ and the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist in the operator norm.*

(ii) *Under the assumptions of Theorem 2.4, for a.e. $\lambda \in \mathbb{R}$ the derivatives $F'_0(\lambda)$, $F'(\lambda)$ and the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist in the operator norm.*

Proof. (i) is one of the key facts of the trace class scattering theory, see e.g. [27, Section 6.1].

(ii) First consider F'_0 and T_0 . Let us apply a standard argument: let $\Delta_1 = (-R, R)$, $\Delta_2 = \mathbb{R} \setminus \Delta_1$ and write $G_j = GE_0(\Delta_j)$, $j = 1, 2$. Then $G_1 \in \mathbf{S}_2$. Thus, by part (i) of the lemma, the derivative

$$\frac{d}{d\lambda} F_0(\lambda) = \frac{d}{d\lambda} G_1 E_0(-\infty, \lambda) G_1^*, \quad \lambda \in \Delta_1,$$

exists in the operator norm. Let us consider $T_0(z)$; we have

$$(4.2) \quad T_0(z) = G_1 R_0(z) G_1^* + G_2 (G_2 R_0(\bar{z}))^*.$$

By part (i) of the lemma, the first term in the r.h.s. of (4.2) has a limit as $z \rightarrow \lambda + i0$ for a.e. $\lambda \in \Delta_1$. Since $R_0(z)E_0(\Delta_2)$ is analytic in $z \in \mathbb{C} \setminus \overline{\Delta_2}$, the second term in the r.h.s. of (4.2) has a limit as $z \rightarrow \lambda + i0$ for all $\lambda \in \Delta_1$. It follows that $T_0(z)$ has boundary values as $z \rightarrow \lambda + i0$ for a.e. $\lambda \in \Delta_1$. Since R in the definition of Δ_1 can be taken arbitrary large, this gives the desired statement for a.e. $\lambda \in \mathbb{R}$.

Consider F' and T . First, exactly as in the proof of [22, Theorem XI.30], using (2.1) and (2.18), one shows that

$$(4.3) \quad G(|H| + I)^{-m} \in \mathbf{S}_2.$$

After this, the proof follows the same argument as above. ■

Lemma 4.3. *Assume (2.1) and let $\Delta \subset \mathbb{R}$ be a bounded interval. Suppose that for a.e. $\lambda \in \Delta$, the derivative $F'_0(\lambda)$ exists in the operator norm. Then for a.e. $\lambda \in \Delta \setminus \hat{\sigma}_{\text{ac}}(H_0)$, one has $F'_0(\lambda) = 0$.*

Proof. 1. Recall the following measure theoretic statement. Let μ be a finite Borel measure on \mathbb{R} and let Z be a Borel support of μ , i.e. $\mu(\mathbb{R} \setminus Z) = 0$. Then

$$(4.4) \quad \frac{d}{d\lambda} \mu(-\infty, \lambda) = 0 \quad \text{for Lebesgue-a.e. } \lambda \in \mathbb{R} \setminus Z.$$

Indeed, let $\mu = \mu_{\text{ac}} + \mu_s$ be the decomposition of μ into the a.c. and singular components with respect to the Lebesgue measure. Let $0 \leq f \in L^1(\mathbb{R})$ be the Radon-Nikodym derivative of μ_{ac} with respect to the Lebesgue measure. Then (see e.g. [24, Section 8.6])

$$\frac{d}{d\lambda} \mu_{\text{ac}}(-\infty, \lambda) = f(\lambda), \quad \frac{d}{d\lambda} \mu_s(-\infty, \lambda) = 0, \quad \text{for Lebesgue-a.e. } \lambda \in \mathbb{R}.$$

The statement $\mu(\mathbb{R} \setminus Z) = 0$ implies that

$$\int_{\mathbb{R} \setminus Z} f(\lambda) d\lambda = 0.$$

Thus, $f(\lambda) = 0$ for Lebesgue-a.e. $\lambda \in \mathbb{R} \setminus Z$. From here we get (4.4).

2. Let Z_s be a Borel support of the singular part of the spectral measure E_0 . Since the Lebesgue measure of Z_s is zero, the set $\hat{\sigma} = \hat{\sigma}_{\text{ac}}(H_0) \cup Z_s$ is again a core of the a.c. spectrum of H_0 . Moreover, $\hat{\sigma}$ is a Borel support of E_0 , i.e. $E_0(\mathbb{R} \setminus \hat{\sigma}) = 0$.

3. Let $G_\Delta = GE_0(\Delta)$; by (2.1), G_Δ is a compact operator. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in \mathcal{K} . Consider the complex valued measures

$$\mu_{nm}(\Lambda) = (E_0(\Lambda)G_\Delta^* e_n, G_\Delta^* e_m), \quad n, m \in \mathbb{N}, \quad \Lambda \subset \Delta.$$

We have $\mu_{nm}(\Delta \setminus \hat{\sigma}) = 0$. Representing each μ_{nm} as a linear combination of four non-negative measures and applying (4.4), we obtain

$$\frac{d}{d\lambda} \mu_{nm}(-\infty, \lambda) = 0, \quad \lambda \in (\Delta \setminus \hat{\sigma}) \setminus \Lambda_{nm}, \quad n, m \in \mathbb{N},$$

where the Lebesgue measure of the set Λ_{nm} is zero. It follows that

$$(4.5) \quad \frac{d}{d\lambda} \mu_{nm}((-\infty, \lambda)) = 0, \quad \lambda \in (\Delta \setminus \hat{\sigma}) \setminus \Lambda, \quad n, m \in \mathbb{N},$$

where $\Lambda = \cup_{n,m} \Lambda_{nm}$ and the Lebesgue measure of Λ is zero.

4. Let $\mathcal{D} \subset \mathcal{K}$ be the dense set of all finite linear combinations of elements of the basis $\{e_n\}_{n=1}^\infty$. It follows from (4.5) that

$$\frac{d}{d\lambda} (E_0(-\infty, \lambda) G_\Delta^* f, G_\Delta^* g) = 0, \quad \forall f, g \in \mathcal{D}, \quad \text{a.e. } \lambda \in \Delta \setminus \hat{\sigma},$$

and therefore $F'_0(\lambda) = 0$ for a.e. $\lambda \in \Delta \setminus \hat{\sigma}$. ■

4.2. Connection between $\alpha(\lambda)$ and $S(\lambda)$. First we establish a connection between $\alpha(\lambda)$ and some auxiliary unitary operator $\tilde{S}(\lambda)$. The idea to use the operator $\tilde{S}(\lambda)$ is due to A. V. Sobolev and D. R. Yafaev [26].

Lemma 4.4. *Assume (2.1) and suppose that the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ and the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist for some $\lambda \in \mathbb{R}$. Then the operator*

$$(4.6) \quad \tilde{S}(\lambda) = I - 2iB_0(\lambda)^{1/2}(J - JT(\lambda + i0)J)B_0(\lambda)^{1/2}$$

in \mathcal{K} is unitary and

$$(4.7) \quad \frac{1}{2} \|\tilde{S}(\lambda) - I\| = \alpha(\lambda).$$

Proof. 1. From (2.16) one easily obtains the identity

$$(4.8) \quad I - T(z)J = (I + T_0(z)J)^{-1}, \quad \text{Im } z > 0.$$

Since the limits $T_0(\lambda + i0)$ and $T(\lambda + i0)$ exist in the operator norm, we conclude that the operator $I + T_0(\lambda + i0)J$ has a bounded inverse and

$$(4.9) \quad I - T(\lambda + i0)J = (I + T_0(\lambda + i0)J)^{-1}.$$

In the same way, one obtains

$$(4.10) \quad I - JT(\lambda + i0) = (I + JT_0(\lambda + i0))^{-1}.$$

Taking adjoints in (4.10) and subtracting from (4.9), after some simple algebra we get

$$JB(\lambda)J = (J - JT(\lambda + i0)J)B_0(\lambda)(J - JT(\lambda + i0)^*J).$$

From here the unitarity of $\tilde{S}(\lambda)$ follows by a direct calculation.

2. Using the unitarity of $\tilde{S}(\lambda)$ and the identity (4.1), we obtain

$$\begin{aligned} (\tilde{S}(\lambda) - I)^*(\tilde{S}(\lambda) - I) &= 2I - 2\text{Re } \tilde{S}(\lambda) = 4 \text{Im} (B_0(\lambda)^{1/2}JT(\lambda + i0)JB_0(\lambda)^{1/2}) \\ &= 4 \text{Im} (B_0(\lambda)^{1/2}JB(\lambda)JB_0(\lambda)^{1/2}) = 4\pi^2 \text{Im} (F'_0(\lambda)^{1/2}JF'(\lambda)JF'_0(\lambda)^{1/2}). \end{aligned}$$

From here, taking into account (2.12), we get

$$\frac{1}{4} \|\tilde{S}(\lambda) - I\|^2 = \pi^2 \|F'_0(\lambda)^{1/2} J F'(\lambda) J F'_0(\lambda)^{1/2}\|^2 = \pi^2 \|F'_0(\lambda)^{1/2} J F'(\lambda)^{1/2}\|^2 = \alpha(\lambda)^2,$$

as required. ■

The following Lemma is essentially contained in [27, Section 7.7].

Lemma 4.5. (i) *Under the assumptions (2.1), (2.17), the local wave operators $W_{\pm}(H_0, H; \Delta)$ exist and are complete, and for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0) \cap \Delta$ we have*

$$(4.11) \quad \|S(\lambda) - I\| = \|\tilde{S}(\lambda) - I\|.$$

(ii) *Under the assumptions of Theorem 2.3 or 2.4, the wave operators $W_{\pm}(H_0, H)$ exist and are complete, and for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0)$, the relation (4.11) holds true.*

Proof. (i) For the existence and completeness of wave operators, we refer to [27, Section 4.5]. Next, for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0) \cap \Delta$, the scattering matrix can be represented as

$$(4.12) \quad S(\lambda) = I - 2\pi i Z(\lambda)(J - J T(\lambda + i0)J)Z(\lambda)^*,$$

where $Z(\lambda) : \mathcal{K} \rightarrow \mathfrak{h}(\lambda)$ is an operator such that

$$(4.13) \quad \pi Z(\lambda)^* Z(\lambda) = B_0(\lambda).$$

This is the well known stationary representation for the scattering matrix, see e.g. [27, Section 5.5(3)]. Let us use the polar decomposition of $Z(\lambda)$, $Z(\lambda) = U|Z(\lambda)|$, where $|Z(\lambda)| = \frac{\sqrt{Z(\lambda)^* Z(\lambda)}}{\pi} = B_0(\lambda)^{1/2}/\pi$, and U is an isometry which maps $\text{Ran } Z(\lambda)^*$ onto $\text{Ran } Z(\lambda)$. Then we get

$$S(\lambda) - I = U(\tilde{S}(\lambda) - I)U^*,$$

and (4.11) follows. This argument is borrowed from [27, Lemma 7.7.1].

(ii) Existence and completeness of wave operators is well known, see e.g. [27, Theorem 6.4.5]. As in the proof of part (i), we have the representation (4.12), (4.13) for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0)$ (see e.g. [27, Section 5.5(3)]) and the required statement follows by the same argument as above. ■

Proof of Theorem 2.2. The existence of the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ follows from Lemma 4.1. Thus, by Theorem 2.1, we obtain (2.11). By Lemma 4.4 and Lemma 4.5, we have

$$\alpha(\lambda) = \frac{1}{2} \|\tilde{S}(\lambda) - I\| = \frac{1}{2} \|S(\lambda) - I\|$$

for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0) \cap \Delta$. Thus, we have (2.13) and therefore (1.1) for a.e. $\lambda \in \hat{\sigma}_{\text{ac}}(H_0) \cap \Delta$. On the other hand, for a.e. $\lambda \in \Delta \setminus \hat{\sigma}_{\text{ac}}(H_0)$, by Lemma 4.3, we have $\alpha(\lambda) = 0$. Thus, according to (2.15), the relations (2.13) and (1.1) hold true also for a.e. $\lambda \in \Delta \setminus \hat{\sigma}_{\text{ac}}(H_0)$. ■

Proof of Theorems 2.3 and 2.4. By Lemma 4.2, the derivatives $F'_0(\lambda)$, $F'(\lambda)$ and the limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ exist for a.e. $\lambda \in \mathbb{R}$. Thus, the identity (2.11) follows from Theorem 2.1. The identities (2.13) and (1.1) follow for a.e. $\lambda \in \mathbb{R}$ as in the proof of Theorem 2.2. ■

4.3. The Fredholm property.

Proof of Theorem 2.6. 1. As in the proof of Lemma 4.4, we get that the operators $I + T_0(\lambda + i0)J$ and $I + JT_0(\lambda + i0)$ have bounded inverses and the identities (4.9), (4.10) hold true.

2. From (4.9), (4.10) we obtain

$$I - A(\lambda)J = (I + T_0(\lambda + i0)J)^{-1}(I + A_0(\lambda)J)(I + T_0(\lambda + i0)^*J)^{-1}.$$

This proves that $\dim \text{Ker}(I - A(\lambda)J) = \dim \text{Ker}(I + A_0(\lambda)J)$ and so (ii) \Leftrightarrow (iii).

3. Let us prove that

$$(4.14) \quad \dim \text{Ker}(I + \tilde{S}(\lambda)) = \dim \text{Ker}(I + A_0(\lambda)J).$$

Using the identity (4.9) and the fact that $\dim \text{Ker}(I + XY) = \dim \text{Ker}(I + YX)$ for any bounded operators X and Y , we obtain:

$$\begin{aligned} \dim \text{Ker}(I + \tilde{S}(\lambda)) &= \dim \text{Ker}(I - iB_0(\lambda)^{1/2}J(I - T(\lambda + i0)J)B_0(\lambda)^{1/2}) \\ &= \dim \text{Ker}(I - iB_0(\lambda)^{1/2}J(I + T_0(\lambda + i0)J)^{-1}B_0(\lambda)^{1/2}) \\ &= \dim \text{Ker}(I - iB_0(\lambda)J(I + T_0(\lambda + i0)J)^{-1}) \\ &= \dim \text{Ker}(I + T_0(\lambda + i0)J - iB_0(\lambda)J) \\ &= \dim \text{Ker}(I + A_0(\lambda)J), \end{aligned}$$

as required.

4. Let us prove that (i) \Leftrightarrow (ii). By the definition (2.20) and by Theorem 2.1, it suffices to prove that $\alpha(\lambda) < 1$ if and only if $\text{Ker}(I + A_0(\lambda)J) = \{0\}$. Suppose that $\text{Ker}(I + A_0(\lambda)J) = \{0\}$. Then by (4.14), we have $\text{Ker}(I + \tilde{S}(\lambda)) = \{0\}$. Since $\tilde{S}(\lambda) - I$ is compact, it follows that $-1 \notin \sigma(\tilde{S}(\lambda))$. Since $\tilde{S}(\lambda)$ is unitary, we get $\|\tilde{S}(\lambda) - I\| < 2$. By (4.7), it follows that $\alpha(\lambda) < 1$.

Conversely, suppose that $\dim \text{Ker}(I + A_0(\lambda)J) > 0$. Then $\dim \text{Ker}(I + \tilde{S}(\lambda)) > 0$ and therefore $\|\tilde{S}(\lambda) - I\| = 2$. By (4.7), it follows that $\alpha(\lambda) = 1$. ■

5. PIECEWISE CONTINUOUS FUNCTIONS φ

We closely follow the proof used by S. Power in his description [15] of the essential spectrum of Hankel operators with piecewise continuous symbols. We use the shorthand notation

$$\delta(\varphi) = \varphi(H) - \varphi(H_0).$$

5.1. Auxiliary statements.

Lemma 5.1. *Assume (2.1) and let $\varphi_1, \varphi_2 \in PC_0(\mathbb{R})$. Suppose that*

$$\text{sing supp } \varphi_1 \cap \text{sing supp } \varphi_2 = \emptyset.$$

Then the operator $\delta(\varphi_1)\delta(\varphi_2)$ is compact.

Proof. 1. For $j = 1, 2$ one can represent φ_j as $\varphi_j = \psi_j + \zeta_j$, where $\zeta_j \in C_0(\mathbb{R})$, $\psi_j \in PC_0(\mathbb{R})$ and $\text{supp } \psi_1 \cap \text{supp } \psi_2 = \emptyset$. By Lemma 3.4, the operators $\delta(\zeta_1)$ and $\delta(\zeta_2)$ are compact. We have

$$\delta(\varphi_1)\delta(\varphi_2) = (\delta(\psi_1) + \delta(\zeta_1))(\delta(\psi_2) + \delta(\zeta_2))$$

and so it suffices to prove that the operator $\delta(\psi_1)\delta(\psi_2)$ is compact.

2. One can choose $\omega \in C_0(\mathbb{R})$ such that $\psi_1\omega = \psi_1$ and $\omega\psi_2 \equiv 0$. Then $\omega(H_0)\psi_2(H_0) = 0$ and

$$\psi_1(H)\psi_2(H_0) = \psi_1(H)\omega(H)\psi_2(H_0) = \psi_1(H)(\omega(H) - \omega(H_0))\psi_2(H_0),$$

and the operator in the r.h.s. is compact by Lemma 3.4. By the same argument, the operator $\psi_1(H_0)\psi_2(H)$ is compact. It follows that the operator

$$\begin{aligned} \delta(\psi_1)\delta(\psi_2) &= (\psi_1(H) - \psi_1(H_0))(\psi_2(H) - \psi_2(H_0)) \\ &= -\psi_1(H_0)\psi_2(H) - \psi_1(H)\psi_2(H_0) \end{aligned}$$

is compact, as required. ■

Lemma 5.2. *Let A_n , $n = 1, \dots, N$, be bounded operators in a Hilbert space. Assume that $A_n A_m$ is compact for all $n \neq m$. Then*

$$(5.1) \quad \sigma_{\text{ess}}(A_1 + \dots + A_N) \cup \{0\} = \left(\bigcup_{j=1}^N \sigma_{\text{ess}}(A_j)\right).$$

See e.g. [14, Section 10.1] for a proof via the Calkin algebra argument. We would like to emphasise that Lemma 5.2 holds true with the definition of the essential spectrum as stated in Section 2.5; it is in general false for some other definitions of the essential spectrum, see e.g. [23, Section XIII.4, Example 1].

5.2. Proof of Theorem 2.5. We start by considering the case of finitely many discontinuities:

Lemma 5.3. *Assume the hypothesis of Theorem 2.5 and suppose in addition that the set $\text{sing supp } \varphi$ is finite. Then the conclusion of Theorem 2.5 holds true.*

Proof. 1. First assume that φ has only one discontinuity, i.e. $\text{sing supp } \varphi = \{\lambda_0\}$. Denote

$$(5.2) \quad \tilde{\varphi}(\lambda) = (\varphi(\lambda_0 + 0) - \varphi(\lambda))/\varkappa_{\lambda_0}(\varphi).$$

Then $\tilde{\varphi}(\lambda_0 - 0) = 1$, $\tilde{\varphi}(\lambda_0 + 0) = 0$. We can write $\tilde{\varphi}$ as

$$\tilde{\varphi}(\lambda) = \chi_{(-\infty, \lambda_0)}(\lambda) + \zeta(\lambda),$$

where $\chi_{(-\infty, \lambda_0)}(\lambda)$ is the characteristic function of $(-\infty, \lambda_0)$ and $\zeta \in C(\mathbb{R})$ is such that the limits of $\zeta(\lambda)$ as $\lambda \rightarrow \pm\infty$ exist. Then by Lemma 3.4, $\zeta(H) - \zeta(H_0)$ is compact, and so

$$\tilde{\varphi}(H) - \tilde{\varphi}(H_0) = E(-\infty, \lambda_0) - E_0(-\infty, \lambda_0) + \text{compact operator}.$$

By Theorem 2.2 and Weyl's theorem on the invariance of the essential spectrum under the compact perturbations, we obtain

$$\sigma_{\text{ess}}(\tilde{\varphi}(H) - \tilde{\varphi}(H_0)) = [-\alpha(\lambda_0), \alpha(\lambda_0)].$$

Recalling the definition (5.2) of φ , we obtain

$$(5.3) \quad \sigma_{\text{ess}}(\varphi(H) - \varphi(H_0)) = [-\alpha(\lambda_0)\varkappa_{\lambda_0}(\varphi), \alpha(\lambda_0)\varkappa_{\lambda_0}(\varphi)].$$

2. Consider the general case; let $\text{sing supp } \varphi = \{\lambda_1, \dots, \lambda_N\} \subset \Delta$. One can represent $\varphi = \sum_{n=1}^N \varphi_n$, where $\varphi_n \in PC_0(\mathbb{R})$, $\text{sing supp } \varphi_n = \{\lambda_n\}$ and $\varkappa_{\lambda_n}(\varphi_n) = \varkappa_{\lambda_n}(\varphi)$ for each n . Then

$$\delta(\varphi) = \sum_{n=1}^N \delta(\varphi_n),$$

and by Lemma 5.1, the operators $\delta(\varphi_n)\delta(\varphi_m)$ are compact for $n \neq m$. Applying Lemma 5.2 and the first step of the proof, we get

$$\sigma_{\text{ess}}(\delta(\varphi)) \cup \{0\} = \cup_{n=1}^N \sigma_{\text{ess}}(\delta(\varphi_n)) = \cup_{n=1}^N [-\alpha(\lambda_n)\varkappa_{\lambda_n}(\varphi), \alpha(\lambda_n)\varkappa_{\lambda_n}(\varphi)].$$

Since σ_{ess} is a closed set, we get $0 \in \sigma_{\text{ess}}(\delta(\varphi))$ and thus the required statement (2.19) follows. ■

Proof of Theorem 2.5. 1. Let

$$\begin{aligned} \Lambda_0 &= \{\lambda \in \Delta \mid |\varkappa_{\lambda}(\varphi)| \geq 1\}, \\ \Lambda_n &= \{\lambda \in \Delta \mid 2^{-n} \leq |\varkappa_{\lambda}(\varphi)| < 2^{-n+1}\}, \quad n = 1, 2, \dots \end{aligned}$$

The set Λ_n is finite for all $n \geq 0$. It is easy to see that for each $n \geq 0$ there exists a function $\varphi_n \in PC_0(\mathbb{R})$ with $\text{sing supp } \varphi_n = \Lambda_n$, $\text{supp } \varphi_n \subset \Delta$, and

$$(5.4) \quad \begin{aligned} \varkappa_{\lambda}(\varphi_n) &= \varkappa_{\lambda}(\varphi) \quad \forall \lambda \in \Lambda_n, \\ \|\varphi_n\|_{\infty} &= \frac{1}{2} \max_{\lambda \in \Lambda_n} |\varkappa_{\lambda}(\varphi)| \leq 2^{-n}. \end{aligned}$$

With this choice, the series $\sum_{n \geq 0} \varphi_n$ converges absolutely and uniformly on \mathbb{R} and defines a function $f = \sum_{n \geq 0} \varphi_n$ such that $f \in PC_0(\mathbb{R})$ and the function $\zeta \stackrel{\text{def}}{=} \varphi - f$ is in the class $C_0(\mathbb{R})$.

2. For a given $N \in \mathbb{N}$, write

$$\varphi = f_N + g_N + \zeta, \quad f_N = \sum_{n=0}^N \varphi_n, \quad g_N = \sum_{n=N+1}^{\infty} \varphi_n.$$

By Lemma 5.1, the operator $\delta(\varphi_m)\delta(\varphi_n)$ is compact for $n \neq m$. By the estimate (5.4), the series in the r.h.s. of

$$\delta(\varphi_m)\delta(g_N) = \sum_{n=N+1}^{\infty} \delta(\varphi_m)\delta(\varphi_n)$$

converges in the operator norm, and so for any $m \leq N$ the operator $\delta(\varphi_m)\delta(g_N)$ is also compact. Applying Lemma 5.2 to the decomposition $\delta(\varphi) = \delta(f_N) + \delta(g_N) + \delta(\zeta)$ and subsequently using Lemma 5.3, we get

$$\begin{aligned} \sigma_{\text{ess}}(\delta(\varphi)) \cup \{0\} &= \sigma_{\text{ess}}(\delta(f_N)) \cup \sigma_{\text{ess}}(\delta(g_N)) \cup \{0\} \\ &= \left(\bigcup_{n=0}^N \sigma_{\text{ess}}(\delta(\varphi_n)) \right) \cup \sigma_{\text{ess}}(\delta(g_N)) \cup \{0\} \\ &= \left(\bigcup_{|\varkappa_\lambda(\varphi)| \leq 2^{1-N}} [-\alpha(\lambda)\varkappa_\lambda(\varphi), \alpha(\lambda)\varkappa_\lambda(\varphi)] \right) \cup \sigma_{\text{ess}}(\delta(g_N)). \end{aligned}$$

Finally, by the estimate (5.4) we have $\|\delta(g_N)\| \leq 2\|g_N\|_\infty \leq 2^{1-N}$ and so $\sigma_{\text{ess}}(\delta(g_N)) \subset \{z \in \mathbb{C} \mid |z| \leq 2^{1-N}\}$. Since N can be taken arbitrary large, we obtain

$$\sigma_{\text{ess}}(\delta(\varphi)) \cup \{0\} = \bigcup_{\lambda \in \Delta} [-\alpha(\lambda)\varkappa_\lambda(\varphi), \alpha(\lambda)\varkappa_\lambda(\varphi)].$$

Since σ_{ess} is a closed set, we get $0 \in \sigma_{\text{ess}}(\delta(\varphi))$ and thus the required statement (2.19) follows. ■

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