# Scattering by anisotropic potential in a constant electric field

E.L.Korotyaev*
Department of Mathematics-2
St-Petersburg Electrotechnical University
197376, St-Petersburg, Russia
A.B.Pushnitski †
Department of Higher Mathematics and Mathematical Physics
St-Petersburg State University
198904, St-Petersburg, Russia

**Abstract**

We consider the scattering of a particle on anisotropic potential in a constant electric field. We obtain conditions on anisotropic decaying of potential, giving the asymptotic completeness without modification of wave operators. We use the mixed approach, applying the Enss method to one part of variables, and the smoothness technique – to the other part.

## 1 Introduction and main results

Consider the scattering of a particle in a constant external electric field $E \in \mathbf{R}^d$. The Hamiltonian is $H = H_E + V$ in $L_2(\mathbf{R}^d)$, where $H_E = -\triangle - (E, x)$, $x \in \mathbf{R}^d$, and real-valued potential $V$ obeys the condition

$$V \in L_\infty(\mathbf{R}^d), \qquad \text{and} \qquad |V(x)| \to 0 \text{ as } |x| \to \infty. \tag{1.1}$$

The wave operators are given by

$$W_\pm(H, H_E) = \text{s-lim} \exp(itH) \exp(-itH_E), \quad t \to \pm\infty.$$

It is usual to assume in addition that $V(x) = O(|(E, x)|^{-\varepsilon_0})$ as $x \to \infty$ with $\varepsilon_0 > 1/2$. Note that this assumption cannot be weakened. For example, if $\varepsilon_0 \le 1/2$, then a modification of

*E-mail: evgeni@niif.spb.su
†E-mail: pushnit@mph.niif.spb.su

the wave operators is required (see [12], [5]). The goal of this paper is to prove asymptotic completeness for anisotropic potential (see condition (1.2) below). Roughly speaking, we consider the case when $\varepsilon_0 \le 1/2$ but $V$ decreases in other directions and modification of the wave operators is not required. To prove our results we use the mixed approach of [7], the Enss method being applied to one part of variables and the smoothness technique to the other part.

With an eye to the three-dimensional physical situation, we consider the decomposition of the coordinate space $\mathbf{R}^d = \mathbf{R}^{d_1} \oplus \mathbf{R}^{d_2} \oplus \mathbf{R}^{d_3}$. For $x \in \mathbf{R}^d$ we write $x = (x_1, x_2, x_3)$ with $x_j \in \mathbf{R}^{d_j}$. Suppose $V$ obeys the condition

$$|V(x)| \le C \langle x_1 \rangle^{-\varepsilon_1} \langle x_2 \rangle^{-\varepsilon_2} \langle x_3 \rangle^{-\varepsilon_3} \tag{1.2}$$

with some $\varepsilon_j \ge 0$, $j = 1, 2, 3$. Here and further we write $\langle x \rangle = (1 + |x|^2)^{1/2}$. We define the nonnegative numbers $e_1$, $e_2$, $e_3$:

$$e_j = \begin{cases} 2\varepsilon_j & \text{if } E_j \ne 0, \\ \frac{1}{4}\min\{2\varepsilon_j, d_j\} & \text{if } E_j = 0 \end{cases} \tag{1.3}$$

First we consider electric field $E$ with non-zero component only in $\mathbf{R}^{d_1}$.

**Theorem 1.1** *Suppose* $E = (E_1, 0, 0)$, $E_1 \in \mathbf{R}^{d_1}$, $E_1 \ne 0$. *Assume that* $V$ *satisfies conditions (1.1), (1.2) with*

$$e_1 + e_2 + e_3 > 1 \qquad \textit{and} \qquad \max\{e_1, e_2, e_3\} > 1/2. \tag{1.4}$$

*Then the wave operators* $W_\pm(H, H_E)$ *are complete, the singular continuous spectrum* $\sigma_{sc}(H)$ *is empty, all eigenvalues of* $H$ *have finite multiplicity and can accumulate only at* $\pm\infty$.

**Remarks.** 1. The existence of the wave operators can be proved by standard application of the stationary phase method (see e.g. [9],[8]) if $2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 > 1$.

2.Following the proof of the theorem, it's easy to see that for $V = \sum_{k=1}^m V_k$, where each $V_k$ obeys the hypothesis of the theorem, its conclusion also holds true.

**Examples.** Below we present the equivalent forms of condition (1.4) for the most important physical cases.

(i) Suppose $d_3 = 0$ (i.e. we have $\mathbf{R}^d = \mathbf{R}^{d_1} \oplus \mathbf{R}^{d_2}$). Then condition (1.4) means that $2\varepsilon_1 + \varepsilon_2/2 > 1$ and, in addition, a) $\varepsilon_1 > 3/8$ if $d_2 = 1$; b) $\varepsilon_1 > 1/4$ if $d_2 = 2$; c) $\varepsilon_1 > 1/8$ if $d_2 = 3$; d) $\varepsilon_1 > 0$ if $d_2 = 4$.

(ii) Suppose $d_2 = d_3 = 1$. Then $e_2 \le 1/4$, $e_3 \le 1/4$ and so (1.4) can be rewritten in the form: $e_1 + e_2 + e_3 > 1$ and $\varepsilon_1 > 1/4$.

Next we consider electric field $E$ with two non-zero components.

**Theorem 1.2** *Suppose* $E = (E_1, E_2, 0)$, $E_j \in \mathbf{R}^{d_j}$, $E_j \ne 0$, $j = 1, 2$. *Assume that* $V$ *satisfies (1.1), (1.2) and one of the two following conditions is fulfilled:*

$$e_1 + e_2 + e_3 > 1 \qquad \textit{and} \qquad e_3 > 1/2 \tag{1.5}$$

*or*

$$e_1 > 1/2 \qquad and \qquad e_2 > 1/2 \tag{1.6}$$

*Then the wave operators $W_{\pm}(H, H_E)$ are complete, the singular continuous specrtum $\sigma_{sc}(H)$ is empty, all eigenvalues of $H$ have finite multiplicity and can accumulate only at $\pm\infty$.*

**Remarks.** 1. The existence of wave operators can be proved by a standard application of the stationary phase method (see e.g. [9],[8]) if $2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 > 1$.

2. For the physical case $d_1 = d_2 = d_3 = 1$ we have $e_3 \leq 1/4$ and so condition (1.5) cannot be fulfilled.

# 2 Proof of Theorems

We recall that conventional proof of completeness for the pair $(H, H_E)$ with $\sigma(H) = \sigma(H_E) = \mathbf{R}$ by Enss method requires fulfillment of the following three conditions for suitable bounded operators $P_{\pm}$ such that $P_+ + P_- = I$.

**Condition 1** *For any $\varphi \in C_0^\infty(\mathbf{R})$ operators $\varphi(H) - \varphi(H_E)$ are compact.*

**Condition 2** s-lim$P_{\mp}^* \exp(-itH_E) = 0$ *as* $t \to \pm\infty$.

**Condition 3** *For any $\varphi \in C_0^\infty(\mathbf{R})$ operators $(I - W_{\pm}(H, H_E))\varphi(H_E)P_{\pm}$ are compact.*

First let us note that by (1.1), $V$ is a relatively compact perturbation of $H_E$ (see [10],[8]) and hence Condition 1 is obviously fulfilled.

Later we shall define operators $P_{\pm}$ in a suitable way for different combinations of conditions on $\varepsilon_j$ (here and further $j \in \{1, 2, 3\}$). We shall prove Theorems 1.1 and 1.2 by directly checking Conditions 2 and 3 for all choices of $P_{\pm}$.

Below we define the standard Enss "projections onto incoming and outgoing states" for Schrödinger and Stark operators.

(i) Let us consider the Schrödinger operator $(-\triangle)$ in $L_2(\mathbf{R}^d)$ and let $J$ be a unitary operator, $J : L_2(\mathbf{R}^d, dx) \to L_2(\mathbf{R}_+, d\lambda) \otimes L_2(\mathbf{S}^{d-1}, d\omega)$, which gives the spectral representation of $(-\triangle)$. Explicitly,

$$(Jf)(\lambda, \omega) = (1/\sqrt{2})\lambda^{(d-2)/4}\hat{f}(\sqrt{\lambda}\omega), \qquad \lambda \geq 0, \qquad \omega = (\omega_1, \dots, \omega_d) \in \mathbf{S}^{d-1},$$

where $\hat{f}$ is the Fourier transform of $f$. We define the operator

$$Q_{\pm}^0 = J^*\theta\left(\pm\frac{1}{i}\frac{d}{d\lambda}\right)J, \tag{2.1}$$

where $\theta$ is the characteristic function of $(0, +\infty)$.

(ii) Let us consider the Stark operator $-\triangle - (E, x)$ in $L_2(\mathbf{R}^d)$. First we suppose that $E = (1, 0, \dots, 0)$, and for $x \in \mathbf{R}^d$ write $x = (x_1, x_\perp)$, $x_\perp = (x_2, \dots, x_d)$. Let $U$ be a unitary operator, $U : L_2(\mathbf{R}^d, dx) \to L_2(\mathbf{R}, d\lambda) \otimes L_2(\mathbf{R}^{d-1}, dx_\perp)$, defined as

$$(Uf)(\lambda, x_\perp) = \int_{-\infty}^{\infty} Ai(-x_1 - \lambda)f(x_1, x_\perp)dx_1, \tag{2.2}$$

where $Ai$ is an Airy function. We define an operator

$$Q_{\pm}^{E} = U^{*}\theta\left(\pm\frac{1}{i}\frac{d}{d\lambda}\right)U. \tag{2.3}$$

For general $E$'s let us change coordinates so that $E$ take the form $E = (1, 0, \dots, 0)$ and then define operators $Q_{\pm}^{E}$ according to (2.2) and (2.3).

Now we proceed to the definition of operators $Q_{\pm}^{(j)}$ for $j = 1, 2, 3$, which will play the role of $P_{\pm}$ in Conditions 2 and 3. In what follows we extensively use the representation $H_E = h_1 + h_2 + h_3$ where $h_j = -\triangle_{x_j} - (E_j, x_j)$ in $L_2(\mathbf{R}^{d_j})$. Here and further we use the same notation for operators $A_j$ in $L_2(\mathbf{R}^{d_j})$, $j = 1, 2, 3$ and operators $A_1 \otimes I \otimes I$, $I \otimes A_2 \otimes I$, $I \otimes I \otimes A_3$ respectively in $L_2(\mathbf{R}^{d_1} \oplus \mathbf{R}^{d_2} \oplus \mathbf{R}^{d_3})$.

Let us define operators $Q_{\pm}^{(j)}$ for $j = 1, 2, 3$:

$$Q_{\pm}^{(j)} = \begin{cases} Q_{\pm}^{E} \text{ for } h_j, & \text{if } E_j \neq 0 \\ Q_{\pm}^{0} \text{ for } h_j, & \text{if } E_j = 0. \end{cases} \tag{2.4}$$

A direct calculation shows that $(Q_{\pm}^{(j)})^{*} = Q_{\pm}^{(j)}$, and using the stationary phase method one easily verifies that

$$\text{s-lim} Q_{\mp}^{(j)} \exp(-itH_E) = 0 \quad \text{as} \quad t \to \pm\infty \tag{2.5}$$

(see, e.g., [9],[8]). As we shall see later, formula (2.5) provides the Condition 2 fulfilled for all choices of $P_{\pm}$.

Now the proof of the Theorems is, roughly speaking, reduced to the check of Condition 3. This check will require some technical results (most of them being well-known) presented below.

The following Lemmas 2.1 and 2.2 give uniform in time estimates (2.8) and (2.9) which are typical for the Enss method. First let us define the function that will realize the high-energy cutoff:

$$\eta \in C^{\infty}(\mathbf{R}), \quad 0 \leq \eta(x) \leq 1, \quad \eta(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ 0 & \text{if } x \geq 0 \end{cases} \tag{2.6}$$

Let us also denote by $F(x_1 < c)$ the characteristic function of the interval $(-\infty, c)$.

**Lemma 2.1** *Suppose* $H_E = -\triangle - (E, x)$ *in* $L_2(\mathbf{R}^d)$, $E = (1, 0, \dots, 0)$. *Then for any* $a \in \mathbf{R}$, $\pm t > 0$, $N > 0$ *the following estimate holds true:*

$$\|F(x_1 < t^2/2 - a) \exp(-itH_E)\eta(H_E - a)Q_{\pm}^{E}\| \leq C_N \langle t \rangle^{-N}, \tag{2.7}$$

*where the constant* $C_N$ *doesn't depend on* $a$.

The proof involves rather cumbersome calculations, and we delay it until the Section 3. Lemma 2.1 expresses the intuitively clear fact that $\{x_1 < t^2/2 - a\}$ is a classically forbidden region for the particle with energy less than $a$. Estimate (2.7) immediately implies part (i) of the next Lemma.

**Lemma 2.2** *(i) Let* $H_E = -\triangle - (E, x)$ *in* $L_2(\mathbf{R}^d)$, $E \neq 0$. *Let* $Q^E_\pm$ *be projections for* $H_E$ *defined by (2.3). Then for any* $\varepsilon > 0$, $a \in \mathbf{R}$ *and* $\pm t > 0$*:*

$$\| \langle x \rangle^{-\varepsilon} \exp(-itH_E)\eta(H_E - a)Q^E_\pm\| \leq C(a) \langle t \rangle^{-e}, \qquad e = 2\varepsilon \tag{2.8}$$

*with some constant* $C(a)$.

*(ii) Let* $H_0 = -\triangle$ *in* $L_2(\mathbf{R}^d)$, *and* $Q^0_\pm$ *be as defined by (2.1). Then for any* $\varepsilon > 0$ *and* $\pm t > 0$*:*

$$\| \langle x \rangle^{-\varepsilon} \exp(-itH_0)Q^0_\pm\| \leq C \langle t \rangle^{-e}, \qquad e = \frac{1}{4}\min\{2\varepsilon, d\} \tag{2.9}$$

*with some constant* $C$.

**Proof** (i) Without loss of generality we suppose that $E = (1, 0, \dots, 0)$. Then, using Lemma 2.1 with $N > 2\varepsilon$ and elementary estimate of $\langle x \rangle^{-\varepsilon}$ for $x_1 > t^2/2 - a$, we obtain:

$$\| \langle x \rangle^{-\varepsilon} e^{-itH_E}\eta(H_E - a)Q^E_\pm\| \leq \| \langle x \rangle^{-\varepsilon} \| \cdot \|F(x_1 < t^2 - a)e^{-itH_E}\eta(H_E - a)Q^E_\pm\|$$

$$+\| \langle x \rangle^{-\varepsilon} F(x_1 > t^2/2 - a)\| \cdot \|e^{-itH_E}\eta(H_E - a)Q^E_\pm\|$$

$$\leq C_N \langle t \rangle^{-N} + C_1(a) \langle t \rangle^{-2\varepsilon} \leq (C_N + C_1(a)) \langle t \rangle^{-2\varepsilon}$$

with some constant $C_1(a)$, and (2.8) follows with $C(a) = C_N + C_1(a)$.

(ii) was proved in [11]. ∎

Next Lemma presents the "weighted" variance of Kato's smoothness.

**Lemma 2.3** *Consider* $H_E = -\triangle - (E, x)$ *in* $L_2(\mathbf{R}^{d_1} \oplus \mathbf{R}^{d_2})$ *with* $E = (E_1, E_2)$, $E_j \in \mathbf{R}^{d_j}$. *For* $\varepsilon_j \geq 0$ *define* $e_j$ *by formula (1.3). Suppose that for some constant* $\gamma \geq 0$ *we have*

$$\gamma + e_1 + e_2 > 1/2.$$

*Then for any* $f \in L_2(\mathbf{R}^{d_1} \oplus \mathbf{R}^{d_2})$*:*

$$\int_{-\infty}^{\infty} \langle t \rangle^{-2\gamma} \| \langle x_1 \rangle^{-\varepsilon_1} \langle x_2 \rangle^{-\varepsilon_2} \exp(-itH_E)f\|^2 dt \leq C\|f\|^2, \tag{2.10}$$

*where the constant* $C$ *doesn't depend on* $f$.

**Proof** First we suppose that $e_1 > 1/2$. If $E_1 \neq 0$, then (2.10) follows from the global smoothness of $\langle x_1 \rangle^{-\varepsilon_1}$, $\varepsilon_1 > 1/4$, with respect to the Stark operator $h_1 = -\triangle_{x_1} - (E_1, x_1)$ (see [6],[4]). If $E_1 = 0$, then (2.10) follows from the global smoothness of $\langle x_1 \rangle^{-\varepsilon_1}$, $\min\{\varepsilon_1, d_1/2\} > 1$, with respect to the Schrödinger operator $h_1 = -\triangle_{x_1}$ (see [7]).

Similarly, if $e_2 > 1/2$, (2.10) holds true for the same reason. If $\gamma > 1/2$, then (2.10) is trivial. The general case is obtained by interpolation between the cases ($e_1 > 1/2$, $e_2 = 0$, $\gamma = 0$), ($e_1 = 0$, $e_2 > 1/2$, $\gamma = 0$) and ($e_1 = 0$, $e_2 = 0$, $\gamma > 1/2$). ∎

Combining Lemma 2.2 and 2.3 we obtain

**Lemma 2.4** *(i) Suppose that* $e_1 + e_2 + e_3 > 1$, $E_j \neq 0$ *and* $e_j > 1/2$ *for some* $j$. *Then for any* $a \in \mathbf{R}$, $r > 0$ *and any* $f \in L_2(\mathbf{R}^d)$*:*

$$\int_r^{\infty} \|V \exp(\pm itH_E)\eta(h_j - a)Q^{(j)}_\pm f\|dt \leq C(r, a)\|f\|, \tag{2.11}$$

*where the constant $C(r,a)$ doesn't depend on $f$ and $C(r,a) \to 0$ as $r \to \infty$ for any fixed $a$.*

*(ii) Suppose that $e_1 + e_2 + e_3 > 1$, $E_j = 0$ and $e_j > 1/2$ for some $j$ . Then for any $r > 0$ and any $f \in L_2(\mathbf{R}^d)$:*

$$\int_r^\infty \|V \exp(\pm itH_E) Q_\pm^{(j)} f\| dt \le C(r)\|f\|, \tag{2.12}$$

*where the constant $C(r)$ doesn't depend on $f$ and $C(r) \to 0$ as $r \to \infty$.*

**Proof** Without loss of generality we take $j = 1$ and consider the sign "+".

(i) Using condition (1.2) and estimate (2.8) from Lemma 2.2, we obtain:

$$\|V \exp(-itH_E)\eta(h_1 - a)Q_+^{(1)} f\| \le C(a) \langle t\rangle^{-e_1} \| \langle x_2\rangle^{-\varepsilon_2} \langle x_3\rangle^{-\varepsilon_3} e^{-it(h_2+h_3)} f\|.$$

Using strict inequality $e_1 + e_2 + e_3 > 1$, let us take $\delta > 0$ so that $e_1 + e_2 + e_3 > 1 + \delta$. Now we rewrite $e_1$ as $e_1 = 1/2 + \delta + \gamma$, $\gamma > 0$ apply the Hölder inequality, and then use Lemma 2.3 with $\gamma = e_1 - 1/2 - \delta$:

$$\int_r^\infty \langle t\rangle^{-e_1} \| \langle x_2\rangle^{-\varepsilon_2} \langle x_3\rangle^{-\varepsilon_3} e^{-it(h_2+h_3)} f\| dt \le$$

$$\left(\int_r^\infty \langle t\rangle^{-1-2\delta}\right)^{1/2} \left(\int_r^\infty \langle t\rangle^{-2\gamma} \| \langle x_2\rangle^{-\varepsilon_2} \langle x_3\rangle^{-\varepsilon_3} e^{-it(h_2+h_3)} f\|^2 dt\right)^{1/2} \le C_1 r^{-\delta}\|f\|,$$

with some constant $C_1$, and (2.11) holds with $C(r,a) = C_1 C(a) r^{-\delta}$.

(ii) As in the case (i), we have the estimate

$$\|V \exp(-itH_E) Q_+^{(1)} f\| \le C \langle t\rangle^{-e_1} \| \langle x_2\rangle^{-\varepsilon_2} \langle x_3\rangle^{-\varepsilon_3} e^{-it(h_2+h_3)} f\|$$

by Lemma 2.2(ii), and further proof is the same. ■

As we shall see later, the following Corollary provides the Condition 3 fulfilled.

**Corollary 2.5** *(i) Suppose that $e_1 + e_2 + e_3 > 1$ and for some $j$: $E_j \ne 0$ and $e_j > 1/2$. Then for any $\varphi \in C_0^\infty(\mathbf{R})$ and any $a \in \mathbf{R}$ operators $(W_\pm - I)\varphi(H_E)\eta(h_j - a)Q_\pm^{(j)}$ are compact.*

*(ii) Suppose that $e_1 + e_2 + e_3 > 1$ and for some $j$: $E_j = 0$ and $e_j > 1/2$. Then for any $\varphi \in C_0^\infty(\mathbf{R})$ operators $(W_\pm - I)\varphi(H_E)Q_\pm^{(j)}$ are compact.*

**Proof** This result is standard in the Enss method (see [8]).

(i) We have:

$$(W_\pm - I)\varphi(H_E)\eta(h_j - a)Q_\pm^{(j)} = \varphi(H)(W_\pm - I)\eta(h_j - a)Q_\pm^{(j)} +$$

$$(\varphi(H) - \varphi(H_E))\eta(h_j - a)Q_\pm^{(j)}.$$

So, by compactness of $\varphi(H) - \varphi(H_E)$ we only need to check compactness of the first summand which follows from compactness of its approximations

$$\varphi(H)(e^{itH} e^{-itH_E} - I)\eta(h_j - a)Q_\pm^{(j)} = \varphi(H) \int_0^t e^{i\tau H} V e^{-i\tau H_E} \eta(h_j - a) Q_\pm^{(j)} d\tau,$$

and from their uniform convergence which is given by Lemma 2.4(i).

(ii) can be proved analogously. ∎

Now we are ready to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1** We shall specify operators $P_\pm$ for several $\{\varepsilon_j\}_{j=1}^3$ and check Conditions 2 and 3 using formula (2.5) and Corollary 2.5 (as we have stated above, Condition 1 is fulfilled by (1.1)).

(i) First we consider the case $e_1 + e_2 + e_3 > 1$ and $e_1 > 1/2$. Here we take $P_\pm = Q_\pm^{(1)}$. Condition 2 is fulfilled by (2.5). Since $H_E = h_1 + h_2 + h_3$ and $\sigma(h_2) = \sigma(h_3) = [0, +\infty)$, we can write for given $\varphi \in C_0^\infty(\mathbf{R})$: $\varphi(H_E) = \varphi(H_E)\eta(h_1 - a)$ with $a \geq 1 + \sup \operatorname{supp}\varphi$ ($\eta$ is defined by (2.6)). Hence,

$$(I - W_\pm)\varphi(H_E)P_\pm = (I - W_\pm)\varphi(H_E)\eta(h_1 - a)Q_\pm^{(1)}$$

and Condition 3 is fulfilled by Corollary 2.5(i) with $j = 1$.

(ii) Next, we suppose that $e_1 + e_2 + e_3 > 1$ and $e_2 > 1/2$. Here we take $P_\pm = Q_\pm^{(2)}$. Condition 2 is fulfilled by (2.5), and Condition 3 follows from Corollary 2.5(ii) with $j = 2$.

(iii) Finally, the case $e_1 + e_2 + e_3 > 1$ and $e_3 > 1/2$ is treated exactly as the previous case, taking $P_\pm = Q_\pm^{(3)}$. ∎

**Proof of Theorem 1.2**

(i) First we consider the case $e_1 + e_2 + e_3 > 1$ and $e_3 > 1/2$. Here we take $P_\pm = Q_\pm^{(3)}$. Condition 2 is fulfilled by (2.5), and compactness of $(I - W_\pm)\varphi(H_E)P_\pm$ follows from Corollary 2.5(ii) with $j = 3$, as in Theorem 1.1, cases (ii) and (iii).

(ii) Suppose that $e_1 > 1/2$ and $e_2 > 1/2$. Here the proof is based on the representation

$$\varphi(H_E) = \varphi(H_E)\eta(h_1 - a) + \varphi(H_E)\eta(h_2 - a)(I - \eta(h_1 - a))$$

for some $a \in \mathbf{R}$, $a = a(\varphi)$. By Corollary 2.5(i) with $j = 1, 2$ operators $(I - W_\pm)\varphi(H_E)\eta(h_1 - a)Q_\pm^{(1)}$ and $(I - W_\pm)\varphi(H_E)\eta(h_2 - a)(I - \eta(h_1 - a))Q_\pm^{(2)}$ are compact. This with (2.5) and usual argument of Enss method (see [2],[8]) gives the proof. ∎

# 3 Proof of Lemma 2.1

In this section we give the proof of Lemma 2.1, which describes the motion of a quantum particle in a constant electric field. Several statements close to this Lemma can be found in [8], [5].

**Proof of Lemma 2.1**

1. First we give the proof for the one-dimensional case. Without loss of generality we take the sign "+" and suppose that $t > 1$. Let us define a $C^\infty$ partition of unity for $(-\infty, a)$ as follows. Take $\eta_k(\lambda) = \eta(\lambda - a + k) - \eta(\lambda - a + k + 1)$, so that $\operatorname{supp}\eta_k \subset [a - k - 2, a - k]$, and $\eta(\lambda - a) = \sum_{k=0}^\infty \eta_k(\lambda)$.

We shall prove the estimate for $t>1$:

$$\|F(x<t^2/2-a)\exp(-itH_E)\eta_k(H_E)Q_\pm^E\|\le C_N(1+k)^{-N}\langle t\rangle^{-N} \tag{3.1}$$

from which (2.7) follows by summing on $k$.

2. To prove (3.1) let us write out $Q_+^E$ in an explicit form. We denote by $\Phi$ the Fourier transform in $L_2(\mathbf{R})$. For any $f\in L_2(\mathbf{R})$ we have

$$Q_+^E f=U^*\Phi^*\theta(p)\Phi U f=U^*\Phi^*\theta(p)g(p),$$

where $g=\Phi Uf$, $\|g\|=\|f\|$, $U$ is defined by (2.2). So, to prove (3.1) it is sufficient to show that for any $g\in C_0^\infty(\mathbf{R}_+)$:

$$\|F(x<t^2/2-a)U^*e^{-it\lambda}\eta_k(\lambda)\Phi^*g\|\le C_N(1+k)^{-N}\langle t\rangle^{-N}\|g\|. \tag{3.2}$$

But for $g\in C_0^\infty(\mathbf{R}_+)$ we have:

$$\left[F(x<t^2/2-a)U^*e^{-it\lambda}\eta_k(\lambda)\Phi^*g\right](x)=\int_0^\infty M_{t,k}(x,p)g(p)dp,$$

with the kernel

$$M_{t,k}(x,p)=F(x<t^2/2-a)\int_{-\infty}^\infty Ai(-x-\lambda)e^{-i(p+t)\lambda}\eta_k(\lambda)d\lambda \tag{3.3}$$

with $t>1$ and $p>0$. Below we prove the estimate for any $N>0$:

$$|M_{t,k}(x,p)|\le F(x<t^2/2-a)C'_N(t^2-x-a+k)^{-N} \tag{3.4}$$

(with some constant $C'_N$) from which (3.2) readily follows.

3. Let us prove (3.4). Changing notations in formula (3.3), we see that (3.4) will follow from the estimate for any $b<t^2/2$:

$$\left|\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\omega(\lambda-b)d\lambda\right|\le C'_N(t^2-b)^{-N}, \tag{3.5}$$

where $\omega\in C_0^\infty(\mathbf{R})$, $\mathrm{supp}\omega\in[-2,0]$. To prove (3.5) let us make the following simple calculation. Suppose $\varphi\in C_0^\infty(\mathbf{R})$. Integrating by parts and using the Airy equation for $Ai(\lambda)$, one easily obtains:

$$\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\varphi(\lambda)d\lambda=(1/t^2)\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\varphi(\lambda)\lambda d\lambda$$

$$+(2i/t)\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\varphi'(\lambda)d\lambda+(1/t^2)\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\varphi''(\lambda)d\lambda.$$

Now let us take $\varphi(\lambda)=\omega(\lambda-b)(1-\lambda/t^2)^{-1}$. Then we have:

$$\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}\omega(\lambda-b)d\lambda=\int_{-\infty}^\infty Ai(-\lambda)e^{-it\lambda}[(2i/t)\omega'(\lambda-b)(1-\lambda/t^2)^{-1}$$

$$+(2i/t^3)\omega(\lambda-b)(1-\lambda/t^2)^{-2}+(1/t^2)\omega''(\lambda-b)(1-\lambda/t^2)^{-1}$$
$$+(2/t^4)\omega'(\lambda-b)(1-\lambda/t^2)^{-2}+(2/t^6)\omega(\lambda-b)(1-\lambda/t^2)^{-3}]d\lambda.$$

Again applying this formula to each of the summands in the right-hand part, and iterating the calculation $2N$ times, we see that (3.5) will follow from the estimate:

$$t^{-m}\left|\int_{-\infty}^{\infty} Ai(-\lambda)e^{-it\lambda}\omega^{(l)}(\lambda-b)\left(1-\lambda/t^2\right)^{-n}d\lambda\right|\leq C_N''(t^2-b)^{-N} \tag{3.6}$$

with some constant $C_N''$ and all $n\geq 2N$, $m\geq 2N$, and $l\leq 4N$. But for $(\lambda-b)\in \mathrm{supp}\omega$ we have:

$$t^2\left(1-\lambda/t^2\right)\geq t^2-b,$$

and (3.6) follows.

4. Now we turn to the multidimensional case. Let $\Phi_\perp$ be the Fourier transform in $x_\perp$ variables in $L_2(\mathbf{R}^d)$. Then

$$H_E=\Phi_\perp^*(h_1+\mathbf{p}_\perp^2)\Phi_\perp,$$

where $h_1=-d^2/dx_1^2-x_1$ in $L_2(\mathbf{R},dx_1)$ and $\mathbf{p}_\perp^2$ is the operator of multiplication by $p_\perp^2$ in $L_2(\mathbf{R}^{d-1},dp_\perp)$. For $f\in L_2(\mathbf{R}^d)$ we write $\hat{f}=\Phi_\perp f$ and $\hat{f}(\cdot,p_\perp)=f_{p_\perp}\in L_2(\mathbf{R},dx_1)$ for a.e. $p_\perp\in\mathbf{R}^{d-1}$. We have for $\pm t>0$, using (2.7) for $d=1$:

$$\|F(x_1<t^2/2-a)\exp(-itH_E)\eta(H_E-a)Q_\pm^E f\|^2$$
$$=\int_{\mathbf{R}^{d-1}}\|F(x_1<t^2/2-a)\exp(-it(h_1+p_\perp^2))\eta(h_1+p_\perp^2-a)Q_\pm^E f_{p_\perp}\|^2_{L_2(\mathbf{R})}dp_\perp$$
$$=\int_{\mathbf{R}^{d-1}}\|F(x_1<t^2/2-a)F(x_1<t^2-(a-p_\perp^2))\exp(-it(h_1+p_\perp^2))$$
$$\times\eta(h_1-(a-p_\perp^2))Q_\pm^E f_{p_\perp}\|^2_{L_2(\mathbf{R})}dp_\perp\leq C_N\langle t\rangle^{-N}\int_{\mathbf{R}^{d-1}}\|f_{p_\perp}\|^2_{L_2(\mathbf{R})}dp_\perp=C_N\langle t\rangle^{-N}\|f\|^2,$$

and Lemma follows. ∎

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