

SPECTRAL THEORY OF DISCONTINUOUS FUNCTIONS OF SELF-ADJOINT OPERATORS AND SCATTERING THEORY

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To the memory of M. Sh. Birman (1928–2009)

ABSTRACT. In the smooth scattering theory framework, we consider a pair of self-adjoint operators H_0 , H and discuss the spectral projections of these operators corresponding to the interval $(-\infty, \lambda)$. The purpose of the paper is to study the spectral properties of the difference $D(\lambda)$ of these spectral projections. We completely describe the absolutely continuous spectrum of the operator $D(\lambda)$ in terms of the eigenvalues of the scattering matrix $S(\lambda)$ for the operators H_0 and H . We also prove that the singular continuous spectrum of the operator $D(\lambda)$ is empty and that its eigenvalues may accumulate only at “thresholds” in the absolutely continuous spectrum.

1. INTRODUCTION

Let H_0 and H be self-adjoint operators in a Hilbert space \mathcal{H} and suppose that the difference $V = H - H_0$ is a compact operator. If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which tends to zero at infinity then a well known simple argument shows that the difference

$$(1.1) \quad \varphi(H) - \varphi(H_0)$$

is compact. On the other hand, if φ has discontinuities on the essential spectrum of H_0 and H , then the difference (1.1) may fail to be compact even for perturbations V of a finite rank; see [12, 11].

The simplest example of a function φ with a discontinuity is the characteristic function of a semi-axis. Thus, for a Borel set $\Lambda \subset \mathbb{R}$ we denote by $E_0(\Lambda)$ (resp. $E(\Lambda)$) the spectral projection of H_0 (resp. H) corresponding to the set Λ and consider the difference

$$(1.2) \quad D(\lambda) = E((-\infty, \lambda)) - E_0((-\infty, \lambda))$$

where λ belongs to the absolutely continuous (a.c.) spectrum of H_0 .

In [12], M. G. Kreĭn has shown that under some assumptions of the trace class type on the pair H_0 and H , the operator $\varphi(H) - \varphi(H_0)$ belongs to the trace class for all sufficiently “nice” functions φ and

$$\mathrm{Tr}(\varphi(H) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(t)\xi(t)dt,$$

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where the function $\xi(\cdot) = \xi(\cdot; H, H_0)$ is known as the spectral shift function. Formally taking the characteristic function $\chi_{(-\infty, \lambda)}$ of the interval $(-\infty, \lambda)$ for φ , we obtain the relation

$$(1.3) \quad \xi(\lambda) = -\text{“Tr”} D(\lambda)$$

where “Tr” is the regularized trace.

The relation between the spectral shift function and the scattering matrix $S(\lambda) = S(\lambda; H, H_0)$ for the pair H_0, H was found in the paper [3] by M. Sh. Birman and M. G. Kreĭn, where it was shown that

$$(1.4) \quad \det S(\lambda) = e^{-2\pi i \xi(\lambda)}$$

for a.e. λ from the core of the a.c. spectrum of H_0 (see e.g. [19, Section 1.3] for the discussion of the notion of the core). The importance of (1.3), (1.4) is in the fact that they give a relation between the key object of *spectral perturbation theory* $D(\lambda)$ and the key object of *scattering theory* $S(\lambda)$.

Our aim here is to discuss the spectral properties of $D(\lambda)$. It turns out that (1.3), (1.4) is not the only link between $D(\lambda)$ and $S(\lambda)$. In fact, the spectral properties of $D(\lambda)$ can be completely described in terms of the eigenvalues $e^{i\theta_n(\lambda)}$ of the scattering matrix $S(\lambda)$. We show that the a.c. spectrum of $D(\lambda)$ consists of the union of the intervals

$$(1.5) \quad \bigcup_n [-\varkappa_n(\lambda), \varkappa_n(\lambda)], \quad \varkappa_n(\lambda) = |e^{i\theta_n(\lambda)} - 1|/2, \quad e^{i\theta_n(\lambda)} \neq 1,$$

where each interval has multiplicity one in the spectrum. We also prove that the singular continuous spectrum of $D(\lambda)$ is empty, the eigenvalues of $D(\lambda)$ can accumulate only to 0 and to the points $\pm \varkappa_n(\lambda)$, and all eigenvalues of $D(\lambda)$ distinct from 0 and $\pm \varkappa_n(\lambda)$ have finite multiplicity. In particular, $D(\lambda)$ is compact if and only if $S(\lambda) = I$. On the other hand, the a.c. spectrum of $D(\lambda)$ covers the interval $[-1, 1]$ if and only if the spectrum of $S(\lambda)$ contains -1 .

The present paper can be considered as a continuation of [15], where the description (1.5) of the a.c. spectrum of $D(\lambda)$ was obtained using a combination of assumptions of trace class and smooth scattering theory. In contrast to [15], here we use only the technique of smooth scattering theory, which yields stronger results.

Our “model” operator is constructed in terms of a certain Hankel integral operator with kernel (3.1) and of the scattering matrix. Using the explicit diagonalization of the Hankel integral operator (3.1) (which we call the “half-Carleman” operator) given by the Mehler-Fock transform (see Section 3.1), we find a class of operators smooth with respect to the “half-Carleman” operator. This allows us to develop scattering theory for the pair consisting of the model operator and the operator $D(\lambda)^2$. To a certain extent, we were inspired by J. S. Howland’s papers [7] where the smooth version of scattering theory was developed for operators of Carleman type via the Mourre commutator method.

There is a close relationship between the properties of the difference $\varphi(H) - \varphi(H_0)$ and the theory of Hankel operators. This fact was exhibited in the work [14] by V. Peller. The problem discussed in this paper gives another example of this relationship.

When this paper was at the final stage of preparation, the authors have learnt that their teacher M. Sh. Birman has passed away. Much of the modern spectral and scattering theory is Birman's legacy. We dedicate this work to his memory.

2. MAIN RESULTS

2.1. Definition of the operator H . Let H_0 be a self-adjoint operator in a Hilbert space \mathcal{H} , and let V be a symmetric operator which we consider as a perturbation of H_0 . Our first goal is to correctly define the sum $H = H_0 + V$. Following the approach which goes back at least to [9] and is developed in more detail in [19, Sections 1.9, 1.10], below we define the operator H in terms of its resolvent. If V is bounded, then the operator H we define coincides with the operator sum $H_0 + V$. In the semi-bounded case the operator H can be defined via its quadratic form.

We suppose that V is factorized as $V = G^* J G$, where G is an operator from \mathcal{H} to an auxiliary Hilbert space \mathcal{K} and J is an operator in \mathcal{K} . We assume that

$$(2.1) \quad \begin{aligned} & J = J^* \text{ is bounded in } \mathcal{K}, \\ & \text{Dom}|H_0|^{1/2} \subset \text{Dom } G \quad \text{and} \quad G(|H_0| + I)^{-1/2} \text{ is compact.} \end{aligned}$$

In applications such a factorization often arises naturally from the structure of the problem. In any case, one can always take $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$ and $J = \text{sign}(V)$.

Let us accept

Definition 2.1. A self-adjoint operator H corresponds to the sum $H_0 + V$ if the following two conditions are satisfied:

- (i) For any regular point $z \in \mathbb{C} \setminus \text{spec}(H)$, its resolvent $R(z) = (H - zI)^{-1}$ admits the representation

$$(2.2) \quad R(z) = (|H_0| + I)^{-1/2} B(z) (|H_0| + I)^{-1/2}$$

where the operator $B(z)$ is bounded. In particular, $\text{Dom } H \subset \text{Dom}|H_0|^{1/2}$.

- (ii) One has

$$(f_0, Hf) = (H_0 f_0, f) + (JGf_0, Gf), \quad \forall f_0 \in \text{Dom } H_0, \quad \forall f \in \text{Dom } H.$$

Only one self-adjoint operator H can satisfy this definition, and under the assumption (2.1) such an operator exists and is defined below via its resolvent. For $z \in \mathbb{C} \setminus \text{spec}(H_0)$, let us denote $R_0(z) = (H_0 - zI)^{-1}$. Formally, we define the operator $T(z)$ (sandwiched resolvent) by

$$(2.3) \quad T(z) = GR_0(z)G^*;$$

more precisely, this means

$$T(z) = (G(|H_0| + I)^{-1/2})(|H_0| + I)R_0(z)(G(|H_0| + I)^{-1/2})^*.$$

By (2.1), the operator $T(z)$ is compact. Under the assumption (2.1), it can be shown (see [19, Sections 1.9, 1.10]) that the operator $I + T(z)J$ has a bounded inverse for all $z \in \mathbb{C} \setminus \mathbb{R}$ and

$$(2.4) \quad R(z) = R_0(z) - (GR_0(\bar{z}))^* J (I + T(z)J)^{-1} GR_0(z)$$

is the resolvent of a self-adjoint operator H which satisfies Definition 2.1. Of course the resolvents of H_0 and H are related by the usual identity

$$(2.5) \quad R(z) - R_0(z) = -(GR_0(\bar{z}))^* JGR(z).$$

If H_0 is semi-bounded from below, then (2.1) means that V is H_0 -form compact, and then H coincides with the operator $H_0 + V$ defined as a quadratic form sum (see the KLMN Theorem in [18]).

2.2. Scattering Theory. Recall that, for a pair of self-adjoint operators H_0 and H and a Borel set $\Lambda \subset \mathbb{R}$, the (local) wave operators are introduced by the relation

$$W_{\pm}(H, H_0; \Lambda) = \text{s-lim}_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0t} E_0(\Lambda) P_0^{(a)}$$

provided these strong limits exist. Here and in what follows we denote by $P_0^{(a)}$ (resp. $P^{(a)}$) the orthogonal projection onto the absolutely continuous subspace of H_0 (resp. H). The wave operators enjoy the intertwining property $W_{\pm}(H, H_0; \Lambda) H_0 = HW_{\pm}(H, H_0; \Lambda)$. The wave operators are called complete if

$$\text{Ran } W_+(H, H_0; \Lambda) = \text{Ran } W_-(H, H_0; \Lambda) = \text{Ran } (E(\Lambda) P^{(a)}).$$

If $\Lambda = \mathbb{R}$, then Λ is omitted from the notation.

We fix a compact interval $\Delta \subset \mathbb{R}$ and assume that the spectrum of H_0 in Δ is purely a.c. with a constant multiplicity $N_0 \leq \infty$. The interior of Δ is denoted by $\text{int}(\Delta)$. We make an assumption typical for smooth scattering theory; in the terminology of [19], we assume that G is strongly H_0 -smooth on Δ with some exponent $\alpha \in (0, 1]$. This means the following. Let \mathcal{F} be a unitary operator from $\text{Ran } E_0(\Delta)$ to $L^2(\Delta, \mathcal{N})$, $\dim \mathcal{N} = N_0$, such that \mathcal{F} diagonalizes H_0 : if $f \in \text{Ran } E_0(\Delta)$ then

$$(2.6) \quad (\mathcal{F}H_0f)(\lambda) = \lambda(\mathcal{F}f)(\lambda), \quad \lambda \in \Delta.$$

The strong H_0 -smoothness of G on the interval Δ means that the operator

$$G_{\Delta} \stackrel{\text{def}}{=} GE_0(\Delta) : \text{Ran } E_0(\Delta) \rightarrow \mathcal{K}$$

satisfies the equation

$$(2.7) \quad (\mathcal{F}G_{\Delta}^* \psi)(\lambda) = Z(\lambda)\psi, \quad \forall \psi \in \mathcal{K}, \quad \lambda \in \Delta,$$

where $Z = Z(\lambda) : \mathcal{K} \rightarrow \mathcal{N}$ is a family of compact operators obeying

$$(2.8) \quad \|Z(\lambda)\| \leq C, \quad \|Z(\lambda) - Z(\lambda')\| \leq C|\lambda - \lambda'|^{\alpha}, \quad \lambda, \lambda' \in \Delta.$$

Note that the notion of strong smoothness is not unitary invariant, as it depends on the choice of the map \mathcal{F} . It follows from (2.7) that the adjoint operator $G_{\Delta} \mathcal{F}^* : L^2(\Delta, \mathcal{N}) \rightarrow \mathcal{K}$ acts by the formula

$$(2.9) \quad G_{\Delta} \mathcal{F}^* f = \int_{\Delta} Z(\lambda)^* f(\lambda) d\lambda.$$

Let us summarize our assumptions:

Assumption 2.2. (A) H_0 has a purely a.c. spectrum with multiplicity N_0 on the interval Δ .

- (B) V admits a factorization $V = G^*JG$ with the operators G and J satisfying (2.1).
 (C) G is strongly H_0 -smooth on Δ .

We need the following well known results (see e.g. [19, Section 4.4]).

Proposition 2.3. *Let Assumption 2.2 hold. Then the operator-valued function $T(z)$ defined by (2.3) is Hölder continuous for $\operatorname{Re} z \in \operatorname{int}(\Delta)$, $\operatorname{Im} z \geq 0$. The set $\mathcal{X} \subset \Delta$ where the equation $f + T(\lambda + i0)Jf = 0$ has a nontrivial solution is closed and has the Lebesgue measure zero. The operator $I + T(\lambda + i0)J$ is invertible for all $\lambda \in \Omega \stackrel{\text{def}}{=} \operatorname{int}(\Delta) \setminus \mathcal{X}$.*

Proposition 2.4. *Let Assumption 2.2 hold. Then the local wave operators $W_{\pm}(H, H_0; \Delta)$ exist and are complete. Moreover, the spectrum of H in Ω is purely absolutely continuous.*

The last statement of Proposition 2.4 is usually formulated under the additional assumption $\operatorname{Ker} G = \{0\}$. Actually, this assumption is not necessary; this is verified in Lemma A.1 of the Appendix.

In terms of the wave operators the (local) scattering operator is defined as

$$\mathbf{S} = \mathbf{S}(H, H_0; \Delta) = W_+(H, H_0; \Delta)^*W_-(H, H_0; \Delta).$$

The scattering operator \mathbf{S} commutes with H_0 and is unitary on the subspace $\operatorname{Ran} E_0(\Delta)$. Thus, we have a representation

$$(\mathcal{F}\mathbf{S}\mathcal{F}^*f)(\lambda) = S(\lambda)f(\lambda), \quad \text{a.e. } \lambda \in \Delta,$$

where the operator $S(\lambda) : \mathcal{N} \rightarrow \mathcal{N}$ is called the scattering matrix for the pair of operators H_0, H . The scattering matrix is a unitary operator in \mathcal{N} .

We need the stationary representation for the scattering matrix (see [19, Chapter 7] for the details).

Proposition 2.5. *Let Assumption 2.2 hold, and let $\lambda \in \Omega$. Then*

$$(2.10) \quad S(\lambda) = I - 2\pi i Z(\lambda)J(I + T(\lambda + i0)J)^{-1}Z(\lambda)^*.$$

This proposition, in particular, implies that $S(\lambda)$ is a Hölder continuous function of $\lambda \in \Omega$.

Since the operator $Y(\lambda) = J(I + T(\lambda + i0)J)^{-1}$ is bounded and $Z(\lambda)$ is compact, it follows that the operator $S(\lambda) - I$ is compact. Thus, the spectrum of $S(\lambda)$ consists of eigenvalues accumulating possibly only to the point 1. All eigenvalues of $S(\lambda)$ distinct from 1 have finite multiplicities. If $N_0 = \infty$ then necessarily 1 is the eigenvalue of infinite multiplicity or the accumulation point (or both).

2.3. Main Result. First note that since $D(\lambda)$ is the difference of two orthogonal projections, the spectrum of $D(\lambda)$ is a subset of $[-1, 1]$.

We denote by $e^{i\theta_n(\lambda)}$, $n = 1, \dots, N$, $N \leq N_0$, the eigenvalues of $S(\lambda)$ distinct from 1. The eigenvalues are enumerated with the multiplicities taken into account. We set $\varkappa_n(\lambda) = |e^{i\theta_n(\lambda)} - 1|/2$.

Theorem 2.6. *Let Assumption 2.2 hold true and let $\lambda \in \Omega$. Then the a.c. spectrum of $D(\lambda)$ consists of the union of intervals (1.5), where each interval has multiplicity one in the spectrum. The operator $D(\lambda)$ has no singular continuous spectrum. The eigenvalues of $D(\lambda)$ can accumulate only to 0 and to the points $\pm\kappa_n(\lambda)$. All eigenvalues of $D(\lambda)$ distinct from 0 and $\pm\kappa_n(\lambda)$ have finite multiplicities.*

The part of the theorem concerning the a.c. spectrum can be equivalently stated as follows: *The a.c. component of $D(\lambda)$ is unitarily equivalent to the operator of multiplication by x in the orthogonal sum*

$$\bigoplus_{n=1}^N L^2([-\kappa_n(\lambda), \kappa_n(\lambda)], dx).$$

In [15], the above characterization of the a.c. spectrum of $D(\lambda)$ was obtained under more restrictive assumptions which combined smooth type and trace class type requirements. The construction of [15] gives no information on either the singular spectrum of $D(\lambda)$ or on its eigenvalues.

2.4. Examples. Let $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$ with $d \geq 1$. Application of the Fourier transform shows that H_0 has a purely a.c. spectrum $[0, \infty)$ with multiplicity $N_0 = 2$ if $d = 1$ and $N_0 = \infty$ if $d \geq 2$.

Let $H = H_0 + V$, where V is the operator of multiplication by a function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ which is assumed to satisfy

$$(2.11) \quad |V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1.$$

Let $G = |V|^{1/2}$, $J = \text{sign } V$ so that $V = G^* J G$. Then Assumption 2.2 is fulfilled on every compact subinterval Δ of $(0, \infty)$. Moreover, by a well known argument involving Agmon's "bootstrap" [1] and Kato's theorem [8] on the absence of positive eigenvalues of H , the operator $I + T(\lambda + i0)J$ is invertible for all $\lambda > 0$ and hence $\Omega = (0, \infty)$. Thus, Proposition 2.4 implies that the wave operators $W_{\pm}(H, H_0)$ exist and are complete (this result was first obtained in [10, 13]). The scattering matrix $S(\lambda)$ is a unitary operator in $L^2(\mathbb{S}^{d-1})$ (here $\mathbb{S}^0 = \{-1, 1\}$) and depends Hölder continuously on $\lambda > 0$. According to Proposition 2.5 the operator $S(\lambda) - I$ is compact, and hence its spectrum consists of eigenvalues $e^{i\theta_n(\lambda)}$.

In this example all the assumptions of Theorem 2.6 hold true with $\Omega = (0, \infty)$. Denoting, as before, $\kappa_n(\lambda) = |e^{i\theta_n(\lambda)} - 1|/2$, we obtain:

Theorem 2.7. *Assume (2.11). Then for any $\lambda > 0$, the a.c. spectrum of $D(\lambda)$ consists of the union of intervals (1.5), where each interval has multiplicity one in the spectrum. The operator $D(\lambda)$ has no singular continuous spectrum. The eigenvalues of $D(\lambda)$ can accumulate only to 0 and to the points $\pm\kappa_n(\lambda)$. All eigenvalues of $D(\lambda)$ distinct from 0 and $\pm\kappa_n(\lambda)$ have finite multiplicities.*

The above characterisation of the a.c. spectrum was obtained earlier in [15] for $d = 1, 2, 3$ under the more restrictive assumption $\rho > d$.

Similar applications are possible in situations where the diagonalization of H_0 is known explicitly. For example, the perturbed Schrödinger operator with a constant magnetic field in dimension three (and probably the perturbed periodic Schrödinger

operator in arbitrary dimension) can be considered. Moreover, in Theorem 2.6, we do not assume the operators H_0, H to be semibounded. Thus, one can apply this theorem to the perturbations of the Dirac operator or the Stark operator (i.e. the Schrödinger operator with a linear electric potential).

2.5. The strategy of the proof of Theorem 2.6. In order to simplify our notation, we will assume without the loss of generality that $\Delta = [-1, 1]$ and $\lambda = 0 \in \Omega$. Clearly, the general case can be reduced to this one by a shift and scaling. We fix $a > 0$ such that $[-a, a] \subset \Omega$. In Section 4 by using a simple operator theoretic argument (borrowed from [15]), we reduce the spectral analysis of $D(0)$ to the spectral analysis of the self-adjoint operators

$$(2.12) \quad M_+ = E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+), \quad M_- = E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-),$$

where as usual $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$. In Section 3, we construct an explicit “model” self-adjoint operator M and analyze its spectrum. After this, in Sections 4 and 5 we prove that the wave operators $W_{\pm}(M_{\pm}, M)$ exist and are complete. This allows us to describe the spectrum of M_{\pm} . The operator M_- is analyzed in a similar way.

The proof of the existence and completeness of the wave operators for the pair M, M_+ is achieved by showing that the difference $M_+ - M$ can be represented as XKX , where the operator X is strongly M -smooth and the operator K is compact, see Section 4.2. In [15] the same aim was achieved, roughly speaking, by showing that (under more stringent assumptions) the difference $M_+ - M$ is a trace class operator.

3. THE MODEL OPERATOR

3.1. The half-Carleman operator \mathcal{C}_a . The Carleman operator is the Hankel integral operator in $L^2(\mathbb{R}_+)$ with the integral kernel $1/(x+y)$. Let \mathcal{C}_a be the integral operator on $L^2(0, a)$ with the Carleman kernel (up to a normalization $1/\pi$):

$$(3.1) \quad \mathcal{C}_a(x, y) = \frac{1}{\pi} \frac{1}{x+y}.$$

We will call \mathcal{C}_a the half-Carleman operator.

Our first task is to recall the explicit diagonalization formula for \mathcal{C}_a . Essentially, this diagonalization is given by Mehler’s formula (see e.g. [5, formula (3.14.6)]):

$$(3.2) \quad \frac{1}{\pi} \int_1^{\infty} \frac{P_{-\frac{1}{2}+it}(y)}{x+y} dy = \frac{1}{\cosh(\pi t)} P_{-\frac{1}{2}+it}(x), \quad t \in \mathbb{R}.$$

Here P_{ν} is the Legendre function.

Let us exhibit the unitary operator which diagonalizes \mathcal{C}_a . Recall that the Mehler-Fock transform (see e.g. [20, Section 3.4] and references therein) is a unitary operator $U : L^2((1, \infty), dx) \rightarrow L^2((0, \infty), dt)$ defined for $g \in C_0^{\infty}(1, \infty)$ by

$$(3.3) \quad (Ug)(t) = \sqrt{t \tanh(\pi t)} \int_1^{\infty} P_{-\frac{1}{2}+it}(x)g(x)dx, \quad t \in (0, \infty).$$

Let us introduce the unitary operators $B_1 : L^2((0, \infty), dt) \rightarrow L^2((0, 1), d\mu)$ and $B_2 : L^2((0, a), du) \rightarrow L^2((1, \infty), dx)$ by the formulas

$$(B_1 h)(\mu) = \frac{\cosh(\pi t)}{\sqrt{\pi \sinh(\pi t)}} h(t), \quad \mu = \frac{1}{\cosh(\pi t)} \in (0, 1),$$

and

$$(B_2 f)(x) = \frac{\sqrt{a}}{x} f(a/x), \quad x \in (1, \infty).$$

Then the operator $\mathcal{U}_a = B_1 U B_2 : L^2((0, a), du) \rightarrow L^2((0, 1), d\mu)$ is also unitary. Using the change of variables $u = a/x$ in (3.3), we see that \mathcal{U}_a acts as

$$(3.4) \quad (\mathcal{U}_a f)(\mu) = \sqrt{\frac{a}{\pi} t \cosh(\pi t)} \int_0^a P_{-\frac{1}{2}+it}(a/u) \frac{f(u)}{u} du, \quad \mu = \frac{1}{\cosh(\pi t)} \in (0, 1).$$

Changing the variables $u = a/x$, $v = a/y$ in Mehler's formula (3.2), we get

$$(3.5) \quad (\mathcal{U}_a \mathcal{C}_a f)(\mu) = \mu (\mathcal{U}_a f)(\mu), \quad \mu \in (0, 1).$$

Let us summarize the above calculations.

Lemma 3.1. *The half-Carleman operator \mathcal{C}_a in $L^2(0, a)$ has a purely a.c. spectrum of multiplicity one, $\text{spec}(\mathcal{C}_a) = [0, 1]$. The explicit diagonalization (3.5) of \mathcal{C}_a is given by the unitary operator \mathcal{U}_a defined by (3.4).*

3.2. The strong \mathcal{C}_a -smoothness. It turns out that the operators of multiplication by functions with a certain logarithmic decay as $x \rightarrow 0+$ in $L^2(0, a)$ are strongly \mathcal{C}_a -smooth. Before discussing this, we need some bounds on the Legendre function:

Lemma 3.2. *For all $R > 0$ and $\delta \in (0, 1]$ there exist constants $C_1(R)$, $C_2(R, \delta)$ such that for any $x \geq 1$ and any $t, t_1, t_2 \in [0, R]$, one has*

$$(3.6) \quad |P_{-\frac{1}{2}-it}(x)| \leq C_1(R) x^{-1/2},$$

$$(3.7) \quad |P_{-\frac{1}{2}-it_2}(x) - P_{-\frac{1}{2}-it_1}(x)| \leq C_2(R, \delta) |t_2 - t_1|^\delta x^{-1/2} (1 + \log x)^\delta.$$

The proof is given in the Appendix.

Let the operator $X_\gamma^{(0)}$ act in the space $L^2(0, a)$ by the formula

$$(3.8) \quad (X_\gamma^{(0)} f)(x) = (1 + |\log x|)^{-\gamma} f(x), \quad x \in (0, a), \quad \gamma > 0.$$

Similarly to (2.7), we define the operator $\mathcal{Z}(\mu) : L^2(0, a) \rightarrow \mathbb{C}$ for $\mu \in (0, 1)$ by the equation

$$(3.9) \quad (\mathcal{U}_a X_\gamma^{(0)} f)(\mu) = \mathcal{Z}(\mu) f.$$

In view of (3.4) and (3.8), the operator $\mathcal{Z}(\mu)$ satisfies the equation

$$\mathcal{Z}(\mu) f = \sqrt{\frac{a}{\pi} t \cosh(\pi t)} \int_0^a P_{-\frac{1}{2}+it}(a/u) (1 + |\log u|)^{-\gamma} \frac{f(u)}{u} du,$$

where $\mu = 1/\cosh(\pi t)$.

Lemma 3.3. *Let $\delta \in (0, 1]$. Then for any $\gamma > \delta + 1/2$, the operator $X_\gamma^{(0)}$ is strongly \mathcal{C}_a -smooth with the exponent δ on any compact subinterval of $(0, 1)$.*

Proof. We have to check the estimates (cf. (2.8))

$$(3.10) \quad \|\mathcal{Z}(\mu)\| \leq C, \quad \|\mathcal{Z}(\mu) - \mathcal{Z}(\mu')\| \leq C|\mu - \mu'|^\delta$$

on any compact subinterval of $(0, 1)$. If μ is bounded away from zero, then the variable t belongs to the interval $[0, R]$ with some $R < \infty$. It follows from Lemma 3.2 that the function

$$P_{-\frac{1}{2}+it}(a/u)(1 + |\log u|)^{-\gamma}(1/u)$$

of $u \in (0, a)$ belongs to the space $L^2((0, a), du)$ for any $\gamma > 1/2$ and as an element of this space is Hölder continuous in $t \in [0, R]$ with the exponent $\delta < \gamma - 1/2$. Since the map $\mu \mapsto t$ is continuously differentiable away from $\mu = 1$, the required claim follows. \square

3.3. The operator M . Here we define the operator M which we consider as a “model operator” for M_+ (recall that M_+ is defined by (2.12)). Our goal will be to prove that the wave operators $W_\pm(M_+, M)$ exist and are complete.

First consider the operator \mathcal{C}_a^2 in $L^2(0, a)$; obviously this operator has the integral kernel

$$(3.11) \quad \mathcal{C}_a^2(x, y) = \frac{1}{\pi^2} \int_0^a \frac{dt}{(x+t)(y+t)}.$$

Lemmas 3.1 and 3.3 yield the following result.

Lemma 3.4. *The operator \mathcal{C}_a^2 has a purely a.c. spectrum $[0, 1]$ of multiplicity one and for any $\delta \in (0, 1]$ and any $\gamma > \delta + 1/2$ the operator $X_\gamma^{(0)}$ is strongly \mathcal{C}_a^2 -smooth with the exponent δ on any compact subinterval of $(0, 1)$.*

Next, in $L^2((0, a), \mathcal{N}) = L^2(0, a) \otimes \mathcal{N}$ consider the operators

$$(3.12) \quad M_1 = \mathcal{C}_a^2 \otimes \Gamma, \quad X_\gamma^{(1)} = X_\gamma^{(0)} \otimes I,$$

where

$$(3.13) \quad \Gamma = \frac{1}{4}(S(0) - I)(S(0)^* - I) = \frac{1}{2}(I - \operatorname{Re} S(0)).$$

At the last step we have used the unitarity of the scattering matrix. The operator Γ has a pure point spectrum with the eigenvalues $\varkappa_n(0)^2$, $n = 1, \dots, N$. From Lemma 3.4 it follows that, apart from the possible zero eigenvalue of infinite multiplicity, M_1 has a purely a.c. spectrum $\cup_{n=1}^N [0, \varkappa_n(0)^2]$ (each interval has multiplicity one). Moreover, using the diagonalization of \mathcal{C}_a^2 and choosing the basis of the eigenfunctions of Γ in \mathcal{N} , we can diagonalize the operator M_1 in an obvious way. With respect to this diagonalization, for any $\delta \in (0, 1]$ and any $\gamma > \delta + 1/2$ the operator $X_\gamma^{(1)}$ is strongly M_1 -smooth with the exponent δ on any compact interval which contains neither 0 nor $\varkappa_n(0)^2$, $n = 1, \dots, N$.

Finally, we “transplant” the operators M_1 and $X_\gamma^{(1)}$ into \mathcal{H} . Recall (see Section 2.2) that $\mathcal{F} : \operatorname{Ran} E_0([-1, 1]) \rightarrow L^2([-1, 1], \mathcal{N})$ is a unitary operator which diagonalizes H_0 . Let $\mathcal{H}_a = \operatorname{Ran} E_0((0, a))$. It will be convenient to consider the restriction $\mathcal{F}_a = \mathcal{F}|_{\mathcal{H}_a}$. Clearly, $\mathcal{F}_a : \mathcal{H}_a \rightarrow L^2((0, a), \mathcal{N})$ is a unitary operator.

Let us define the operators M, X_γ in \mathcal{H} by

$$(3.14) \quad M = \mathcal{F}_a^* M_1 \mathcal{F}_a \oplus 0, \quad X_\gamma = \mathcal{F}_a^* X_\gamma^{(1)} \mathcal{F}_a \oplus I$$

with respect to the orthogonal sum decomposition $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_a^\perp$. Clearly,

$$(3.15) \quad \begin{aligned} X_\gamma &= \omega_\gamma(H_0), \quad \text{where} \\ \omega_\gamma(x) &= (1 + |\log x|)^{-\gamma} \chi_{(0,a)}(x) + \chi_{[a,\infty)}(x) + \chi_{(-\infty,0]}(x). \end{aligned}$$

From the above analysis we obtain:

Theorem 3.5. *Besides the eigenvalue at 0 (possibly, of infinite multiplicity), the spectrum of M is absolutely continuous. The a.c. spectrum of M consists of the union $\cup_{n=1}^N [0, \varkappa_n(0)^2]$, where each interval has multiplicity one. For any $\delta \in (0, 1]$ and any $\gamma > \delta + 1/2$ the operator X_γ is strongly M -smooth with the exponent δ on any compact interval which contains neither 0 nor $\varkappa_n(0)^2$, $n = 1, \dots, N$.*

4. PROOF OF THEOREM 2.6

4.1. Reduction to the products of spectral projections. Let us denote

$$D = D(0) = E_0(\mathbb{R}_+) - E(\mathbb{R}_+)$$

and

$$\mathcal{H}_+ = \text{Ker}(D - I), \quad \mathcal{H}_- = \text{Ker}(D + I), \quad \mathcal{H}_0 = (\mathcal{H}_- \oplus \mathcal{H}_+)^\perp.$$

Set $F = I - E_0(\mathbb{R}_+) - E(\mathbb{R}_+)$. A simple algebra shows that $FD = -DF$ and $F^2 = I - D^2$. It follows that $\text{Ker } F = \mathcal{H}_- \oplus \mathcal{H}_+$, and hence the operator F is invertible on the subspace \mathcal{H}_0 . From here one obtains (see e.g. [2] or [6] for the details) that on the invariant subspace \mathcal{H}_0 ,

$$(4.1) \quad D|_{\mathcal{H}_0} \text{ is unitarily equivalent to } (-D)|_{\mathcal{H}_0}.$$

Thus, the spectral analysis of D reduces to the spectral analysis of D^2 and to the calculation of the dimensions of \mathcal{H}_+ and \mathcal{H}_- .

Recall that by our assumptions, the operator $I + T(\lambda + i0)J$ is invertible for all $|\lambda| \leq a$, and H has a purely a.c. spectrum on $[-a, a]$; in particular,

$$E(\{0\}) = E_0(\{0\}) = 0.$$

Using the last relation and employing the notation M_+, M_- (see (2.12)), by a simple algebra one obtains

$$(4.2) \quad D^2 = M_+ + M_- = E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+) + E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-).$$

Clearly, the r.h.s. provides a block-diagonal decomposition of D^2 with respect to the orthogonal sum $\mathcal{H} = \text{Ran } E_0(\mathbb{R}_-) \oplus \text{Ran } E_0(\mathbb{R}_+)$.

Denote $\varkappa_n = \varkappa_n(0)$. Below we prove

Theorem 4.1. *Let Assumption 2.2 hold true and $\Delta = [-1, 1]$, $\lambda = 0$. Then the a.c. spectrum of M_\pm consists of the union of intervals $\cup_{n=1}^N [0, \varkappa_n^2]$, where each interval has multiplicity one in the spectrum. The operators M_+ and M_- have no singular continuous spectrum. The eigenvalues of M_\pm can accumulate only to 0 and to the points \varkappa_n^2 . All eigenvalues of M_\pm distinct from 0 and \varkappa_n^2 have finite multiplicities.*

From Theorem 4.1 and the decomposition (4.2) we immediately obtain that D^2 has no singular continuous spectrum; the a.c. spectrum of D^2 consists of the union of intervals $\cup_{n=1}^N [0, \varkappa_n^2]$, where each interval has multiplicity two; the eigenvalues of D^2 can accumulate only to 0 and to the points \varkappa_n^2 , and all eigenvalues of D^2 distinct from 0 and \varkappa_n^2 have finite multiplicities.

From the above description of the spectrum of D^2 and from (4.1) we obtain the description of the spectrum of $D|_{\mathcal{H}_0}$. In order to complete the proof of Theorem 2.6, it remains to show that the points 1 and -1 cannot be eigenvalues of D of infinite multiplicity unless $\varkappa_n = 1$ for some n . This fact follows again from Theorem 4.1 because if $\varkappa_n < 1$ for all n then according to (4.2) the point 1 is not an eigenvalue of D^2 of infinite multiplicity.

Thus, for the proof of Theorem 2.6 it suffices to prove Theorem 4.1. We consider the operator M_+ ; the proof for M_- is analogous.

4.2. Application of scattering theory. Our proof of Theorem 4.1 is based on the following well known fact from scattering theory, see e.g. [19, Theorems 4.6.4, 4.7.9, 4.7.10].

Proposition 4.2. *Suppose that a bounded self-adjoint operator M has a purely a.c. spectrum of constant multiplicity on an open interval Λ . Suppose that a bounded operator X is strongly M -smooth with an exponent $\delta > 0$ on every compact subinterval of Λ . Let K be a compact self-adjoint operator and $\widetilde{M} = M + X^*KX$. Then the local wave operators $W_{\pm}(\widetilde{M}, M; \Lambda)$ for M , \widetilde{M} and the interval Λ exist and are complete. Thus, the a.c. spectrum of \widetilde{M} on Λ has the same multiplicity as that of M . Moreover, if $\delta > 1/2$ then \widetilde{M} has no singular continuous spectrum or eigenvalues of infinite multiplicity on Λ . The eigenvalues of \widetilde{M} in Λ can accumulate only to the endpoints of Λ .*

In what follows we prove

Theorem 4.3. *Let Assumption 2.2 hold true. Then for any $\gamma > 0$, the difference $M_+ - M$ can be represented as $X_{\gamma}KX_{\gamma}$ where K is a compact self-adjoint operator.*

Given Theorem 4.3, we are in a position to prove Theorem 4.1 (for M_+). Let us assume that \varkappa_n are enumerated such that $\varkappa_n \geq \varkappa_{n+1}$ for all n . Take any n such that $\varkappa_n > \varkappa_{n+1}$ and let us apply Proposition 4.2 to the pair M , $\widetilde{M} = M_+$ and the interval $\Lambda_n = (\varkappa_{n+1}^2, \varkappa_n^2)$. If $N < \infty$, then we also consider the interval $\Lambda_N = (0, \varkappa_N^2)$. By Theorem 3.5, the operator X_{γ} for $\gamma > 1$ is strongly M -smooth with some $\delta > 1/2$ on all compact subintervals of Λ_n . Thus, it follows from Proposition 4.2 that the local wave operators $W_{\pm}(M_+, M; \Lambda_n)$ for all n exist and are complete. This implies (see e.g. [19, Theorem 4.6.5]) that the global wave operators $W_{\pm}(M_+, M)$ also exist and are complete. In particular, the a.c. parts of M and M_+ are unitarily equivalent. Furthermore, since $\delta > 1/2$ the conclusions of Theorem 4.1 about the singular spectrum of M_+ and its eigenvalues also follow from Proposition 4.2. Thus, we have proven Theorem 4.1 for M_+ ; the proof for M_- is analogous.

4.3. The proof of Theorem 4.3. The proof of Theorem 4.3 consists of several steps which we proceed to outline. In this subsection, we state four lemmas; the proofs will be given in Section 5. The first two lemmas show that only a neighborhood of the point $\lambda = 0$ is essential for the analysis of the operator M_+ .

Define

$$(4.3) \quad M_2 = E_0(\mathbb{R}_+)E((-a, 0))E_0(\mathbb{R}_+),$$

$$(4.4) \quad M_3 = E_0((0, a))E((-a, 0))E_0((0, a)).$$

Lemma 4.4. *For any $\gamma > 0$, the difference $M_+ - M_2$ can be represented as $X_\gamma K X_\gamma$ where K is a compact self-adjoint operator.*

Lemma 4.5. *For any $\gamma > 0$, the difference $M_2 - M_3$ can be represented as $X_\gamma K X_\gamma$ where K is a compact self-adjoint operator.*

Below we use the fact that the operator $R_0(z)E_0(\mathbb{R}_+)$ is analytic in $z \in \mathbb{C} \setminus [0, \infty)$ and so for any $\lambda < 0$ the operator $R_0(\lambda)E_0(\mathbb{R}_+)$ is well defined, bounded and self-adjoint. Let $\mathcal{D} \subset \mathcal{H}$ be the dense set

$$(4.5) \quad \mathcal{D} = \{f \in \mathcal{H} \mid \exists \delta = \delta(f) : E_0((-\delta, \delta))f = 0\}.$$

Recall our notation $Y(\lambda) = J(I + T(\lambda + i0)J)^{-1}$ (see Section 2.2) and set $\text{Im } Y(\lambda) = (Y(\lambda) - Y(\lambda)^*)/2i$. Let us introduce an auxiliary operator M_4 in terms of its quadratic form

$$(4.6) \quad (M_4 f, f) = -\frac{1}{\pi} \int_{-a}^0 ((\text{Im } Y(0))GR_0(\lambda)E_0((0, a))f, GR_0(\lambda)E_0((0, a))f)d\lambda$$

for $f \in \mathcal{D}$.

Lemma 4.6. *Formula (4.6) defines a bounded self-adjoint operator M_4 on \mathcal{H} . For any $\gamma > 0$, the difference $M_3 - M_4$ can be represented as $X_\gamma K X_\gamma$ where K is a compact self-adjoint operator.*

Lemma 4.7. *For any $\gamma > 0$, the difference $M_4 - M$ can be represented as $X_\gamma K X_\gamma$ where K is a compact self-adjoint operator.*

Clearly, Theorem 4.3 and hence Theorem 4.1 follow from Lemmas 4.4, 4.5, 4.6, and 4.7.

5. PROOFS OF LEMMAS 4.4–4.7

5.1. Auxiliary estimates. Let \mathcal{D} be as in (4.5). It is straightforward to see that $\mathcal{D} \subset \text{Dom}(X_\gamma^{-1})$ for all $\gamma > 0$. Denote $G_a = GE_0((0, a))$.

Lemma 5.1. *Let Assumption 2.2 hold true. Then for any $\gamma > 0$, the operator $GR_0(i)X_\gamma^{-1}$, defined initially on \mathcal{D} , extends to a compact operator from \mathcal{H} to \mathcal{K} .*

Proof. By the definition (3.14) of X_γ , we have to prove the compactness of the two operators

$$(5.1) \quad GR_0(i)\mathcal{F}_a^*(X_\gamma^{(1)})^{-1}\mathcal{F}_aE_0((0, a)) \text{ and } GR_0(i)E_0(\mathbb{R} \setminus (0, a)).$$

The second operator is compact by assumption (2.1). Consider the first one. Since the operators H_0 and X_γ commute and \mathcal{F}_a is unitary, it suffices to prove the compactness of the operator $G_a\mathcal{F}_a^*(X_\gamma^{(1)})^{-1} : L^2((0, a), \mathcal{N}) \rightarrow \mathcal{K}$. According to formula (2.9) this operator acts as

$$(5.2) \quad G_a\mathcal{F}_a^*(X_\gamma^{(1)})^{-1}f = \int_0^a (1 + |\log x|)^\gamma Z(x)^* f(x) dx, \quad f \in \mathcal{D}.$$

By the strong smoothness assumption the operator $Z(x) : \mathcal{K} \rightarrow \mathcal{N}$ is compact and depends continuously on x . From here and the fact that $(1 + |\log x|)^\gamma$ is in $L^2((0, a), dx)$, the required statement follows. \square

Using the above lemma, we immediately obtain that for all $\lambda < 0$ the operators $GR_0(\lambda)E_0(\mathbb{R}_+)X_\gamma^{-1}$ defined initially on \mathcal{D} extend to compact operators from \mathcal{H} to \mathcal{K} .

Lemma 5.2. *Under Assumption 2.2 for any $\gamma > 0$ we have:*

- (i) $\|GR_0(\lambda)E_0((a, \infty))X_\gamma^{-1}\| = O(1)$, as $\lambda \rightarrow 0^-$;
- (ii) $\|GR_0(\lambda)E_0(\mathbb{R}_+)X_\gamma^{-1}\| = O(|\lambda|^{-1/2}|\log|\lambda|^\gamma)$, as $\lambda \rightarrow 0^-$.

Proof. (i) Since (in view of (3.15))

$$GR_0(\lambda)E_0((a, \infty))X_\gamma^{-1} = GR_0(i)(H_0 - i)R_0(\lambda)E_0((a, \infty))$$

and the operator $GR_0(i)$ is bounded, the required statement follows from the trivial estimate

$$\|(H_0 - i)R_0(\lambda)E_0((a, \infty))\| \leq C, \quad \forall \lambda < 0.$$

(ii) It follows from (3.14) that the problem reduces (cf. (5.1)) to estimating the norms of the two operators:

$$G_aR_0(\lambda)\mathcal{F}_a^*(X_\gamma^{(1)})^{-1} \text{ and } GR_0(\lambda)E_0((a, \infty)).$$

The norm of the second operator has already been estimated in (i). Consider the first operator. According to (2.9) this operator acts from $L^2((0, a), \mathcal{N})$ to \mathcal{K} as

$$G_aR_0(\lambda)\mathcal{F}_a^*(X_\gamma^{(1)})^{-1}f = \int_0^a \frac{(1 + |\log x|)^\gamma}{x - \lambda} Z(x)^* f(x) dx, \quad f \in \mathcal{D}, \quad \lambda < 0.$$

The norm of this operator can be explicitly estimated:

$$\begin{aligned} \left\| \int_0^a \frac{(1 + |\log x|)^\gamma}{x - \lambda} Z(x)^* f(x) dx \right\| &\leq \|f\| \left(\int_0^a \frac{(1 + |\log x|)^{2\gamma}}{(x - \lambda)^2} \|Z(x)\|^2 dx \right)^{1/2} \\ &\leq C\|f\| \left(\int_0^a \frac{(1 + |\log x|)^{2\gamma}}{(x - \lambda)^2} dx \right)^{1/2} \leq C_1\|f\| |\lambda|^{-1/2} |\log|\lambda|^\gamma, \end{aligned}$$

for all $\lambda < 0$, and the required statement follows. \square

5.2. Compactness properties of auxiliary operators.

Lemma 5.3. *Let Assumption 2.2 hold true. Then for any $\varphi \in C_0^\infty(\mathbb{R})$ and any $\gamma > 0$ the operator*

$$X_\gamma^{-1}(\varphi(H) - \varphi(H_0))$$

is compact.

Proof. 1. First note that

$$X_\gamma^{-1}(GR_0(z))^* = X_\gamma^{-1}(GR_0(i))^* + (\bar{z} + i)X_\gamma^{-1}(GR_0(i)R_0(z))^*,$$

for $\text{Im } z \neq 0$. Using Lemma 5.1, from here we get

$$(5.3) \quad \|X_\gamma^{-1}(GR_0(z))^*\| \leq C \frac{|z| + 1}{|\text{Im } z|}, \quad \text{Im } z \neq 0.$$

Next, from (2.1) and (2.2) it follows that $GR(i)$ is bounded. Therefore, similarly to (5.3), we get

$$(5.4) \quad \|GR(z)\| \leq C \frac{|z| + 1}{|\text{Im } z|}, \quad \text{Im } z \neq 0.$$

2. We use the technique of functional calculus via the almost analytic extension, see e.g. [4, Section 8]. Let $\tilde{\varphi} \in C_0^\infty(\mathbb{C})$ be the almost analytic extension of φ , i.e. $\tilde{\varphi}|_{\mathbb{R}} = \varphi$ and

$$(5.5) \quad \left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| = O(|\text{Im } z|^k) \quad \text{as } \text{Im } z \rightarrow 0$$

for any $k > 0$. Then

$$(5.6) \quad \varphi(H) = \int_{\mathbb{C}} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) R(z) dL(z),$$

where $L(z)$ is the Lebesgue measure in \mathbb{C} . Note that this integral is norm convergent due to (5.5) and the trivial estimate $\|R(z)\| \leq |\text{Im } z|^{-1}$.

3. Using the resolvent identity (2.5) and the representation (5.6), we get

$$X_\gamma^{-1}(\varphi(H) - \varphi(H_0)) = - \int_{\mathbb{C}} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) X_\gamma^{-1}(GR_0(\bar{z}))^* JGR(z) dL(z).$$

The integrand in the r.h.s. is compact for any $\text{Im } z \neq 0$ by Lemma 5.1. By (5.3), (5.4) and (5.5), the integral converges in the operator norm. From here we get the required statement. \square

Lemma 5.4. *Let part (B) of Assumption 2.2 hold true. Then the operator $\psi(H) - \psi(H_0)$ is compact for any function $\psi \in C(\mathbb{R})$ such that the limits $\lim_{x \rightarrow \pm\infty} \psi(x)$ exist and are finite.*

Proof. As is well known (and can easily be deduced from the compactness of $R(z) - R_0(z)$ for $\text{Im } z \neq 0$), the operator $\psi(H) - \psi(H_0)$ is compact for any function $\psi \in C(\mathbb{R})$ such that $\psi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Therefore, it suffices to prove that $\psi(H) - \psi(H_0)$ is compact for at least one function $\psi \in C(\mathbb{R})$ such that $\lim_{x \rightarrow \infty} \psi(x) \neq \lim_{x \rightarrow -\infty} \psi(x)$ and both limits exist. The latter fact is provided by [16, Theorem 7.3]

where it has been proven that if part (B) of Assumption 2.2 holds true then the difference $\tan^{-1}(H) - \tan^{-1}(H_0)$ is compact. \square

5.3. Proofs of Lemmas 4.4, 4.5 and 4.6.

Proof of Lemma 4.4. 1. Comparing (2.12) and (4.3), we see that $M_+ - M_2 = X_\gamma K X_\gamma$, where

$$K = X_\gamma^{-1} E_0(\mathbb{R}_+) E((-\infty, -a)) E_0(\mathbb{R}_+) X_\gamma^{-1}.$$

It suffices to prove that the operator

$$\begin{aligned} X_\gamma^{-1} E_0(\mathbb{R}_+) E((-\infty, -a)) \\ = X_\gamma^{-1} E_0((0, a)) E((-\infty, -a)) + X_\gamma^{-1} E_0((a, \infty)) E((-\infty, -a)) \end{aligned}$$

is compact. We will prove the compactness of the two terms in the r.h.s. separately.

2. Consider the first term. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(x) = 1$ for $x \in [0, a]$ and $\varphi(x) = 0$ for $x \leq -a$. Then

$$X_\gamma^{-1} E_0((0, a)) E((-\infty, -a)) = X_\gamma^{-1} E_0((0, a)) (\varphi(H_0) - \varphi(H)) E((-\infty, -a)).$$

Since the operators $E_0((0, a))$ and X_γ^{-1} commute, the r.h.s. is compact by Lemma 5.3.

3. Consider the second term. Let $\psi \in C(\mathbb{R})$ be such that $\psi(x) = 1$ for $x \geq a$ and $\psi(x) = 0$ for $x \leq -a$. Then, using (3.15), we find that

$$X_\gamma^{-1} E_0((a, \infty)) E((-\infty, -a)) = E_0((a, \infty)) (\psi(H_0) - \psi(H)) E((-\infty, -a)),$$

and the r.h.s. is compact by Lemma 5.4. \square

Proof of Lemma 4.5. 1. First we need to obtain an integral representation for M_2 similar to (4.6). By using Stone's formula (see e.g. [17, Theorem VII.13]) and the fact that the spectra of H_0 and H on $[-a, a]$ are purely a.c., we obtain for any $f \in \mathcal{H}$:

$$\begin{aligned} (M_2 f, f) &= ((E((-a, 0)) - E_0((-a, 0))) E_0(\mathbb{R}_+) f, E_0(\mathbb{R}_+) f) \\ &= \frac{1}{\pi} \int_{-a}^0 \lim_{\varepsilon \rightarrow +0} \text{Im} ((R(\lambda + i\varepsilon) - R_0(\lambda + i\varepsilon)) E_0(\mathbb{R}_+) f, E_0(\mathbb{R}_+) f) d\lambda. \end{aligned}$$

Substituting the resolvent identity (2.4) into this formula and using the notation $Y(\lambda) = J(I + T(\lambda + i0)J)^{-1}$, we obtain:

$$(5.7) \quad (M_2 f, f) = -\frac{1}{\pi} \int_{-a}^0 ((\text{Im } Y(\lambda)) G R_0(\lambda) E_0(\mathbb{R}_+) f, G R_0(\lambda) E_0(\mathbb{R}_+) f) d\lambda.$$

2. Comparing (4.3) and (4.4) and taking into account (3.15), we find that $M_2 - M_3 = X_\gamma K X_\gamma$, where

$$(5.8) \quad K = E_0((a, \infty)) M_2 X_\gamma^{-1} + X_\gamma^{-1} M_2 E_0((a, \infty)) + E_0((a, \infty)) M_2 E_0((a, \infty)).$$

Since X_γ is a bounded operator, it suffices to check the compactness of the first operator in the r.h.s. By (5.7), it can be represented as

$$(5.9) \quad -\frac{1}{\pi} \int_{-a}^0 (G R_0(\lambda) E_0((a, \infty)))^* \text{Im } Y(\lambda) G R_0(\lambda) E_0(\mathbb{R}_+) X_\gamma^{-1} d\lambda,$$

where *a priori* the integral converges weakly on the dense set \mathcal{D} . Applying Lemma 5.2, we see that the norm of integrand in (5.9) is bounded by

$$\|GR_0(\lambda)E_0((a, \infty))\| \|\operatorname{Im} Y(\lambda)\| \|GR_0(\lambda)E_0(\mathbb{R}_+)X_\gamma^{-1}\| \leq C|\lambda|^{-1/2}|\log|\lambda||^\gamma.$$

Hence the integral in (5.9) converges actually in the operator norm. By Lemma 5.1, the integrand is compact for all $\lambda < 0$. Thus, the above integral is compact, as required. \square

Proof of Lemma 4.6. Similarly to (5.7), we have the representation

$$(M_3f, f) = -\frac{1}{\pi} \int_{-a}^0 ((\operatorname{Im} Y(\lambda))GR_0(\lambda)E_0((0, a))f, GR_0(\lambda)E_0((0, a))f) d\lambda.$$

Thus, recalling the definition (4.6) of M_4 and setting $\tilde{Y}(\lambda) = \operatorname{Im}(Y(\lambda) - Y(0))$, we get $M_3 - M_4 = X_\gamma K X_\gamma$, where

$$(5.10) \quad (Kf, f) = -\frac{1}{\pi} \int_{-a}^0 (\tilde{Y}(\lambda)GR_0(\lambda)E_0((0, a))X_\gamma^{-1}f, GR_0(\lambda)E_0((0, a))X_\gamma^{-1}f) d\lambda, \quad f \in \mathcal{D}.$$

Since $Y(\lambda)$ is Hölder continuous, we have $\|\tilde{Y}(\lambda)\| \leq C|\lambda|^\beta$, $\beta > 0$. Combining this with the estimate of Lemma 5.2(ii), we see that

$$\int_{-a}^0 \|\tilde{Y}(\lambda)\| \|GR_0(\lambda)E_0((0, a))X_\gamma^{-1}\|^2 d\lambda < \infty.$$

Recalling Lemma 5.1, we obtain that the operator K is compact. This result also shows that the operator M_4 is bounded. \square

5.4. Proof of Lemma 4.7. First we need the following simple auxiliary statement.

Lemma 5.5. *Let $p > q > 0$. Then the operator K in $L^2(0, a)$ with the integral kernel*

$$K(x, y) = (1 + |\log x|)^{-p}(x + y)^{-1}(1 + |\log y|)^q$$

is compact.

The proof is given in the Appendix.

Proof of Lemma 4.7. 1. First recall the definitions (3.14) of M and X_γ and (4.6) of M_4 . Next, note that both M and M_4 vanish on \mathcal{H}_a^\perp . Thus, applying a unitary transformation \mathcal{F}_a , it suffices to prove that the operator $M_1 - \mathcal{F}_a M_4 \mathcal{F}_a^*$ in $L^2((0, a), \mathcal{N})$ can be represented as $X_\gamma^{(1)} K X_\gamma^{(1)}$ with a compact operator K .

2. Consider M_1 and $\mathcal{F}_a M_4 \mathcal{F}_a^*$ as integral operators in $L^2((0, a), \mathcal{N})$. Set $Q = -\pi \operatorname{Im} Y(0)$. It follows from the representations (2.10), (3.13) that

$$\Gamma = -\pi Z(0) \operatorname{Im} Y(0) Z(0)^* = Z(0) Q Z(0)^*.$$

Therefore formula (3.12) shows that the integral kernel of M_1 can be represented as

$$M_1(x, y) = \mathcal{C}_a^2(x, y) Z(0) Q Z(0)^*$$

where $\mathcal{C}_a^2(x, y)$ is defined by (3.11). Next, it follows from (2.9) that

$$GR_0(\lambda)E_0((0, a))\mathcal{F}_a^*f = \int_0^a \frac{1}{x - \lambda} Z(x)^* f(x) dx, \quad \lambda < 0.$$

From here and the definition (4.6) of M_4 it is clear that the integral kernel of $\mathcal{F}_a M_4 \mathcal{F}_a^*$ is

$$\begin{aligned} (\mathcal{F}_a M_4 \mathcal{F}_a^*)(x, y) &= \frac{1}{\pi^2} \int_{-a}^0 Z(x) \frac{1}{x - \lambda} Q \frac{1}{y - \lambda} Z(y)^* d\lambda \\ &= \mathcal{C}_a^2(x, y) Z(x) Q Z(y)^*. \end{aligned}$$

Using the definition (3.8), (3.12) of $X_\gamma^{(1)}$, let us represent the integral kernel of the difference

$$(X_\gamma^{(1)})^{-1} (M_1 - \mathcal{F}_a M_4 \mathcal{F}_a^*) (X_\gamma^{(1)})^{-1}$$

as

$$\begin{aligned} \mathcal{C}_a^2(x, y) (\omega_\gamma(x) \omega_\gamma(y))^{-1} (Z(x) - Z(0)) Q Z(y)^* \\ + \mathcal{C}_a^2(x, y) (\omega_\gamma(x) \omega_\gamma(y))^{-1} Z(0) Q (Z(y)^* - Z(0)^*) \end{aligned}$$

where $\omega_\gamma(x) = (1 + |\log x|)^{-\gamma}$.

3. Let us prove that the first kernel represents a compact operator; the second kernel is considered in the same way. We have

$$\begin{aligned} (5.11) \quad \mathcal{C}_a^2(x, y) (\omega_\gamma(x) \omega_\gamma(y))^{-1} (Z(x) - Z(0)) Q Z(y)^* \\ = \frac{1}{\pi^2} \int_0^a \omega_\gamma(x)^{-1} (Z(x) - Z(0)) \frac{1}{x + t} Q \frac{1}{t + y} Z(y)^* \omega_\gamma(y)^{-1} dt. \end{aligned}$$

Choose $\sigma > \gamma$. The above formula defines a factorization of the operator with integral kernel (5.11) as $K_2 K_1$, where

$$\begin{aligned} K_1 : L^2((0, a), \mathcal{N}) &\rightarrow L^2((0, a), \mathcal{K}) \\ K_2 : L^2((0, a), \mathcal{K}) &\rightarrow L^2((0, a), \mathcal{N}), \end{aligned}$$

are the integral operators with the kernels

$$\begin{aligned} K_1(t, y) &= \frac{1}{\pi} \omega_\sigma(t) \frac{1}{t + y} Z(y)^* \omega_\gamma(y)^{-1} \\ K_2(x, t) &= \frac{1}{\pi} \omega_\gamma(x)^{-1} (Z(x) - Z(0)) \frac{1}{x + t} \omega_\sigma(t)^{-1} Q, \end{aligned}$$

Let us prove that K_1 is compact and K_2 is bounded (in fact, K_2 is also compact, but we will not need this fact).

Since the operator-valued function $Z(y)^* : \mathcal{N} \rightarrow \mathcal{K}$ is continuous and its values are compact operators, we can approximate this function in the operator norm uniformly in y by a step function with compact values. This yields an approximation of the operator K_1 in the operator norm by a finite sum of operators $K_1^{(i)}$ acting from $L^2((0, a), \mathcal{N}) = L^2(0, a) \otimes \mathcal{N}$ to $L^2((0, a), \mathcal{K}) = L^2(0, a) \otimes \mathcal{K}$. Each of the operators $K_1^{(i)}$ can be represented as a tensor product of an operator in $L^2((0, a))$ and a compact operator from \mathcal{N} to \mathcal{K} . The operator in $L^2((0, a))$ is compact by

Lemma 5.5. Thus, each of the operators $K_1^{(i)}$ is compact. This argument proves that the operator K_1 is compact.

Finally, since $\|Z(x) - Z(0)\| < C|x|^\alpha$, the operator K_2 is bounded by Lemma 5.5. \square

APPENDIX A

Here we prove three elementary statements.

The first of them constitutes a part of Proposition 2.4. The set Ω has been defined in Proposition 2.3.

Lemma A.1. *Let Assumption 2.2 hold true. Then the spectrum of H on Ω is purely a.c.*

Proof. Let $\Delta_n = (a_n, b_n)$ be one of the component intervals of the open set Ω . It suffices to prove that for every $\varepsilon > 0$ and for a dense set of elements $f \in \mathcal{H}$, the function $(R(z)f, f)$ is bounded on the set

$$\Pi_{n,\varepsilon} := \{z \in \mathbb{C} \mid \operatorname{Re} z \in [a_n + \varepsilon, b_n - \varepsilon], \operatorname{Im} z \in (0, 1)\}.$$

Let $L_n \subset \operatorname{Ran} E_0(\Delta_n)$ be the set of elements f_1 such that $\mathcal{F}f_1 \in C_0^\infty(\Delta_n, \mathcal{N})$ (recall that \mathcal{F} is defined in (2.6)). It is clear that L_n is dense in $\operatorname{Ran} E_0(\Delta_n)$. We will prove the boundedness of $(R(z)f, f)$ on the dense set of elements of the form $f = f_1 + f_2$, where $f_1 \in L_n$, $f_2 \in \operatorname{Ran} E_0(\mathbb{R} \setminus \Delta_n)$.

Using (2.4), write

$$(A.1) \quad (R(z)f, f) = (R_0(z)f, f) + (J(I + T(z)J)^{-1}GR_0(z)f, GR_0(z)f).$$

By the definition of Ω , the norms of $(I + T(z)J)^{-1}$ are uniformly bounded for all $z \in \Pi_{n,\varepsilon}$. First let $f = f_2$. Then it is obvious that $R_0(z)f$ and hence the r.h.s. of (A.1) is bounded for $z \in \Pi_{n,\varepsilon}$. Next, let $f = f_1$. Then it follows from (2.6), (2.7) that

$$(A.2) \quad (R_0(z)f, f) = \int_{\Delta_n} \frac{\|(\mathcal{F}f)(\lambda)\|_{\mathcal{N}}^2}{\lambda - z} d\lambda,$$

and for all $g \in \mathcal{H}$

$$(A.3) \quad (GR_0(z)f, g) = \int_{\Delta_n} \frac{((\mathcal{F}f)(\lambda), Z(\lambda)g)_{\mathcal{N}}}{\lambda - z} d\lambda.$$

According to (2.8), $((\mathcal{F}f)(\lambda), Z(\lambda)g)_{\mathcal{N}}$ is a Hölder continuous function of $\lambda \in \Delta_n$; moreover, the corresponding constant in the definition of Hölder continuity is bounded by $C\|g\|$. Therefore, by the Privalov theorem, integral (A.3) is bounded by $C\|g\|$ for all $z \in \Pi_{n,\varepsilon}$. Hence the function $\|GR_0(z)f\|$ is bounded on $\Pi_{n,\varepsilon}$. Integral (A.2) is considered in a similar but simpler way. Thus, for $f = f_1$ the r.h.s. of (A.1) is bounded on $\Pi_{n,\varepsilon}$. These arguments show also that $(R(z)f_2, f_1)$ is bounded on $\Pi_{n,\varepsilon}$. This proves that the quadratic form $(R(z)f, f)$ is bounded on $\Pi_{n,\varepsilon}$ for all f of the form $f = f_1 + f_2$, as required. \square

Proof of Lemma 3.2. We recall that the Legendre function can be expressed in terms of the hypergeometric function as

$$(A.4) \quad P_\nu(x) = F(-\nu, \nu + 1; 1; \frac{1-x}{2}), \quad |x - 1| < 2.$$

The hypergeometric function $F(a, b; c; z)$ is defined by the hypergeometric series

$$(A.5) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1).$$

For $|z| < 1$, this series is absolutely convergent and analytic in a, b, c, z . For $x > 1$, formulas (9) and (23) of Section 3.2 of [5] yield the representation

$$(A.6) \quad P_{-\frac{1}{2}+it}(x) = \frac{\Gamma(-it)}{\sqrt{\pi}\Gamma(\frac{1}{2}-it)} (2x)^{-\frac{1}{2}-it} F(\frac{1}{4} + i\frac{t}{2}, \frac{3}{4} + i\frac{t}{2}; 1 + it; x^{-2}) \\ + \frac{\Gamma(it)}{\sqrt{\pi}\Gamma(\frac{1}{2}+it)} (2x)^{-\frac{1}{2}+it} F(\frac{1}{4} - i\frac{t}{2}, \frac{3}{4} - i\frac{t}{2}; 1 - it; x^{-2}).$$

Let us split the interval $[1, \infty)$ into $[1, 2)$ and $[2, \infty)$. For $x \in [1, 2)$, we can use (A.4); then $|\frac{1-x}{2}| < 1/2$ and so the hypergeometric series converges uniformly which shows that the estimates (3.6), (3.7) are trivially true in this range of x .

For $x \in [2, \infty)$, we can use (A.6) and expand the hypergeometric function in the r.h.s. in the hypergeometric series. The series converges uniformly in $x \in [2, \infty)$. Observing that $F(a, b; c; 0) = 1$ and using the elementary estimate

$$|x^{it_1} - x^{it_2}| \leq C(\delta) |t_2 - t_1|^\delta (\log x)^\delta,$$

we obtain the estimates (3.6), (3.7) for $x \geq 2$. \square

Proof of Lemma 5.5. 1. For $\delta \in (0, a)$, let χ_δ be the characteristic function of the interval $(0, \delta)$ and let $\tilde{\chi}_\delta = 1 - \chi_\delta$. Along with K , consider the integral operator \tilde{K}_δ with the integral kernel $\tilde{K}_\delta(x, y) = K(x, y)\tilde{\chi}_\delta(y)$. A direct inspection shows that the kernel $\tilde{K}_\delta(x, y)$ is uniformly bounded in $(x, y) \in [0, a] \times [0, a]$ and therefore the operator \tilde{K}_δ is in the Hilbert-Schmidt class. Thus, it suffices to show that

$$(A.7) \quad \|K - \tilde{K}_\delta\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

2. Let $K_\delta = K - \tilde{K}_\delta$ and $f, g \in L^2(0, a)$. Using Cauchy-Schwartz, we have

$$(A.8) \quad |(K_\delta f, g)| \leq \frac{1}{\pi} \int_0^a dx \int_0^\delta dy \frac{1}{x+y} (1 + |\log x|)^{-p} (1 + |\log y|)^q \sqrt[4]{\frac{x}{y}} \sqrt[4]{\frac{y}{x}} |f(y)| |g(x)| \\ \leq \frac{1}{\pi} \left(\int_0^a dx \int_0^\delta dy \frac{1}{x+y} \sqrt{\frac{y}{x}} |f(y)|^2 \right)^{1/2} \\ \times \left(\int_0^a dx \int_0^\delta dy \frac{1}{x+y} \sqrt{\frac{x}{y}} \frac{(1 + |\log y|)^{2q}}{(1 + |\log x|)^{2p}} |g(x)|^2 \right)^{1/2}.$$

Next, we have

$$\sup_{0 < y < a} \int_0^a \frac{1}{x+y} \sqrt{\frac{y}{x}} dx \leq \int_0^\infty \frac{1}{x+1} \frac{1}{\sqrt{x}} dx = C < \infty,$$

and therefore the first term in the r.h.s. of (A.8) is bounded by $C\|f\|$. In order to estimate the second term in the r.h.s. of (A.8), we first note the elementary estimate

$$(1 + |\log(xy)|) \leq (1 + |\log x|)(1 + |\log y|).$$

Using this, we have:

$$\begin{aligned} & \sup_{0 < x < \delta} \int_0^\delta \frac{1}{x+y} \sqrt{\frac{x}{y}} \frac{(1 + |\log y|)^{2q}}{(1 + |\log x|)^{2p}} dy \\ & \leq (1 + |\log \delta|)^{2q-2p} \sup_{0 < x < \delta} \int_0^\delta \frac{1}{x+y} \sqrt{\frac{x}{y}} \frac{(1 + |\log y|)^{2q}}{(1 + |\log x|)^{2q}} dy \\ & = (1 + |\log \delta|)^{2q-2p} \sup_{0 < x < \delta} \int_0^{\delta/x} \frac{1}{1+t} \frac{1}{\sqrt{t}} \frac{(1 + |\log(xt)|)^{2q}}{(1 + |\log x|)^{2q}} dt \\ & \leq (1 + |\log \delta|)^{2q-2p} \int_0^\infty \frac{1}{1+t} \frac{1}{\sqrt{t}} (1 + |\log t|)^{2q} dt = C(1 + |\log \delta|)^{2q-2p}. \end{aligned}$$

From here we get the estimate for the second term in the r.h.s. of (A.8) by $C(1 + |\log \delta|)^{q-p}\|g\|$. Thus, we have

$$|(K_\delta f, g)| \leq C(1 + |\log \delta|)^{q-p}\|f\|\|g\|,$$

and (A.7) follows. \square

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