

Operator theoretic methods for the eigenvalue counting function in spectral gaps

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Abstract. Using the notion of spectral flow, we suggest a simple approach to various asymptotic problems involving eigenvalues in the gaps of the essential spectrum of self-adjoint operators. Our approach employs some elements of the theory of the spectral shift function. Using this approach, we provide generalisations and streamlined proofs of two results in this area already existing in the literature. We also give a new proof of the generalised Birman-Schwinger principle.

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1. Introduction

1.1. The spectral flow

Since the pioneering work [1], problems involving counting functions of eigenvalues in the gaps of the essential spectrum of self-adjoint operators have attracted a considerable amount of attention in the mathematical physics literature. Let us recall the set-up of the problem. Let M and A be self-adjoint operators in a Hilbert space such that the spectrum of M has a gap and A is M -compact. Then, for any $t \in \mathbb{R}$, the essential spectra of M and $M + tA$ coincide and the eigenvalues of $M + tA$ in the spectral gaps of M are analytic in t . If $A \geq 0$ or $A \leq 0$ in the quadratic form sense, then these eigenvalues are monotone in t ; in general, they may not be monotone.

Let us fix a coupling constant $t > 0$ and a spectral parameter λ in a spectral gap of M and consider one of the variants of the eigenvalue counting function, known as the spectral flow of the family $\{M + \tau A\}_{\tau=0}^t$, through λ . This is defined as follows. Assume that neither the spectrum of M nor the spectrum of $M + tA$ contains λ . As τ increases monotonically from 0 to t , some eigenvalues of $M + \tau A$

may cross λ . By analyticity in τ , there will only be a finite number of such crossings. Some eigenvalues will cross λ from left to right, others from right to left. The spectral flow is defined as

$$\begin{aligned} & \text{sf}(\lambda; M + tA, M) \\ &= \langle \text{the number of eigenvalues of } M + \tau A, 0 \leq \tau \leq t, \text{ which cross } \lambda \text{ rightwards} \rangle \\ &- \langle \text{the number of eigenvalues of } M + \tau A, 0 \leq \tau \leq t, \text{ which cross } \lambda \text{ leftwards} \rangle. \end{aligned} \tag{1.1}$$

Some eigenvalues may “turn around” at λ (i.e. for some $\tau_0 \in (0, t)$, the function $\lambda_n(\tau)$ may have a local minimum = λ or a local maximum = λ at $\tau = \tau_0$); these eigenvalues do not contribute to (1.1). The eigenvalues are counted with multiplicities taken into account.

The asymptotics of $\text{sf}(\lambda; M + tA, M)$ as $t \rightarrow \infty$ and related issues have been extensively studied both for concrete differential operators $M + tA$ and in an abstract setting; see e.g. the survey [16] for the history and a recent paper [17] for extensive bibliography. Most relevant to our approach are the operator theoretic constructions of M. Birman (see [7] and references therein) and O. Safronov [26, 27, 28]. We also note that there is a large family of index theorems (see e.g. [22] and references therein) which use the notion of the spectral flow; these are not directly related to the topic of this paper.

1.2. Spectral flow, Fredholm index, and spectral shift function

Let us start by mentioning two other interpretations of the spectral flow; the precise statements will be given in Section 2. First, if A is a trace class operator, then

$$\text{sf}(\lambda; M + A, M) = \xi(\lambda - 0; M + A, M), \quad \forall \lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M), \tag{1.2}$$

where $\xi(\cdot; M + A, M)$ is M. Krein’s spectral shift function. Next, one has (see Section 2 for definitions and precise statements)

$$\text{sf}(\lambda; M + A, M) = \Xi(\lambda; M + A, M),$$

where the right hand side is defined as the Fredholm index (see Section 2.1) of the pair of spectral projections of $M + A$ and M , associated with the interval $(-\infty, \lambda)$:

$$\Xi(\lambda; M + A, M) = \text{index}(E_M(-\infty, \lambda), E_{M+A}(-\infty, \lambda)). \tag{1.3}$$

These interpretations of spectral flow have now become folklore; they have also been used in the abstract operator theoretic context, in particular in works on operator algebras, see e.g. [10, 4] and references therein. However, the methods emerging from these interpretations have not yet been used to the full extent in the mathematical physics literature. This paper presents a step in this direction.

We consider the function Ξ defined by (1.3); the precise definition is given in Section 2. We use the intuition coming from the theory of the spectral shift function

to provide elementary proofs of a number of simple yet very useful properties of this function. Most importantly, one has the “chain rule”

$$\Xi(\lambda; M + A_1 + A_2, M) = \Xi(\lambda; M + A_1, M) + \Xi(\lambda; M + A_1 + A_2, M + A_1) \quad (1.4)$$

and the estimates (see Theorem 3.1)

$$-\text{rank } A_- \leq \Xi(\lambda; M + A, M) \leq \text{rank } A_+, \quad A_{\pm} = \frac{1}{2}(|A| \pm A). \quad (1.5)$$

In particular,

$$\pm A \geq 0 \quad \Rightarrow \quad \pm \Xi(\lambda; M + A, M) \geq 0. \quad (1.6)$$

These properties are well known in the theory of the spectral shift function. In our approach, they provide a basis for various monotonicity arguments typical for the variational technique.

Next, the function Ξ is related to the eigenvalue counting function by

$$\Xi(\lambda_1; M + A, M) - \Xi(\lambda_2; M + A, M) = N([\lambda_1, \lambda_2]; M + A) - N([\lambda_1, \lambda_2]; M); \quad (1.7)$$

here $N(\delta; M)$ is the number of eigenvalues of M in the interval δ , and we assume that $\sigma_{\text{ess}}(M) \cap [\lambda_1, \lambda_2] = \emptyset$. A particular case of (1.7) is

$$\Xi(\lambda; M + A, M) = -N((-\infty, \lambda); M + A), \quad \lambda < \inf \sigma(M). \quad (1.8)$$

Because of these properties, Ξ is a useful tool in analysing the eigenvalue counting function in the gaps of the essential spectrum.

Further, in Section 3 we consider the behaviour of Ξ with respect to decompositions of the Hilbert space into direct sums. Finally, in Section 4, we discuss and provide a new proof of an identity which can be interpreted as the Birman-Schwinger principle stated in terms of Ξ .

Most of these properties of Ξ have already appeared in the literature in various guises, see e.g. [14, 13, 8, 17], mainly (but not exclusively) in the framework of the theory of the spectral shift function, which requires some trace class assumptions. The novelty of this paper is in collecting these properties together in a unified and rather general form and putting them to work in problems involving eigenvalues in the gaps outside the trace class scheme. We also provide streamlined and self-contained proofs of these properties.

We do not make any attempt here to review the literature on the eigenvalue counting function, as it is enormously wide. Where appropriate, we only mention the works most directly related to our approach. More references and history can be found in the survey [16] and the recent paper [17]. We also note that an interesting approach to the analysis of eigenvalues in the spectral gaps has been developed in [15, 12]. It doesn't seem to be directly related to the approach of this paper. Some discussion of the numerical aspects of calculation of eigenvalues in the gaps and appropriate references can be found e.g. in [11].

1.3. Applications

To illustrate the efficiency of our approach, we apply it to provide simple proofs of two results already present in the literature. The first one is a theorem of O. Safronov from [27] which deals with the asymptotics of $\Xi(\lambda; M + tA, M)$ as $t \rightarrow \infty$. A typical application of this theorem is to the spectral flow of the Schrödinger operator M with a periodic potential, perturbed by the operator A of multiplication by a potential which decays at infinity. This is discussed in Section 5.

The second result is a theorem of G. Rozenblum and A. Sobolev [23] which describes the asymptotic eigenvalue distribution of the Landau operator perturbed by an expanding potential. In Sections 6 and 7, we provide a streamlined proof and a generalisation of this result.

In conclusion, we list other potential areas of application of our technique:

(i) Theorem 6.1 can be applied to the analysis of a periodic operator perturbed by an expanding potential.

(ii) Analysis of eigenvalues in the gap of the Dirac operator. This will require a generalisation of our technique to the case of operators which are not semi-bounded.

1.4. Notation

For a self-adjoint operator A , the symbols $\sigma(A)$, $\sigma_{\text{ess}}(A)$, $E_A(a, b)$, and $N(\delta; A)$ denote, respectively, the spectrum of A , the essential spectrum of A , the spectral projection of A associated with $(a, b) \subset \mathbb{R}$, and the total number of eigenvalues (counting multiplicity) of A in the interval δ . The symbols \mathfrak{S}_∞ and \mathfrak{S}_2 denote the classes of compact and Hilbert-Schmidt operators in a Hilbert space. We denote by $C_0^\infty(\mathbb{R})$ the class of all complex valued infinitely smooth compactly supported functions on \mathbb{R} .

2. The function Ξ

In this section we introduce the function Ξ and discuss its relationship with the spectral shift function, the spectral flow and the eigenvalue counting function. We also discuss the stability of Ξ .

2.1. The index of a pair of projections

Let us recall some background material from [3]. A pair of orthogonal projections P, Q in a Hilbert space \mathcal{H} is called Fredholm, if

$$\{1, -1\} \cap \sigma_{\text{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q$ is compact, then the pair P, Q is Fredholm. The index of a Fredholm pair is defined by the formula

$$\text{index}(P, Q) = \dim \text{Ker}(P - Q - I) - \dim \text{Ker}(P - Q + I).$$

Using $0 \leq P \leq I$ and $0 \leq Q \leq I$, this can be alternatively written as

$$\text{index}(P, Q) = \dim(\text{Ran } P \cap \text{Ker } Q) - \dim(\text{Ran } Q \cap \text{Ker } P). \quad (2.1)$$

It is well known (see e.g. [3, Theorem 4.2]) that

$$\dim \text{Ker}(P - Q - \lambda I) = \dim \text{Ker}(P - Q + \lambda I), \quad \lambda \neq \pm 1;$$

the proof of this is based on the identity

$$(P - Q)(I - P - Q) = (I - P - Q)(Q - P).$$

Thus, if $P - Q$ is a trace class operator, then

$$\text{index}(P, Q) = \text{Tr}(P - Q), \quad (2.2)$$

since all the eigenvalues of $P - Q$ apart from 1 and -1 in the series $\text{Tr}(P - Q) = \sum_k \lambda_k(P - Q)$ cancel out (here $\lambda_k(P - Q)$ are the eigenvalues of $P - Q$).

If both (P, Q) and (Q, R) are Fredholm pairs and at least one of the differences $P - Q$ or $Q - R$ is compact, then the pair (P, R) is also Fredholm and the following identity holds true:

$$\text{index}(P, R) = \text{index}(P, Q) + \text{index}(Q, R). \quad (2.3)$$

See e.g. [3] for the proof of the last statement and the details.

2.2. Definition of Ξ

Let M and \widetilde{M} be self-adjoint operators in a Hilbert space \mathcal{H} . If $E_M(-\infty, \lambda)$, $E_{\widetilde{M}}(-\infty, \lambda)$ is a Fredholm pair, we will say that $\Xi(\lambda; \widetilde{M}, M)$ exists and define

$$\Xi(\lambda; \widetilde{M}, M) := \text{index}(E_M(-\infty, \lambda), E_{\widetilde{M}}(-\infty, \lambda)).$$

It is obvious that $\Xi(\lambda; \widetilde{M}, M)$ is constant on the intervals of the set $\mathbb{R} \setminus (\sigma(M) \cup \sigma(\widetilde{M}))$ and is left continuous in λ on the set $\mathbb{R} \setminus (\sigma_{\text{ess}}(M) \cup \sigma_{\text{ess}}(\widetilde{M}))$. One has

$$\Xi(\lambda; \widetilde{M}, M) = -\Xi(\lambda; M, \widetilde{M}).$$

We will often use the following simple sufficient condition for the existence of $\Xi(\lambda; \widetilde{M}, M)$. Let M be a self-adjoint lower semi-bounded operator in \mathcal{H} and let the self-adjoint operator A in \mathcal{H} be relatively form-compact with respect to M . This means that

$$\text{Dom}|A|^{1/2} \supset \text{Dom}(M + \gamma I)^{1/2} \quad \text{and} \quad |A|^{1/2}(M + \gamma I)^{-1/2} \in \mathfrak{S}_\infty$$

for all sufficiently large $\gamma > 0$. Under this assumption, by the KLMN Theorem (see [25, Theorem X.17]) the operator $\widetilde{M} = M + A$ is well defined in terms of the corresponding quadratic form. Using the resolvent identity, we get

$$(\widetilde{M} - zI)^{-1} - (M - zI)^{-1} \in \mathfrak{S}_\infty \quad \forall z \in \mathbb{C} \setminus (\sigma(M) \cup \sigma(\widetilde{M})). \quad (2.4)$$

By Weyl's theorem on the stability of the essential spectrum under compact perturbations, this implies $\sigma_{\text{ess}}(\widetilde{M}) = \sigma_{\text{ess}}(M)$. If $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M)$, then, representing the spectral projections by Riesz integrals and using (2.4), it is easy to see that the difference $E_{\widetilde{M}}(-\infty, \lambda) - E_M(-\infty, \lambda)$ is compact and therefore $\Xi(\lambda; \widetilde{M}, M)$ exists.

If both A_1 and A_2 are form-compact with respect to M and λ is not in the essential spectrum of M , then, by the above argument and (2.3), the ‘‘chain rule’’ (1.4) holds true.

- Remark.** 1. In this paper, we discuss the function $\Xi(\lambda; \widetilde{M}, M)$ in the gaps of the essential spectrum of M . In the forthcoming publication [21], we will consider this function on the essential spectrum of M , provided that some assumptions typical for the scattering theory hold true for the pair of operators M, \widetilde{M} .
2. In this paper, we assume most of the time that M is lower semi-bounded. It is possible to generalise our results to the case of the operators M which are not semi-bounded. However, this would make our construction and particularly the proofs considerably more complicated.

The function Ξ , with various notation and in various guises, has appeared in the literature many times. Without any attempts at being exhaustive, let us mention a few sources. In [14, 13, 8], Ξ was used in the context of the theory of the spectral shift function. There is extensive literature on Ξ in the theory of operator algebras, see e.g. [5, 4, 19] and references therein.

2.3. Ξ and the spectral shift function

Proposition 2.1. *If M and A are self-adjoint operators and A is a trace class operator, then the identity*

$$\Xi(\lambda; M + A, M) = \xi(\lambda - 0; M + A, M), \quad \forall \lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M) \quad (2.5)$$

holds true, where $\xi(\cdot; M + A, M)$ is M. Krein’s spectral shift function.

Proof. The key point is the M. Krein’s theorem that if $\varphi' \in C_0^\infty(\mathbb{R})$, then the difference $\varphi(M + A) - \varphi(M)$ belongs to the trace class and the trace formula

$$\text{Tr}(\varphi(M + A) - \varphi(M)) = \int_{-\infty}^{\infty} \varphi'(t) \xi(t; M + A, M) dt$$

holds true. Since λ is not in the essential spectrum of M , there exists $\delta > 0$ such that $(\lambda - \delta, \lambda) \cap \sigma(M) = \emptyset$ and $(\lambda - \delta, \lambda) \cap \sigma(M + A) = \emptyset$. We can choose φ with $\text{supp } \varphi' \subset (\lambda - \delta, \lambda)$ and $\varphi(\lambda - \delta) = 1$, $\varphi(\lambda) = 0$. Then, using (2.2), we get

$$\begin{aligned} \text{Tr}(\varphi(M + A) - \varphi(M)) &= \text{Tr}(E_{M+A}(-\infty, \lambda) - E_M(-\infty, \lambda)) \\ &= \text{index}(E_{M+A}(-\infty, \lambda), E_M(-\infty, \lambda)) = -\Xi(\lambda; M + A, M). \end{aligned}$$

On the other hand, since $\xi(\cdot; M + A, M)$ is constant on $(\lambda - \delta, \lambda)$, we get

$$\int_{-\infty}^{\infty} \xi(t; M + A, M) \varphi'(t) dt = -\xi(\lambda - 0; M + A, M);$$

this proves the claim. ■

As mentioned in the introduction, this statement can be regarded as folklore; it was explicitly stated and used e.g. in [28, 4, 19].

2.4. Ξ and the eigenvalue counting function.

Theorem 2.2. *Let M be a lower semi-bounded self-adjoint operator, let A be a self-adjoint operator which is form-compact with respect to M and $[\lambda_1, \lambda_2] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(M)$. Then the identities (1.7), (1.8) hold true.*

Proof. Using (2.3) and (2.2), we get

$$\begin{aligned} \Xi(\lambda_1; M + A, M) &= \text{index}(E_M(-\infty, \lambda_1), E_{M+A}(-\infty, \lambda_1)) \\ &= \text{index}(E_M(-\infty, \lambda_1), E_M(-\infty, \lambda_2)) + \text{index}(E_M(-\infty, \lambda_2), E_{M+A}(-\infty, \lambda_2)) \\ &\quad + \text{index}(E_{M+A}(-\infty, \lambda_2), E_{M+A}(-\infty, \lambda_1)) \\ &= \text{Tr}(E_M(-\infty, \lambda_1) - E_M(-\infty, \lambda_2)) + \Xi(\lambda_2; M + A, M) \\ &\quad + \text{Tr}(E_{M+A}(-\infty, \lambda_2) - E_{M+A}(-\infty, \lambda_1)) \\ &= -N([\lambda_1, \lambda_2]; M) + \Xi(\lambda_2; M + A, M) + N([\lambda_1, \lambda_2]; M + A); \end{aligned}$$

this proves (1.7). Identity (1.8) follows by taking $\lambda_1 \rightarrow -\infty$ and $\lambda_2 = \lambda$. ■

This statement is well known and (if stated in terms of the spectral flow) is intuitively obvious.

2.5. Stability of Ξ

The following result is essentially well known; see [13, Theorem 3.12] for a very similar statement. However, in order to make this text self-contained, we provide a proof (which is not significantly different from the proof of [13]).

Theorem 2.3. *Let M and \widetilde{M} be lower semi-bounded self-adjoint operators and suppose that $\Xi(\lambda; \widetilde{M}, M)$ exists for some $\lambda \in \mathbb{R} \setminus (\sigma(M) \cup \sigma(\widetilde{M}))$. Let M_n and \widetilde{M}_n be two sequences of self-adjoint operators such that $M_n \rightarrow M$ and $\widetilde{M}_n \rightarrow \widetilde{M}$ in the norm resolvent sense and such that M_n, \widetilde{M}_n are uniformly bounded from below: $\gamma I \leq M_n, \gamma I \leq \widetilde{M}_n$ for some $\gamma \in \mathbb{R}$ and all n . Then for all sufficiently large n , $\Xi(\lambda; \widetilde{M}_n, M_n)$ exists and equals $\Xi(\lambda; \widetilde{M}, M)$.*

Remark. It is not difficult to construct an example showing that the assumption of the existence of a uniform lower bound for M_n and \widetilde{M}_n cannot be dropped from the hypothesis of this theorem.

Proof. 1. Let us denote

$$P = E_M(-\infty, \lambda), \quad P_n = E_{M_n}(-\infty, \lambda), \quad Q = E_{\widetilde{M}}(-\infty, \lambda), \quad Q_n = E_{\widetilde{M}_n}(-\infty, \lambda).$$

By our assumptions, $P = E_M(\gamma - 1, \lambda)$, $P_n = E_{M_n}(\gamma - 1, \lambda)$. Since λ and $\gamma - 1$ are not in the spectrum of M , by [24, Theorem VIII.23(b)] it follows that $\|P_n - P\| \rightarrow 0$ as $n \rightarrow \infty$. In the same way, $\|Q_n - Q\| \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$\|(P_n - Q_n) - (P - Q)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

2. Since (P, Q) is a Fredholm pair, there exists $\delta > 0$ such that

$$\sigma(P - Q) \cap (-1, 1) \subset [-1 + 2\delta, 1 - 2\delta]. \quad (2.7)$$

Then $-1 + \delta$ and $1 - \delta$ are not in the spectrum of $P - Q$ and so, using (2.6) and applying [24, Theorem VIII.23(b)] again, we get

$$\|E_{P_n - Q_n}(1 - \delta, 2) - E_{P - Q}(1 - \delta, 2)\| \rightarrow 0, \quad (2.8)$$

$$\|E_{P_n - Q_n}(-2, -1 + \delta) - E_{P - Q}(-2, -1 + \delta)\| \rightarrow 0 \quad (2.9)$$

as $n \rightarrow \infty$. In particular, $\text{rank } E_{P_n - Q_n}(1 - \delta, 2)$ and $\text{rank } E_{P_n - Q_n}(-2, -1 + \delta)$ are finite for large n and therefore $\Xi(\lambda; \widetilde{M}_n, M_n)$ exists.

3. By the definition of index and (2.7), we have

$$\Xi(\lambda; \widetilde{M}, M) = \text{rank } E_{P - Q}(1 - \delta, 2) - \text{rank } E_{P - Q}(-2, -1 + \delta). \quad (2.10)$$

By (2.6) and (2.7), we have $\sigma(P_n - Q_n) \cap (-1, 1) \subset (-1 + \delta, 1 - \delta)$ for all sufficiently large n , and then

$$\Xi(\lambda; \widetilde{M}_n, M_n) = \text{rank } E_{P_n - Q_n}(1 - \delta, 2) - \text{rank } E_{P_n - Q_n}(-2, -1 + \delta). \quad (2.11)$$

Combining (2.8)–(2.11), we get the required statement. ■

2.6. Ξ as spectral flow

Let M be a lower semi-bounded self-adjoint operator and let A be form-compact with respect to M . Recall the definition of the spectral flow (1.1); here we prove that

$$\text{sf}(\lambda; M + A, M) = \Xi(\lambda; M + A, M), \quad \lambda \in \mathbb{R} \setminus (\sigma(M) \cap \sigma(M + A)). \quad (2.12)$$

This statement is not used elsewhere in the paper and is given here only in order to provide some motivation and in order to help comparison with other results in the area.

Let us choose an interval $(a, b) \subset \mathbb{R} \setminus \sigma_{\text{ess}}(M)$ such that $\lambda \in (a, b)$. First suppose that there exists $\lambda_0 < \lambda$, $\lambda_0 \in (a, b)$ such that

$$\lambda_0 \notin \sigma(M + tA) \quad \text{for all } t \in [0, 1]. \quad (2.13)$$

It is easy to see that the resolvent $(M + tA - zI)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, is continuous in $t \in [0, 1]$ in the operator norm. By the stability Theorem 2.3, we conclude that $\Xi(\lambda_0; M + tA, M)$ is independent of $t \in [0, 1]$ and therefore $\Xi(\lambda_0; M + A, M) = 0$. Using (1.7), we get

$$\Xi(\lambda; M + A, M) = N([\lambda_0, \lambda]; M) - N([\lambda_0, \lambda]; M + A).$$

The r.h.s. equals the net flux of eigenvalues of $M + tA$ outward from the interval (λ_0, λ) . By (2.13), the flux through λ_0 equals zero. Thus, it is clear that (2.12) holds true.

In general, there may be no point λ_0 as in (2.13), but we can always find a finite open cover of $[0, 1]$ by sufficiently small subintervals δ_i such that for each family $\{M_t \mid t \in \delta_i\}$, the point λ_0 can be chosen appropriately. Then formula (2.12) can be obtained by combining the formulas corresponding to all the subintervals.

3. Further properties of Ξ

In this section we prove variational estimates for Ξ which will be crucial for our further analysis. We also discuss the behaviour of Ξ with respect to the decomposition of the Hilbert space into direct sums.

3.1. Variational estimates for Ξ

Theorem 3.1. *Let M be a lower semi-bounded self-adjoint operator and let the self-adjoint operator A be relatively form-bounded with respect to M with a relative bound less than one. Let \widetilde{M} be defined as the form sum $\widetilde{M} = M + A$; assume that $\Xi(\lambda; \widetilde{M}, M)$ exists. Then the estimates (1.5) and (1.6) hold true.*

Proof. Let us prove the second inequality in (1.5); the proof of the first one is analogous. If $\text{rank } A_+ = \infty$, there is nothing to prove; so let us assume $\text{rank } A_+ < \infty$. By (2.1), the desired statement will follow if we prove that

$$\dim(\text{Ran } E_M(-\infty, \lambda) \cap \text{Ker } E_{\widetilde{M}}(-\infty, \lambda)) \leq \text{rank } A_+.$$

Suppose to the contrary that

$$\dim(\text{Ran } E_M(-\infty, \lambda) \cap \text{Ker } E_{\widetilde{M}}(-\infty, \lambda)) > \text{rank } A_+.$$

Then there exists $\psi \neq 0$, $\psi \in \text{Ran } E_M(-\infty, \lambda) \cap \text{Ker } E_{\widetilde{M}}(-\infty, \lambda)$ such that $A_+\psi = 0$. Denote by m and \widetilde{m} the sesquilinear forms corresponding to M and \widetilde{M} . Since $\psi \in \text{Ran } E_M(-\infty, \lambda)$, it follows that $\psi \in \text{dom}(m) = \text{dom}(\widetilde{m})$. We have

$$m[\psi, \psi] < \lambda \|\psi\|^2 \quad \text{and} \quad \widetilde{m}[\psi, \psi] \geq \lambda \|\psi\|^2. \quad (3.1)$$

On the other hand, since $A_+\psi = 0$, we have $\widetilde{m}[\psi, \psi] \leq m[\psi, \psi]$, which is a contradiction with (3.1). ■

This result immediately implies the following monotonicity principle, variants of which have been used before, e.g. in [28, 17]:

Corollary 3.2. *Let M be a self-adjoint lower semi-bounded operator and let the operators A_1 and A_2 be self-adjoint and form-compact with respect to M . Let $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M)$ and suppose that $M + A_2 \geq M + A_1$; then*

$$\Xi(\lambda; M + A_2, M) \geq \Xi(\lambda; M + A_1, M).$$

Note that $M + A_2 \geq M + A_1$ can often be written in a simpler form $A_2 \geq A_1$, but this requires that the quadratic forms corresponding to A_2 and A_1 are well defined.

Proof. By the ‘‘chain rule’’ (1.4),

$$\Xi(\lambda; M + A_2, M) = \Xi(\lambda; M + A_2, M + A_1) + \Xi(\lambda; M + A_1, M);$$

by Theorem 3.1, the first term in the r.h.s. is non-negative. ■

A simple example of the application of this monotonicity principle is an estimate of the number of eigenvalues in a gap of M when the perturbation A can be represented as $A = B - C$ with $B \geq 0$ and $C \geq 0$. Indeed, if $[\lambda_1, \lambda_2] \cap \sigma(M) = \emptyset$, then by (1.7) we have

$$\begin{aligned} N([\lambda_1, \lambda_2]; M + B - C) &= \Xi(\lambda_1; M + B - C) - \Xi(\lambda_2; M + B - C) \\ &\leq \Xi(\lambda_1; M + B, M) - \Xi(\lambda_2; M - C, M); \end{aligned}$$

now the right hand side can be evaluated, for example, by using the Birman-Schwinger principle, see (4.4) and (4.5) below. This argument has been used before, see e.g. [17].

Corollary 3.3. *Let M be a lower semi-bounded self-adjoint operator such that $[\lambda - a, \lambda + a] \cap \sigma(M) = \emptyset$ for some $\lambda \in \mathbb{R}$ and $a > 0$. Let A and B be compact self-adjoint operators. Then*

$$\begin{aligned} \Xi(\lambda; M + A + B, M) &\leq \Xi(\lambda - a; M + A, M) + N((a, \infty); B), \\ \Xi(\lambda; M + A + B, M) &\geq \Xi(\lambda + a; M + A, M) - N((-\infty, -a); B). \end{aligned}$$

In particular,

$$-N((-\infty, -a); B) \leq \Xi(\lambda; M + B, M) \leq N((a, \infty); B). \quad (3.2)$$

Note that (3.2) is an improvement of (1.5), given some information on the width of the spectral gap of M around λ .

Proof. Let us write $B = B_1 + B_2$, where $\|B_1\| \leq a$ and $\text{rank}(B_2)_+ = N((a, \infty); B)$, $\text{rank}(B_2)_- = N((-\infty, -a); B)$. Then, by (2.3) and Theorem 3.1, one has

$$\begin{aligned} \Xi(\lambda; M + A + B, M) &= \Xi(\lambda; M + A + B_1 + B_2, M + A + B_1) \\ &\quad + \Xi(\lambda; M + A + aI + (B_1 - aI), M + A + aI) + \Xi(\lambda; M + A + aI, M) \\ &\leq \text{rank}(B_2)_+ + \Xi(\lambda; M + A + aI, M) \\ &= N((a, \infty); B) + \Xi(\lambda; M + A + aI, M + aI) = N((a, \infty); B) + \Xi(\lambda - a; M + A, M). \end{aligned}$$

This proves the upper bound for $\Xi(\lambda; M + A + B, M)$; the lower bound is proven in an analogous way. ■

Theorem 3.1 and Corollary 3.3 appeared before in [8] in a somewhat less general form.

3.2. Orthogonal sums and a “diagonalisation trick”

Here we discuss the behaviour of Ξ with respect to orthogonal sum decompositions of the Hilbert space \mathcal{H} . First we state a trivial yet useful observation. Let M be a lower semi-bounded self-adjoint operator in \mathcal{H} and let A be a self-adjoint operator which is form-compact with respect to M . Next, let P and Q be orthogonal projections in \mathcal{H} such that $P + Q = I$. Assume that M is reduced by the orthogonal decomposition $\mathcal{H} = \text{Ran } P \oplus \text{Ran } Q$; this means that

$$P \text{Dom}(M) \subset \text{Dom}(M), \quad Q \text{Dom}(M) \subset \text{Dom}(M)$$

and

$$PMP\psi = MP\psi, \quad QMQ\psi = MQ\psi, \quad \text{for all } \psi \in \text{Dom}(M).$$

Then it is easy to see that the operators PAP and QMQ are also form-compact with respect to M and therefore the form sums $M + PAP$, $M + QMQ$, $M + PAP + QMQ$ are well defined. Moreover, one has

$$\Xi(\lambda; M + PAP + QMQ, M) = \Xi(\lambda; M + PAP, M) + \Xi(\lambda; M + QMQ, M) \quad (3.3)$$

for all $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M)$. This follows directly from the fact that both M and $PAP + QMQ$ are reduced by the orthogonal decomposition $\mathcal{H} = \text{Ran } P \oplus \text{Ran } Q$.

Next, we apply a trick from [18, Lemma 1.1] to the analysis of Ξ . This trick is not specific to the function Ξ but rather is a general variational consideration. The usefulness of this trick is illustrated by the construction of Section 6.

Theorem 3.4. *Let M be a lower semi-bounded self-adjoint operator in \mathcal{H} and let A be a self-adjoint operator which is form-compact with respect to M . Let P and Q be orthogonal projections in \mathcal{H} such that $P + Q = I$. Assume that M is reduced by the orthogonal decomposition $\mathcal{H} = \text{Ran } P \oplus \text{Ran } Q$. Then for any $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(M)$ and for any $\varepsilon > 0$, one has*

$$\Xi(\lambda; M + A, M) \leq \Xi(\lambda; M + P(A + \varepsilon|A|)P, M) + \Xi(\lambda; M + Q(A + \frac{1}{\varepsilon}|A|)Q, M), \quad (3.4)$$

$$\Xi(\lambda; M + A, M) \geq \Xi(\lambda; M + P(A - \varepsilon|A|)P, M) + \Xi(\lambda; M + Q(A - \frac{1}{\varepsilon}|A|)Q, M). \quad (3.5)$$

Proof. For $\phi, \psi \in \text{Dom}(|A|^{1/2})$, denote

$$a[\psi, \phi] = (\text{sign}(A)|A|^{1/2}\psi, |A|^{1/2}\phi), \quad |a|[\psi, \phi] = (|A|^{1/2}\psi, |A|^{1/2}\phi).$$

Let $\psi \in \text{Dom}(|M|^{1/2})$; by assumption, we have $\psi \in \text{Dom}(|A|^{1/2})$ and $P\psi, Q\psi \in \text{Dom}(|M|^{1/2}) \subset \text{Dom}(|A|^{1/2})$, and

$$\begin{aligned} 2|a[P\psi, Q\psi]| &= 2 \left| (\text{sign}(A)|A|^{1/2}P\psi, |A|^{1/2}Q\psi) \right| \\ &\leq 2\| |A|^{1/2}P\psi \| \| |A|^{1/2}Q\psi \| \leq \varepsilon |a|[P\psi, P\psi] + \frac{1}{\varepsilon} |a|[Q\psi, Q\psi]. \end{aligned}$$

It follows that

$$\begin{aligned} a[\psi, \psi] &= a[P\psi, P\psi] + a[Q\psi, Q\psi] + 2\text{Re } a[P\psi, Q\psi] \\ &\leq a[P\psi, P\psi] + a[Q\psi, Q\psi] + \varepsilon |a|[P\psi, P\psi] + \frac{1}{\varepsilon} |a|[Q\psi, Q\psi], \end{aligned}$$

and therefore

$$M + A \leq M + P(A + \varepsilon|A|)P + Q(A + \frac{1}{\varepsilon}|A|)Q$$

in the quadratic form sense. Denote $K = P(A + \varepsilon|A|)P + Q(A + \frac{1}{\varepsilon}|A|)Q$; using Corollary 3.2 and the identity (3.3), we get

$$\Xi(\lambda; M + A, M) \leq \Xi(\lambda; M + K, M) = \Xi(\lambda; M + PKP, M) + \Xi(\lambda; M + QKQ, M),$$

which yields (3.4). The estimate (3.5) is obtained in a similar way. ■

4. Generalised Birman-Schwinger principle

4.1. Statement and discussion

Let M be a lower semi-bounded self-adjoint operator in \mathcal{H} , let A be a self-adjoint operator which is form-compact with respect to M , and let \widetilde{M} be defined as a form sum $\widetilde{M} = M + A$. Suppose that A is represented as $A = G^*JG$, where G is a closed operator from \mathcal{H} to an auxiliary Hilbert space \mathcal{K} such that for some $\gamma > 0$, $\text{Dom}(M + \gamma I)^{1/2} \subset \text{Dom} G$ and $G(M + \gamma I)^{-1/2}$ is compact, and J is self-adjoint, bounded in \mathcal{K} and has a bounded inverse. (The simplest case of such a factorisation is $\mathcal{K} = \mathcal{H}$, $A = |A|^{1/2} \text{sign}(A) |A|^{1/2}$.) For $\lambda \in \mathbb{R} \setminus \sigma(M)$, define the compact self-adjoint operator $T(\lambda)$ in \mathcal{K} by setting

$$T(\lambda)f = G(M - \lambda I)^{-1}G^*f, \quad f \in \text{Dom} G^*,$$

and taking closures. The following result is essentially due to [13, Theorem 5.5], but it has many precursors in the literature, see the discussion below.

Theorem 4.1. *Under the above assumptions, for any $\lambda \in \mathbb{R} \setminus \sigma(M)$ one has*

$$\Xi(\lambda; \widetilde{M}, M) = \Xi(0; -J^{-1} - T(\lambda), -J^{-1}). \quad (4.1)$$

Remark. 1. The importance of this formula in this context is that the variational estimates for Ξ from Section 2 can now be applied to the r.h.s. of (4.1).

2. If $J^{-1} + T(\lambda)$ is invertible, then the identity (4.1) can be rewritten as

$$\Xi(\lambda; \widetilde{M}, M) = -\Xi(0; J^{-1} + T(\lambda), J^{-1}). \quad (4.2)$$

3. Suppose $\lambda < \inf \sigma(M)$. Then the identity (4.1) can be rewritten as the usual Birman-Schwinger principle:

$$N((-\infty, \lambda); \widetilde{M}) = N((-\infty, -1); X^*JX), \quad X = G(M - \lambda I)^{-1/2} \in \mathfrak{S}_\infty, \quad (4.3)$$

see [6, 30]. Indeed, by (1.8) the l.h.s. of (4.3) coincides with minus the l.h.s. of (4.1). In order to see that the r.h.s. of (4.3) coincides with minus the r.h.s. of (4.1), let us apply the identity (4.1) to the r.h.s. of itself with $\lambda = 0$, $M = -J^{-1}$, $A = -T(\lambda)$, $J = -I$, and $G = X^*$:

$$\begin{aligned} \Xi(0; (-J^{-1}) + X(-I)X^*, (-J^{-1})) &= \Xi(0; I - X^*(-J^{-1})^{-1}X, I) \\ &= -N((-\infty, -1); X^*JX). \end{aligned}$$

4. If $J = I$ or $J = -I$, then (4.1) can be rewritten in the following simpler form:

$$\Xi(\lambda; M + G^*G, M) = N((-\infty, -1]; T(\lambda)), \quad (4.4)$$

$$\Xi(\lambda; M - G^*G, M) = -N((1, \infty); T(\lambda)). \quad (4.5)$$

These identities are essentially due to [29, Theorem 3.5], where they were stated in the framework of the theory of the spectral shift function (cf. (2.5)). In particular, if $\lambda < \inf \sigma(M)$, then (4.5) becomes

$$N((-\infty, \lambda); M - G^*G) = N((1, \infty); T(\lambda)). \quad (4.6)$$

This is perhaps the simplest and the best known case of the Birman-Schwinger principle.

5. Setting $\widetilde{T}(\lambda) = G(\widetilde{M} - \lambda I)^{-1}G^*$, by symmetry between M and \widetilde{M} we also obtain

$$\Xi(\lambda; \widetilde{M}, M) = -\Xi(0; -J^{-1} - \widetilde{T}(\lambda), -J^{-1})$$

for any $\lambda \in \mathbb{R} \setminus \sigma(\widetilde{M})$.

6. If G is a Hilbert-Schmidt class operator (and so A is trace class), then (4.1) reduces to a representation for the spectral shift function from [13].
7. For a discussion of the Birman-Schwinger principle in the context of operator algebras, see [19] and references therein.
8. In a forthcoming publication [21], in the scattering theory framework we will also prove an analogue of (4.1) on the continuous spectrum.

It is not difficult to prove Theorem 4.1 by using the above mentioned result from [13] and an approximation argument. However, [13] uses some highly non-trivial constructions from the theory of the spectral shift function. For this reason, below we give an alternative, perhaps more direct proof.

The following corollary is not used in this paper but might be useful elsewhere.

Corollary 4.2. *Suppose that under the assumptions of Theorem 4.1, one has $[\lambda_1, \lambda_2] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(M)$ and $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \sigma(M)$. Then*

$$N([\lambda_1, \lambda_2]; \widetilde{M}) - N([\lambda_1, \lambda_2]; M) = \Xi(0; -J^{-1} - T(\lambda_1), -J^{-1} - T(\lambda_2)).$$

Proof. By (1.7), Theorem 4.1, and the “chain rule” (1.4), one has

$$\begin{aligned} N([\lambda_1, \lambda_2]; \widetilde{M}) - N([\lambda_1, \lambda_2]; M) &= \Xi(\lambda_1; \widetilde{M}, M) - \Xi(\lambda_2; \widetilde{M}, M) \\ &= \Xi(0; -J^{-1} - T(\lambda_1), -J^{-1}) - \Xi(0; -J^{-1} - T(\lambda_2), -J^{-1}) \\ &= \Xi(0; -J^{-1} - T(\lambda_1), -J^{-1} - T(\lambda_2)), \end{aligned}$$

as required. ■

4.2. Two lemmas

Here we prove two lemmas which are used in our proof of Theorem 4.1; they might also be of independent interest.

Lemma 4.3. *Let $M = M^*$ be a bounded operator which has a bounded inverse. Let X be a bounded operator which has a bounded inverse and suppose that $X - I$ is compact. Then $\Xi(0; XMX^*, M)$ exists and equals zero.*

Proof. By assumptions, $XMX^* - M$ is compact; it follows that $\Xi(0; XMX^*, M)$ exists.

Next, since the operator $X - I$ is compact, one can find a continuous function $f : [0, 1] \rightarrow \mathbb{C}$, $f(0) = 0$, $f(1) = 1$, such that $X_t = I + f(t)(X - I)$ is invertible for all $t \in [0, 1]$. Then the family X_t , $t \in [0, 1]$ is continuous in the operator norm and satisfies conditions $X_0 = I$, $X_1 = X$, $X_t - I$ is compact and X_t is invertible for all t .

Let $M_t = X_t M X_t^*$. Then M_t depends continuously on t in the operator norm, M_t is invertible for all t , and $M_t - M$ is compact for all t . By the stability Theorem 2.3, it follows that $\Xi(0; M_t, M)$ exists for all t , depends continuously on t and therefore is constant. Finally, for $t = 0$ we have $\Xi(0; M_0, M) = \Xi(0; M, M) = 0$. ■

The following well known statement provides some insight into the identity (4.1).

Lemma 4.4. *Under the assumptions of Theorem 4.1, one has*

$$\dim \operatorname{Ker}(\widetilde{M} - \lambda I) = \dim \operatorname{Ker}(J^{-1} + T(\lambda)). \quad (4.7)$$

Proof. Without loss of generality, assume $\lambda = 0$. Denote $M_0 = \operatorname{sign}(M)$ and $Y = G|M|^{-1/2} \in \mathfrak{S}_\infty$. First note that since $\operatorname{Ker} M = \{0\}$, we have

$$\dim \operatorname{Ker} \widetilde{M} = \dim \operatorname{Ker} |M|^{-1/2} \widetilde{M} |M|^{-1/2} = \dim \operatorname{Ker}(M_0 + Y^* J Y). \quad (4.8)$$

Next, recall the well known fact that for any two compact operators K_1, K_2 one has $\dim \operatorname{Ker}(K_1 K_2 + I) = \dim \operatorname{Ker}(K_2 K_1 + I)$. It follows that

$$\dim \operatorname{Ker}(Y^* J Y M_0 + I) = \dim \operatorname{Ker}(Y M_0 Y^* J + I). \quad (4.9)$$

The r.h.s. of (4.9) equals

$$\dim \operatorname{Ker}((Y M_0 Y^* + J^{-1})J) = \dim \operatorname{Ker}(T(0) + J^{-1}).$$

Similarly, the l.h.s. of (4.9) equals $\dim \operatorname{Ker}(M_0 + Y^* J Y)$. Together with (4.8), this proves the claim. ■

4.3. The proof of Theorem 4.1

1. Without the loss of generality, let us assume $\lambda = 0$. First let us prove (4.1) under the additional assumptions that M is bounded and $0 \notin \sigma(\widetilde{M})$. If M is bounded then our original assumption $G(M + \gamma I)^{-1/2} \in \mathfrak{S}_\infty$ simply means that G is compact.

Consider the following bounded operator in $\mathcal{H} \oplus \mathcal{K}$:

$$X = \begin{pmatrix} I & -G^* J \\ GM^{-1} & I \end{pmatrix}.$$

It is straightforward to see that $X - I$ is compact. One can directly verify the identity

$$X \begin{pmatrix} M & 0 \\ 0 & J^{-1} \end{pmatrix} X^* = \begin{pmatrix} \widetilde{M} & 0 \\ 0 & J^{-1} + T(0) \end{pmatrix}. \quad (4.10)$$

By our assumption $0 \notin \sigma(\widetilde{M})$ and Lemma 4.4, the operator on the r.h.s. has a bounded inverse. From here and the compactness of $X - I$ it follows that X has a bounded inverse.

Next, from (4.10) and Lemma 4.3 it follows that

$$\Xi(0; \widetilde{M}, M) + \Xi(0; J^{-1} + T(0), J^{-1}) = 0.$$

This is the same as (4.2).

2. While still assuming that $0 \notin \sigma(\widetilde{M})$, let us lift the assumption of boundedness of M . Let M be unbounded, and let $P_n = E_M(-n, n)$, $G_n = GP_n \in \mathfrak{S}_\infty$, $A_n = G_n^* J G_n$, $T_n(z) = G_n(M - zI)^{-1} G_n^*$. In what follows, we will prove that

$$\|T_n(z) - T(z)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \forall z \notin \sigma(M), \quad (4.11)$$

$$M + A_n \rightarrow \widetilde{M} \quad \text{as } n \rightarrow \infty \text{ in the norm resolvent sense.} \quad (4.12)$$

Since \widetilde{M} and (by Lemma 4.4) $J^{-1} + T(0)$ are invertible, it follows from (4.11), (4.12), in particular, that $M + A_n$ and $J^{-1} + T_n(0)$ are invertible for all sufficiently large n .

Next, note that the orthogonal decomposition $\mathcal{H} = \text{Ran } P_n \oplus \text{Ran}(I - P_n)$ reduces both M and $M + A_n$. The components of both M and $M + A_n$ in $\text{Ran } P_n$ are bounded. The components of M and $M + A_n$ in $\text{Ran}(I - P_n)$ coincide. By the first step of the proof, it follows that

$$\Xi(0; M + A_n, M) = -\Xi(0; J^{-1} + T_n(0), J^{-1}) \quad (4.13)$$

for all sufficiently large n . Now by (4.11) and (4.12) and the stability Theorem 2.3, we can pass to the limit as $n \rightarrow \infty$ in (4.13), which yields (4.2). Note that it is easy to see that the uniform lower bound assumption from Theorem 2.3 is satisfied in our case.

3. Let us prove the convergence (4.11) and (4.12). First note that since $G(M + \gamma I)^{-1/2}$ is compact and $P_n \rightarrow I$ strongly as $n \rightarrow \infty$, we obtain

$$\|G_n(M + \gamma I)^{-1/2} - G(M + \gamma I)^{-1/2}\| \rightarrow 0, \quad n \rightarrow \infty, \quad (4.14)$$

$$\|G_n(M \pm i)^{-1} - G(M \pm i)^{-1}\| \rightarrow 0, \quad n \rightarrow \infty. \quad (4.15)$$

Writing

$$T(z) = [G(M + \gamma I)^{-1/2}](M + \gamma I)(M - zI)^{-1}[G(M + \gamma I)^{-1/2}]^*,$$

$$T_n(z) = [G_n(M + \gamma I)^{-1/2}](M + \gamma I)(M - zI)^{-1}[G_n(M + \gamma I)^{-1/2}]^*$$

and using (4.14), we obtain (4.11). In order to prove the convergence (4.12), we use the iterated resolvent identities in the form

$$(\widetilde{M} - i)^{-1} = (M - i)^{-1} - [G(M + i)^{-1}]^*(J^{-1} + T(i))^{-1}[G(M - i)^{-1}], \quad (4.16)$$

and similarly

$$(M + A_n - i)^{-1} = (M - i)^{-1} - [G_n(M + i)^{-1}]^*(J^{-1} + T_n(i))^{-1}[G_n(M - i)^{-1}]. \quad (4.17)$$

Subtracting and using (4.11) and (4.15), we obtain (4.12).

4. It remains to lift the assumption $0 \notin \sigma(\widetilde{M})$. Suppose $0 \in \mathbb{R} \setminus \sigma(M)$ and $0 \in \sigma(\widetilde{M})$; then 0 is an isolated eigenvalue of \widetilde{M} . It suffices to prove that both sides of (4.1) are left continuous in λ at $\lambda = 0$. For the l.h.s. of (4.1), this is true directly by the definition of Ξ . Let us consider the r.h.s.

We claim that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\Xi(0; -J^{-1} - T(\lambda), -J^{-1}) = \Xi(-\delta; -J^{-1} - T(\lambda), -J^{-1}), \quad \forall \lambda \in [-\varepsilon, 0]. \quad (4.18)$$

Indeed, it is easy to see that $T(\lambda)$ is continuous in λ in the operator norm at $\lambda = 0$ and $T(\lambda) \leq T(0)$ for small $\lambda \leq 0$. Since $T(\lambda)$ is compact, it follows that for some $\delta > 0$ and all sufficiently small $\lambda \leq 0$, one has

$$[-\delta, 0] \cap \sigma(-J^{-1} - T(\lambda)) = \emptyset \quad \text{and} \quad [-\delta, 0] \cap \sigma(-J^{-1}) = \emptyset.$$

Then (4.18) follows.

Now by the stability Theorem 2.3, we have

$$\Xi(-\delta; -J^{-1} - T(\lambda), -J^{-1}) \rightarrow \Xi(-\delta; -J^{-1} - T(0), -J^{-1})$$

as $\lambda \rightarrow 0$. It follows that the r.h.s. of (4.1) is left continuous in λ at $\lambda = 0$, as required. ■

5. Safronov's theorem

5.1. The key estimates

Here we state and prove a result (see (5.10), (5.11) below) which is a slight generalisation of [27] (see also [26]). Let H_0 be a lower semi-bounded self-adjoint operator in \mathcal{H} ; choose $\gamma \in \mathbb{R}$ such that $H_0 + \gamma I \geq I$. Let $V_+ \geq 0$ and $V_- \geq 0$ be self-adjoint operators in \mathcal{H} which are form-compact with respect to H_0 . Then for any $t \in \mathbb{R}$ the operators $H_0 + tV_+ - tV_-$ are well defined in terms of the corresponding quadratic forms. Fix $\lambda \in \mathbb{R} \setminus \sigma(H_0)$; our aim is to consider

$$\Xi(\lambda; H_0 + tV_+ - tV_-, H_0) \quad \text{for } t \rightarrow \infty. \quad (5.1)$$

Let us define the auxiliary compact operators $T_{\alpha\beta}$, $\alpha, \beta \in \{+, -\}$, by setting

$$T_{\alpha\beta}f = \sqrt{V_\alpha}(H_0 - \lambda I)^{-1}\sqrt{V_\beta}f, \quad f \in \text{Dom}(H_0 + \gamma I)^{1/2}$$

and taking closures. Clearly, T_{++} and T_{--} are self-adjoint and $T_{+-}^* = T_{-+}$. We note that, by a well-known identity,

$$N((a, \infty); T_{-+}T_{+-}) = N((a, \infty); T_{+-}T_{-+}) \quad (5.2)$$

for any $a > 0$. The key fact is the following

Theorem 5.1. *For any $a \in (0, 1)$, one has*

$$\begin{aligned} \Xi(\lambda; H_0 + V_+ - V_-, H_0) &\leq \Xi(\lambda; H_0 + \frac{1}{1-a}V_+, H_0) \\ &\quad + \Xi(\lambda; H_0 - \frac{1}{1+a}V_-, H_0) + N((a^2, \infty); T_{-+}T_{+-}), \end{aligned} \quad (5.3)$$

$$\begin{aligned} \Xi(\lambda; H_0 + V_+ - V_-, H_0) &\geq \Xi(\lambda; H_0 + \frac{1}{1+a}V_+, H_0) \\ &\quad + \Xi(\lambda; H_0 - \frac{1}{1-a}V_-, H_0) - N((a^2, \infty); T_{-+}T_{+-}). \end{aligned} \quad (5.4)$$

Proof. 1. Writing

$$\frac{1}{1-a}V_+ = \sqrt{V_+}(1-a)^{-1}\sqrt{V_+}, \quad -\frac{1}{1+a}V_- = \sqrt{V_-}(-1-a)^{-1}\sqrt{V_-},$$

and using the generalised Birman-Schwinger principle (4.1), we get

$$\begin{aligned} \Xi(\lambda; H_0 + \frac{1}{1-a}V_+, H_0) \\ = \Xi(0; -I + aI - T_{++}, -I + aI) = \Xi(-a; -I - T_{++}, -I), \end{aligned} \quad (5.5)$$

$$\Xi(\lambda; H_0 - \frac{1}{1+a}V_-, H_0) = \Xi(0; I + aI - T_{--}, I + aI) = \Xi(-a; I - T_{--}, I). \quad (5.6)$$

2. Let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ and let $G : \mathcal{H} \rightarrow \mathcal{K}$ be the closure of the operator defined by

$$Gf = (\sqrt{V_+}f, \sqrt{V_-}f), \quad f \in \text{Dom}(H_0 + \gamma I)^{1/2}.$$

Let $J : \mathcal{K} \rightarrow \mathcal{K}$ be the operator $(f_+, f_-) \mapsto (f_+, -f_-)$. Then one has

$$\|\sqrt{V_+}f\|_{\mathcal{H}}^2 - \|\sqrt{V_-}f\|_{\mathcal{H}}^2 = (JGf, Gf)_{\mathcal{K}}, \quad f \in \text{Dom}(H_0 + \gamma I)^{1/2},$$

and so $H_0 + V_+ - V_- = H_0 + G^*JG$. Let us define a compact operator \mathbb{T} in \mathcal{K} by setting

$$\mathbb{T}f = G(H_0 - \lambda I)^{-1}G^*f, \quad f \in \text{Dom}(H_0 + \gamma I)^{1/2},$$

and taking closures. An application of the generalised Birman-Schwinger principle (4.1) yields

$$\Xi(\lambda; H_0 + V_+ - V_-, H_0) = \Xi(0; -J - \mathbb{T}, -J); \quad (5.7)$$

note that here $J^{-1} = J$.

3. Let $P_{\pm} : \mathcal{K} \rightarrow \mathcal{K}$ be the orthogonal projections,

$$P_+ : (f_+, f_-) \mapsto (f_+, 0), \quad P_- : (f_+, f_-) \mapsto (0, f_-),$$

and let $\mathbb{T}_{\alpha\beta} = P_\alpha \mathbb{T} P_\beta$, $\alpha, \beta \in \{-, +\}$. Applying Corollary 3.3 followed by (3.3) and then using (5.5), (5.6), we get

$$\begin{aligned} \Xi(0; -J - \mathbb{T}, -J) &\leq \Xi(-a; -J - \mathbb{T}_{++} - \mathbb{T}_{--}, -J) + N((a, \infty); -\mathbb{T}_{+-} - \mathbb{T}_{-+}) \\ &= \Xi(-a; -I - T_{++}, -I) + \Xi(-a; I - T_{--}, I) + N((a, \infty); -\mathbb{T}_{+-} - \mathbb{T}_{-+}) \\ &= \Xi(\lambda; H_0 + \frac{1}{1-a} V_+, H_0) + \Xi(\lambda; H_0 - \frac{1}{1+a} V_-, H_0) + N((a, \infty); -\mathbb{T}_{+-} - \mathbb{T}_{-+}). \end{aligned} \quad (5.8)$$

4. By a direct calculation, $\mathbb{T}_{+-} + \mathbb{T}_{-+} = -J(\mathbb{T}_{+-} + \mathbb{T}_{-+})J$ and therefore

$$N((a, \infty); \mathbb{T}_{+-} + \mathbb{T}_{-+}) = N((a, \infty); -\mathbb{T}_{+-} - \mathbb{T}_{-+}).$$

Using this fact and (5.2), we get

$$\begin{aligned} 2N((a, \infty); \mathbb{T}_{+-} + \mathbb{T}_{-+}) &= N((a^2, \infty); (\mathbb{T}_{+-} + \mathbb{T}_{-+})^2) \\ &= N((a^2, \infty); T_{-+} T_{+-}) + N((a^2, \infty); T_{+-} T_{-+}) = 2N((a^2, \infty); T_{-+} T_{+-}). \end{aligned} \quad (5.9)$$

Combining (5.7), (5.8), (5.9), we obtain the upper bound (5.3). The lower bound (5.4) is proven in the same manner. ■

5.2. Applications

From Theorem 5.1 one easily obtains the main result of [27]. In [27], the asymptotics (5.1) was studied under the assumption that

$$\limsup_{t \rightarrow \infty} t^{-p} N((1, \infty); t^2 T_{-+} T_{+-}) = 0$$

for some exponent $p > 0$. Combining Theorem 5.1 with this assumption, we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+ - tV_-, H_0) &\leq \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + \frac{t}{1-a} V_+, H_0) \\ &+ \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 - \frac{t}{1+a} V_-, H_0) + \limsup_{t \rightarrow \infty} t^{-p} N((1, \infty); \frac{t^2}{a^2} T_{-+} T_{+-}) \\ &= (1-a)^{-p} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+, H_0) \\ &+ (1+a)^{-p} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 - tV_-, H_0). \end{aligned}$$

Letting $a \rightarrow 0$, we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+ - tV_-, H_0) &\leq \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+, H_0) \\ &+ \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 - tV_-, H_0), \end{aligned} \quad (5.10)$$

and in the same way

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+ - tV_-, H_0) &\geq \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + tV_+, H_0) \\ &+ \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 - tV_-, H_0). \end{aligned} \quad (5.11)$$

The estimates (5.10), (5.11) were obtained in [27] (see also [26]) by a different method. These estimates were then applied in [26, 27] to various cases when H_0 is a differential operator and V is the operator of multiplication by a function from an appropriate L^q class. Let us quote a typical application:

$$\lim_{t \rightarrow \infty} t^{-d/2} \Xi(\lambda; -\Delta + V_0 + tV, -\Delta + V_0) = -(2\pi)^{-d} \omega_d \int_{\{x: V(x) \leq 0\}} |V(x)|^{d/2} dx,$$

where $d \geq 3$, $V \in L^{d/2}(\mathbb{R}^d)$, V_0 is a bounded potential in \mathbb{R}^d , λ is in a gap of the spectrum of $-\Delta + V_0$ and ω_d is the volume of a unit ball in \mathbb{R}^d . Here the first terms in the r.h.s. of (5.10), (5.11) vanish and the second terms coincide and are independent of λ . The analysis of the second terms uses the Birman-Schwinger principle in the form (4.5) and the technique of [7].

We note that in [26, 27] the results are stated in terms of the spectral flow of the family $H_0 + tV_+ - tV_-$, see (2.12). We also note that arguments similar to the ones of the proof of Theorem 5.1 were used in [28] in the analysis of the asymptotics of the spectral shift function.

6. An abstract theorem

Here we state and prove a theorem which will be used in the next section in application to the study of the eigenvalues of the perturbed Landau Hamiltonian. This theorem (or the method of its proof) might also be useful in applications to the perturbed periodic operator.

6.1. The statement of the Theorem

Let H_0 be a lower semi-bounded self-adjoint operator in a Hilbert space \mathcal{H} . Let P_0, P_1, P_2, \dots be a sequence of orthogonal projections in \mathcal{H} such that $P_n P_m = 0$ for all $n \neq m$ and $\sum_{n=0}^{\infty} P_n = I$. Assume also that the orthogonal decomposition $\mathcal{H} = \bigoplus_{n=0}^{\infty} \text{Ran } P_n$ reduces H_0 and

$$\inf \sigma(H_0|_{\text{Ran } P_n}) \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (6.1)$$

Let V_t , $t > 0$ be a family of self-adjoint operators such that for all $t > 0$, V_t is form-compact with respect to H_0 . Below we consider the asymptotics of

$$\Xi(\lambda; H_0 + V_t, H_0), \quad \lambda \notin \sigma(H_0), \quad \text{as } t \rightarrow \infty. \quad (6.2)$$

The perturbation V_t can be regarded as a ‘‘matrix’’ $\{P_n V_t P_m\}$ with respect to the orthogonal sum decomposition $\mathcal{H} = \bigoplus_{n=0}^{\infty} \text{Ran } P_n$. Under the appropriate assumptions, below we prove that, roughly speaking, only the diagonal terms of this ‘‘matrix’’ contribute to the asymptotics (6.2).

For some exponent $p > 0$, we assume

$$\limsup_{t \rightarrow \infty} t^{-p} N((a, \infty); \pm(P_n |V_t| P_m + P_m |V_t| P_n)) = 0 \quad \text{for any } a > 0 \text{ and } n \neq m, \quad (6.3)$$

$$\limsup_{t \rightarrow \infty} t^{-p} N((a, \infty); \pm(P_n V_t P_m + P_m V_t P_n)) = 0 \quad \text{for any } a > 0 \text{ and } n \neq m, \quad (6.4)$$

$$\lim_{E \rightarrow \infty} \limsup_{t \rightarrow \infty} t^{-p} N((-\infty, -E); H_0 - s|V_t|) = 0 \quad \text{for any } s > 0. \quad (6.5)$$

Theorem 6.1. *Let the above assumptions hold true. Then for any $\lambda \in \mathbb{R} \setminus \sigma(H_0)$, any $\varepsilon > 0$ and any $a > 0$ sufficiently small such that $[\lambda - a, \lambda + a] \cap \sigma(H_0) = \emptyset$, one has*

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + V_t, H_0) \\ \leq \sum_{n=0}^{\infty} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda - a; H_0 + P_n(V_t + \varepsilon|V_t|)P_n, H_0), \end{aligned} \quad (6.6)$$

and

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + V_t, H_0) \\ \geq \sum_{n=0}^{\infty} \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda + a; H_0 + P_n(V_t - \varepsilon|V_t|)P_n, H_0). \end{aligned} \quad (6.7)$$

Remark. 1. If n is sufficiently large so that $\lambda - a < \inf \sigma(H_0|_{\text{Ran } P_n})$, then, using the orthogonal decomposition $\mathcal{H} = \text{Ran } P_n \oplus \text{Ran}(I - P_n)$ and (1.8), we easily obtain

$$\Xi(\lambda - a; H_0 + P_n(V_t \pm \varepsilon|V_t|)P_n, H_0) \leq 0. \quad (6.8)$$

Thus, all terms in the series (6.6), (6.7) with sufficiently large n are non-positive.

2. Assumption (6.5) is used only in the proof of (6.7).

6.2. Proof of Theorem 6.1

For $r \geq 1$, denote $P^{(r)} = \sum_{n=0}^r P_n$, $Q^{(r)} = I - P^{(r)}$.

Upper bound: 1. Using the ‘‘diagonalisation trick’’ (Theorem 3.4), we obtain

$$\begin{aligned} \Xi(\lambda; H_0 + V_t, H_0) \\ \leq \Xi(\lambda; H_0 + P^{(r)}(V_t + \varepsilon|V_t|)P^{(r)}, H_0) + \Xi(\lambda; H_0 + Q^{(r)}(V_t + \frac{1}{\varepsilon}|V_t|)Q^{(r)}, H_0) \end{aligned} \quad (6.9)$$

for any r . Next, as in (6.8), we see that if r is sufficiently large, then

$$\Xi(\lambda; H_0 + Q^{(r)}(V_t + \frac{1}{\varepsilon}|V_t|)Q^{(r)}, H_0) \leq 0.$$

Thus, from (6.9) we obtain the estimate

$$\Xi(\lambda; H_0 + V_t, H_0) \leq \Xi(\lambda; H_0 + P^{(r)}(V_t + \varepsilon|V_t|)P^{(r)}, H_0). \quad (6.10)$$

2. In what follows, we use the iterated Weyl's inequality (see e.g. [9, Section 11.1]) for the eigenvalues of compact selfadjoint operators K_1, K_2, \dots, K_ℓ :

$$N((a, \infty); K_1 + \dots + K_\ell) \leq \sum_{j=1}^{\ell} N((a/\ell, \infty); K_j). \quad (6.11)$$

Let us write $W_t = V_t + \varepsilon|V_t|$ and

$$P^{(r)}W_tP^{(r)} = W_t^{(diag)} + W_t^{(off)},$$

where

$$W_t^{(diag)} = \sum_{n \leq r} P_n W_t P_n, \quad W_t^{(off)} = \sum_{\substack{n \neq m \\ n, m \leq r}} P_n W_t P_m.$$

By (6.11), we get for any $a > 0$:

$$\begin{aligned} N((a, \infty); W_t^{(off)}) &\leq \sum_{n < m \leq r} N\left(\left(\frac{a}{r(r-1)}, \infty\right); P_n V_t P_m + P_m V_t P_n\right) \\ &\quad + \sum_{n < m \leq r} N\left(\left(\frac{a}{\varepsilon r(r-1)}, \infty\right); P_n |V_t| P_m + P_m |V_t| P_n\right). \end{aligned}$$

From here, using (6.3), (6.4), we get

$$\limsup_{t \rightarrow \infty} t^{-p} N((a, \infty); W_t^{(off)}) = 0. \quad (6.12)$$

3. Using Corollary 3.3 and (3.3), for any sufficiently small $a > 0$ we obtain

$$\begin{aligned} \Xi(\lambda; H_0 + P^{(r)}(V_t + \varepsilon|V_t|)P^{(r)}, H_0) \\ \leq \Xi(\lambda - a; H_0 + W_t^{(diag)}, H_0) + N((a, \infty), W_t^{(off)}) \\ = \sum_{n \leq r} \Xi(\lambda - a; H_0 + P_n W_t P_n, H_0) + N((a, \infty), W_t^{(off)}). \end{aligned}$$

Using (6.12), this yields

$$\limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + V_t, H_0) \leq \sum_{n \leq r} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda - a; H_0 + P_n (V_t + \varepsilon|V_t|) P_n, H_0).$$

Since r can be chosen arbitrarily large, we get the upper bound (6.6).

Lower bound: As in (6.9), we get

$$\begin{aligned} \Xi(\lambda; H_0 + V_t, H_0) &\geq \Xi(\lambda; H_0 + P^{(r)}(V_t - \varepsilon|V_t|)P^{(r)}, H_0) \\ &\quad + \Xi(\lambda; H_0 + Q^{(r)}(V_t - \frac{1}{\varepsilon}|V_t|)Q^{(r)}, H_0). \quad (6.13) \end{aligned}$$

Consider the two terms in the r.h.s. of (6.13). For the first term, as in the proof of the upper bound, we get

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_0 + P^{(r)}(V_t - \varepsilon|V_t|)P^{(r)}, H_0) \\ \geq \sum_{n \leq r} \liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda + a; H_0 + P_n(V_t - \varepsilon|V_t|)P_n, H_0). \end{aligned} \quad (6.14)$$

Consider the second term. Denote $H_0^{(r)} = H_0|_{\text{Ran } Q^{(r)}}$, $V_t^{(r)} = Q^{(r)}V_tQ^{(r)}|_{\text{Ran } Q^{(r)}}$, $|V_t|^{(r)} = Q^{(r)}|V_t|Q^{(r)}|_{\text{Ran } Q^{(r)}}$, $\Lambda_r = \inf \sigma(H_0^{(r)})$. By the assumption (6.1), we have $\Lambda_r \rightarrow \infty$ as $r \rightarrow \infty$. If r is sufficiently large so that $\lambda < \Lambda_r$, then (similarly to (6.8)),

$$\Xi(\lambda; H_0 + Q^{(r)}(V_t - \frac{1}{\varepsilon}|V_t|)Q^{(r)}, H_0) = -N((-\infty, \lambda); H_0^{(r)} + V_t^{(r)} - \frac{1}{\varepsilon}|V_t|^{(r)}).$$

Next, by variational considerations, we have

$$\begin{aligned} N((-\infty, \lambda); H_0^{(r)} + V_t^{(r)} - \frac{1}{\varepsilon}|V_t|^{(r)}) &\leq N((-\infty, \lambda); H_0^{(r)} - \frac{1+\varepsilon}{\varepsilon}|V_t|^{(r)}) \\ &= N((-\infty, 2\lambda); 2H_0^{(r)} - 2\frac{1+\varepsilon}{\varepsilon}|V_t|^{(r)}) \leq N((-\infty, 2\lambda); H_0^{(r)} + \Lambda_r - 2\frac{1+\varepsilon}{\varepsilon}|V_t|^{(r)}) \\ &\leq N((-\infty, 2\lambda - \Lambda_r); H_0^{(r)} - 2\frac{1+\varepsilon}{\varepsilon}|V_t|^{(r)}). \end{aligned}$$

From here, by assumption (6.5), we obtain

$$\lim_{r \rightarrow \infty} \limsup_{t \rightarrow \infty} t^{-p} |\Xi(\lambda; H_0 + Q^{(r)}(V_t - \frac{1}{\varepsilon}|V_t|)Q^{(r)}, H_0)| = 0. \quad (6.15)$$

Combining (6.13), (6.14), and (6.15), we obtain the lower bound (6.7). ■

7. A theorem of Rozenblum and Sobolev

7.1. Introduction

In $L^2(\mathbb{R}^2, dx_1 dx_2)$, consider the Landau operator

$$H_0 = \left(-i \frac{\partial}{\partial x_1} + \frac{1}{2} B x_1 \right)^2 + \left(-i \frac{\partial}{\partial x_2} - \frac{1}{2} B x_2 \right)^2, \quad B > 0.$$

It is well known that the spectrum of H_0 consists of a sequence of infinitely degenerate eigenvalues (Landau levels) $\Lambda_n = B(2n+1)$, $n = 0, 1, \dots$; we set $\Lambda_{-1} = -\infty$ for notational convenience. We denote by P_n the orthogonal projection onto the eigenspace $\text{Ker}(H_0 - \Lambda_n I)$.

Let $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ be a real valued function. It is easy to see that the operator of multiplication by V is form-compact with respect to H_0 . For $\alpha > 0$ and $\beta > 0$, we consider the spectral asymptotics of the operator

$$H_t = H_0 + V_t, \quad V_t(x_1, x_2) = V(x_1 t^{-\alpha}, x_2 t^{-\beta})$$

as $t \rightarrow \infty$. We use the notation $p = \alpha + \beta$ and

$$A(a, V) = \frac{B}{2\pi} \text{meas}\{x \in \mathbb{R}^2 \mid V(x) > a\}, \quad A[a, V] = \frac{B}{2\pi} \text{meas}\{x \in \mathbb{R}^2 \mid V(x) \geq a\},$$

where $a > 0$ and meas is the Lebesgue measure in \mathbb{R}^2 . Our main result in this section is

Theorem 7.1. *Under the above assumptions, for any $q \geq -1$ and any $\lambda \in (\Lambda_q, \Lambda_{q+1})$, one has*

$$\limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_t, H_0) \leq \sum_{0 \leq n \leq q} A[\lambda - \Lambda_n, V] - \sum_{n \geq q+1} A(\Lambda_n - \lambda, -V), \quad (7.1)$$

$$\liminf_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_t, H_0) \geq \sum_{0 \leq n \leq q} A(\lambda - \Lambda_n, V) - \sum_{n \geq q+1} A[\Lambda_n - \lambda, -V]. \quad (7.2)$$

Further, for any interval $[\lambda_1, \lambda_2] \subset \mathbb{R} \setminus \sigma(H_0)$, one has

$$\limsup_{t \rightarrow \infty} t^{-p} N([\lambda_1, \lambda_2]; H_t) \leq \frac{B}{2\pi} \sum_{n \geq 0} \text{meas}\{x \in \mathbb{R}^2 \mid \lambda_1 - \Lambda_n \leq V(x) \leq \lambda_2 - \Lambda_n\}, \quad (7.3)$$

$$\liminf_{t \rightarrow \infty} t^{-p} N([\lambda_1, \lambda_2]; H_t) \geq \frac{B}{2\pi} \sum_{n \geq 0} \text{meas}\{x \in \mathbb{R}^2 \mid \lambda_1 - \Lambda_n < V(x) < \lambda_2 - \Lambda_n\}. \quad (7.4)$$

Note that the r.h.s. of (7.1)–(7.4) are independent of α and β .

In [23], the asymptotic estimates (7.3), (7.4) were proven for $\alpha = \beta = 1$. Our construction is somewhat more direct than the one of [23]. The operator theoretic component of our construction is Theorem 6.1. The other component is the analysis of the spectral asymptotics of the operators $P_n V_t P_m$ in Section 7.2.

7.2. Spectral asymptotics of the operators $P_n V_t P_m$

Here we prove

Proposition 7.2. *For any $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and any $a > 0$, one has*

$$\limsup_{t \rightarrow \infty} t^{-p} N([a, \infty); P_n V_t P_n) \leq A[a, V], \quad \forall n \geq 0, \quad (7.5)$$

$$\liminf_{t \rightarrow \infty} t^{-p} N((a, \infty); P_n V_t P_n) \geq A(a, V), \quad \forall n \geq 0, \quad (7.6)$$

$$\limsup_{t \rightarrow \infty} t^{-p} N((a, \infty); P_m V_t P_n + P_n V_t P_m) = 0, \quad \forall n \neq m. \quad (7.7)$$

The proof of this Proposition follows an unpublished remark by A. Laptev and Yu. Safarov and their earlier work [20]. This Proposition was also proved in [23] by a different method.

The analysis below uses the well known explicit formula for the integral kernel of P_n :

$$P_n(x, y) = \frac{B}{2\pi} L_n\left(\frac{B|x-y|^2}{2}\right) \exp\left(-\frac{B}{4}(|x-y|^2 + 2i[x, y])\right), \quad (7.8)$$

where L_n is the Laguerre polynomial and $[x, y] = x_1y_2 - x_2y_1$.

Lemma 7.3. *For any $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and any $n = 0, 1, 2, \dots$, one has*

$$\|P_n V_t - V_t P_n\|_{\mathfrak{S}_2}^2 = o(t^p), \quad t \rightarrow \infty.$$

Proof. The integral kernel of $P_n V_t - V_t P_n$ is $P_n(x, y)(V_t(y) - V_t(x))$. Using this fact, formula (7.8), and the obvious estimate

$$|L_n(s)|e^{-s/4} \leq C = C(n), \quad s \geq 0$$

for the Laguerre polynomial, we obtain

$$\begin{aligned} \|P_n V_t - V_t P_n\|_{\mathfrak{S}_2}^2 &\leq \left(\frac{B}{2\pi}\right)^2 C^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dx dy |V_t(x) - V_t(y)|^2 \exp(-\frac{B}{4}|x-y|^2) \\ &= \left(\frac{B}{2\pi}\right)^2 C^2 t^{2p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dx dy |V(x) - V(y)|^2 \exp(-\frac{B}{4}t^{2\alpha}(x_1-y_1)^2 - \frac{B}{4}t^{2\beta}(x_2-y_2)^2) \\ &= t^{2p} \int_{\mathbb{R}^2} dz f(z) \exp(-\frac{B}{4}t^{2\alpha}z_1^2 - \frac{B}{4}t^{2\beta}z_2^2), \end{aligned} \quad (7.9)$$

where

$$f(z) = \left(\frac{B}{2\pi}\right)^2 C^2 \int dx |V(x) - V(x+z)|^2.$$

It is easy to see that $f(z)$ is continuous in z , $f(0) = 0$, and

$$f(z) \leq 4 \left(\frac{B}{2\pi}\right)^2 C^2 \int_{\mathbb{R}^2} V(x)^2 dx.$$

Given $\varepsilon > 0$, let us choose $\delta > 0$ such that $|f(z)| \leq \varepsilon$ for $|z| \leq \delta$. Then, splitting the integral in the r.h.s. of (7.9) into the sum of the integrals over $\{z : |z| \leq \delta\}$ and over $\{z : |z| > \delta\}$, we readily obtain the estimate

$$\|P_n V_t - V_t P_n\|_{\mathfrak{S}_2}^2 \leq C_1 \varepsilon t^p + C_2 t^{2p} \int_{|z| > \delta} \exp(-\frac{B}{4}t^{2\alpha}z_1^2 - \frac{B}{4}t^{2\beta}z_2^2) dz.$$

Since $\varepsilon > 0$ is arbitrary and the integral in the r.h.s. tends to zero faster than any power of t as $t \rightarrow \infty$, this yields the required estimate. ■

Lemma 7.4. *Let ϕ be a function from the Sobolev class $W_\infty^2(\mathbb{R})$ (i.e. $\phi'' \in L^\infty(\mathbb{R})$) such that $\phi(0) = 0$. Then*

$$\mathrm{Tr} \phi(P_n V_t P_n) = \frac{B}{2\pi} t^p \int_{\mathbb{R}^2} \phi(V(x)) dx + o(t^p), \quad t \rightarrow \infty.$$

Proof. By [20, Theorem 1.2], we have an estimate

$$|\mathrm{Tr}(P_n \phi(P_n V_t P_n) P_n) - \mathrm{Tr}(P_n \phi(V_t) P_n)| \leq \frac{1}{2} \|\phi''\|_{L^\infty(\mathbb{R})} \|P_n V_t (I - P_n)\|_{\mathfrak{S}_2}^2.$$

This estimate has a general operator theoretic nature and depends only on the facts that P_n is an orthogonal projection, V_t is self-adjoint and $P_n V_t$ is a Hilbert-Schmidt operator. Using formula (7.8) for the integral kernel of P_n and the fact that $L_n(0) = 1$, we get

$$\mathrm{Tr}(P_n \phi(V_t) P_n) = \mathrm{Tr}(P_n \phi(V_t)) = t^p \frac{B}{2\pi} \int_{\mathbb{R}^2} \phi(V(x)) dx.$$

Next, we have

$$\|P_n V_t (I - P_n)\|_{\mathfrak{S}_2}^2 = \|(P_n V_t - V_t P_n)(I - P_n)\|_{\mathfrak{S}_2}^2 \leq \| (P_n V_t - V_t P_n) \|_{\mathfrak{S}_2}^2 = o(t^p)$$

as $t \rightarrow \infty$ by Lemma 7.3. Finally, we note that $\phi(0) = 0$ and so $P_n \phi(P_n V_t P_n) P_n = \phi(P_n V_t P_n)$. Putting this together yields the required asymptotics. ■

Proof of Proposition 7.2. 1. Let us first prove (7.7). One has

$$\begin{aligned} N((a, \infty); P_m V_t P_n + P_n V_t P_m) &= N((1, \infty); \frac{1}{a} P_m V_t P_n + \frac{1}{a} P_n V_t P_m) \\ &\leq \frac{1}{a^2} \|P_m V_t P_n + P_n V_t P_m\|_{\mathfrak{S}_2}^2 \leq \frac{1}{a^2} (\|P_m V_t P_n\|_{\mathfrak{S}_2} + \|P_n V_t P_m\|_{\mathfrak{S}_2})^2 \\ &= \frac{4}{a^2} \|P_n V_t P_m\|_{\mathfrak{S}_2}^2 = \frac{4}{a^2} \|(P_n V_t - V_t P_n) P_m\|_{\mathfrak{S}_2}^2 \leq \frac{4}{a^2} \|P_n V_t - V_t P_n\|_{\mathfrak{S}_2}^2 = o(t^p), \end{aligned}$$

as $t \rightarrow \infty$ by Lemma 7.3.

2. Let us prove (7.5) and (7.6). Let $\phi, \psi \in W_\infty^2(\mathbb{R})$ be such that $\phi(0) = \psi(0) = 0$ and

$$0 \leq \phi(s) \leq \chi_{(a, \infty)}(s) \leq \chi_{[a, \infty)}(s) \leq \psi(s), \quad s \geq 0.$$

Then

$$N((a, \infty), P_n V_t P_n) \geq \mathrm{Tr} \phi(P_n V_t P_n), \quad (7.10)$$

$$N([a, \infty), P_n V_t P_n) \leq \mathrm{Tr} \psi(P_n V_t P_n). \quad (7.11)$$

The asymptotics of the traces in the r.h.s. of (7.10) and (7.11) is given by Lemma 7.4. Choosing appropriate sequences of functions ϕ_n and ψ_n which converge to $\chi_{(a, \infty)}$ pointwise on $\mathbb{R} \setminus \{a\}$, we obtain the required result. ■

7.3. Eigenvalues below the essential spectrum

Lemma 7.5. *One has*

$$\lim_{E \rightarrow \infty} \limsup_{t \rightarrow \infty} t^{-p} N((-\infty, -E); H_t) = 0. \quad (7.12)$$

Proof. By the diamagnetic inequality (see, e.g., [2, Section 2] and references therein), one has the following estimate involving the integral kernels of the resolvents of H_0 and $-\Delta$:

$$|(H_0 + E)^{-1}(x, y)| \leq (-\Delta + E)^{-1}(x, y).$$

It follows that the Hilbert-Schmidt norm of $V(H_0 + E)^{-1}$ can be estimated by the Hilbert-Schmidt norm of $V(-\Delta + E)^{-1}$:

$$\begin{aligned} \|V(H_0 + E)^{-1}\|_{\mathfrak{S}_2}^2 &\leq \|V(-\Delta + E)^{-1}\|_{\mathfrak{S}_2}^2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} |V(x)|^2 dx \int_{\mathbb{R}^2} (p^2 + E)^{-2} dp = \frac{C}{E} \int_{\mathbb{R}^2} |V(x)|^2 dx. \end{aligned} \quad (7.13)$$

Next, recall the following well known estimate. If $L \geq 0$ and $M \geq 0$ are bounded self-adjoint operators such that LM is Hilbert-Schmidt, then

$$\|L^{1/2}ML^{1/2}\|_{\mathfrak{S}_2}^2 = \text{Tr}(L^{1/2}MLM^{1/2}) = \text{Tr}(LMLM) \leq \|LM\|_{\mathfrak{S}_2}^2. \quad (7.14)$$

Using the Birman-Schwinger principle in the form (4.6) and the estimates (7.13), (7.14) (with $L = |V_t|$, $M = (H_0 + E)^{-1}$), we get

$$\begin{aligned} N((-\infty, -E), H_t) &\leq N((-\infty, -E), H_0 - |V_t|) \\ &= N((1, \infty); |V_t|^{1/2}(H_0 + E)^{-1}|V_t|^{1/2}) \leq \| |V_t|^{1/2}(H_0 + E)^{-1}|V_t|^{1/2} \|_{\mathfrak{S}_2}^2 \\ &\leq \| |V_t|(H_0 + E)^{-1} \|_{\mathfrak{S}_2}^2 = \frac{C}{E} \int_{\mathbb{R}^2} |V_t(x)|^2 dx = t^p \frac{C}{E} \int_{\mathbb{R}^2} |V(x)|^2 dx, \end{aligned}$$

which yields the required result. ■

7.4. Proof of Theorem 7.1

1. We first note the following elementary continuity properties of the asymptotic coefficients $A(a, V)$, $A[a, V]$ for any $a > 0$:

$$A(a + 0, V) = A(a, V), \quad A[a - 0, V] = A[a, V], \quad (7.15)$$

$$\lim_{\varepsilon \rightarrow +0} A(a; V - \varepsilon|V|) = A(a; V), \quad \lim_{\varepsilon \rightarrow +0} A[a; V + \varepsilon|V|] = A[a; V]. \quad (7.16)$$

2. In order to prove (7.1), (7.2), let us apply Theorem 6.1. Assumption (6.1) is clearly fulfilled; (6.3), (6.4) hold true by (7.7) and (6.5) holds true by Lemma 7.5. Thus, Theorem 6.1 is applicable.

Let us prove the upper bound (7.1); the lower bound (7.2) can be proven in the same way. Denote $W = V + \varepsilon|V|$; by (6.6) and (7.5), (7.6), we have for all

sufficiently small $a > 0$:

$$\begin{aligned}
\limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_t, H_0) &\leq \sum_{n=0}^{\infty} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda - a; H_0 + P_n W_t P_n, H_0) \\
&= \sum_{n=0}^{\infty} \limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda - a - \Lambda_n; P_n W_t P_n, 0) \\
&= \sum_{n=0}^q \limsup_{t \rightarrow \infty} t^{-p} N([\lambda - a - \Lambda_n, \infty); P_n W_t P_n) \\
&\quad - \sum_{n=q+1}^{\infty} \liminf_{t \rightarrow \infty} t^{-p} N((\Lambda_n + a - \lambda, \infty); -P_n W_t P_n) \\
&\leq \sum_{n=0}^q A[\lambda - a - \Lambda_n, W] - \sum_{n=q+1}^{\infty} A(\Lambda_n + a - \lambda, -W) \\
&\leq \sum_{n=0}^q A[\lambda - a - \Lambda_n, V + \varepsilon|V|] - \sum_{n=q+1}^N A(\Lambda_n + a - \lambda, -V - \varepsilon|V|)
\end{aligned}$$

for any $N \geq q + 1$. Letting $a \rightarrow +0$, $\varepsilon \rightarrow +0$ and using the continuity properties (7.15), (7.16), we obtain

$$\limsup_{t \rightarrow \infty} t^{-p} \Xi(\lambda; H_t, H_0) \leq \sum_{n=0}^q A[\lambda - \Lambda_n, V] - \sum_{n=q+1}^N A(\Lambda_n - \lambda, -V).$$

Since N here is arbitrary, we obtain (7.1).

3. Let us prove (7.3). Combining (7.1), (7.2) with identities (1.7), (1.8), one obtains

$$\limsup_{t \rightarrow \infty} t^{-p} N([\lambda_1, \lambda_2]; H_t) \leq \frac{B}{2\pi} \sum_{n \geq 0} \text{meas}\{x \in \mathbb{R}^2 \mid \lambda_1 - \Lambda_n \leq V(x) \leq \lambda_2 - \Lambda_n\}.$$

Replacing λ_2 by $\lambda_2 + \varepsilon$, letting $\varepsilon \rightarrow +0$ and using the continuity properties (7.15), one obtains (7.3). In the same way one obtains the lower bound (7.4). ■

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