

THE SCATTERING MATRIX AND THE DIFFERENCES OF SPECTRAL PROJECTIONS

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ABSTRACT

In the scattering theory framework, we point out a connection between the spectrum of the scattering matrix of two operators and the spectrum of the difference of spectral projections of these operators.

1. Introduction and results

1.1. Motivation and an informal description of results

Let H_0 and H be self-adjoint operators in a Hilbert space \mathcal{H} and suppose that the difference $V = H - H_0$ is a compact operator. For $\lambda \in \mathbb{R}$, we denote by $E_0(\lambda)$ and $E(\lambda)$ the spectral projections of H_0 and H , corresponding to the interval $(-\infty, \lambda)$. Our aim is to discuss the spectral properties of the operators

$$D(\lambda) := E(\lambda) - E_0(\lambda), \quad \lambda \in \mathbb{R} \tag{1.1}$$

and to point out the connection between these properties and the scattering matrix $S(\lambda)$ for the pair of operators H_0, H .

It is well known that due to the compactness of V , for any continuous function φ which tends to zero at infinity, the difference

$$\varphi(H) - \varphi(H_0) \tag{1.2}$$

is compact. However, the difference (1.2) in general fails to be compact if φ has discontinuities on the essential spectrum of H_0 and H . This observation goes back to M. G. Krein [10] and was recently revisited in [9]; we will say more on this in section 1.3. An attempt to understand Krein's example was part of the motivation for this paper.

The first question we address is the nature of the essential spectrum of the operators $D(\lambda)$, as these are the simplest operators of the type (1.2) when φ has a discontinuity. We consider this problem in the scattering theory framework, i.e. we make certain assumptions of the Kato smoothness type that are typical in scattering theory. These assumptions, in particular, ensure that the scattering matrix $S(\lambda)$ for the pair H_0, H is well defined. Under these assumptions, we prove (see Theorem 1.1) that

$$\sigma_{ess}(D(\lambda)) = [-a, a], \quad a = \frac{1}{2} \|S(\lambda) - I_\lambda\|. \tag{1.3}$$

Here the scattering matrix $S(\lambda)$ acts in the fiber Hilbert space $\mathfrak{h}(\lambda)$, which appears in the diagonalisation of the absolutely continuous part of H_0 (see (1.6) below) and I_λ is the identity operator in $\mathfrak{h}(\lambda)$. In particular, (1.3) says that $D(\lambda)$ is compact if and only if $S(\lambda) = I_\lambda$.

Next, we consider the difference $D(\lambda)$ in the framework of the trace class scattering theory. Assuming that a certain trace class condition on V is fulfilled, we describe the absolutely

continuous (a.c.) spectrum of the operator $D(\lambda)$ in terms of the spectrum of the scattering matrix. See Theorem 1.2 for the precise statement.

Note that the question of the spectral analysis of the difference $D(\lambda)$ is well posed regardless of any scattering theory type assumptions on the pair of operators H_0, H . Thus, the observations presented here might offer an insight into possible extensions of some elements of the scattering theory framework to wider classes of pairs of operators.

In this paper, we do not aim to prove our results under the optimal assumptions on H_0 and H . Our aim is rather to point out the connection between the spectral properties of $D(\lambda)$ and $S(\lambda)$ while keeping the technical details simple.

Our construction borrows several ideas from the spectral theory of Hankel operators; see [11, 5, 6, 7].

We denote by \mathfrak{S}_∞ the class of all compact operators and by \mathfrak{S}_1 and \mathfrak{S}_2 the trace class and the Hilbert-Schmidt class respectively. Along with the notation $E_0(\lambda), E(\lambda)$ for $\lambda \in \mathbb{R}$, we also use the notation $E_0(\delta), E(\delta)$ for the spectral projections of H_0 and H associated with a Borel set $\delta \subset \mathbb{R}$. For a subspace \mathcal{M} of a Hilbert space \mathcal{H} , we denote by $\mathcal{H} \ominus \mathcal{M}$ the orthogonal complement of \mathcal{M} in \mathcal{H} .

1.2. Statement of Results

Let $H - H_0 = V = G^*V_0G$, where G is a bounded operator from \mathcal{H} to an auxiliary Hilbert space \mathcal{K} , and V_0 is a bounded self-adjoint operator in \mathcal{K} . The simplest case of such a factorisation is when $\mathcal{K} = \mathcal{H}$, $G = |V|^{1/2}$ and $V_0 = \text{sign}(V)$. Let us define

$$F_0(\lambda) = GE_0(\lambda)G^*, \quad F(\lambda) = GE(\lambda)G^*, \quad \lambda \in \mathbb{R}. \quad (1.4)$$

Next, let $\delta \subset \sigma_{ac}(H_0)$ be an open interval.

HYPOTHESIS 1. The operator G is compact. For all $\lambda \in \delta$, the derivatives $F'_0(\lambda) = \frac{d}{d\lambda}F_0(\lambda)$ and $F'(\lambda) = \frac{d}{d\lambda}F(\lambda)$ exist in operator norm. The maps $\delta \ni \lambda \mapsto F'_0(\lambda)$ and $\delta \ni \lambda \mapsto F'(\lambda)$ are Hölder continuous (with some positive exponent) in the operator norm.

Hypothesis 1 is close to (but stronger than) the local Kato smoothness assumption in scattering theory (see [13] or [15]). In fact, one can make the required assumption concerning $F'_0(\lambda)$ and in addition assume that

$$\lim_{\epsilon \rightarrow +0} (I + V_0G(H_0 - \lambda - i\epsilon)^{-1}G^*) \text{ is invertible for all } \lambda \in \delta. \quad (1.5)$$

This will ensure that the required assumption holds true also for $F'(\lambda)$.

Next, we recall the definition of the scattering matrix. Let $\mathcal{H}_0^{(ac)}(\delta) \subset \text{Ran } E_0(\delta)$ be the absolutely continuous subspace of the operator $H_0E_0(\delta)$ and $\mathcal{H}^{(ac)}(\delta)$ be the absolutely continuous subspace of $HE(\delta)$; let $P_0^{(ac)}$ be the orthogonal projection onto $\mathcal{H}_0^{(ac)}(\delta)$ in \mathcal{H} . Hypothesis 1 ensures that the local wave operators

$$W_\pm := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} P_0^{(ac)}$$

exist and are complete: $\text{Ran } W_\pm = \mathcal{H}^{(ac)}(\delta)$. The local scattering operator $S = W_+^* W_-$ is unitary in $\mathcal{H}_0^{(ac)}(\delta)$ and commutes with $H_0|_{\mathcal{H}_0^{(ac)}(\delta)}$. Consider the direct integral decomposition

$$\mathcal{H}_0^{(ac)}(\delta) = \int_\delta^\oplus \mathfrak{h}(\lambda) d\lambda \quad (1.6)$$

which diagonalises $H_0|_{\mathcal{H}_0^{(ac)}(\delta)}$. Then

$$S = \int_\delta^\oplus S(\lambda) d\lambda, \quad S(\lambda) : \mathfrak{h}(\lambda) \rightarrow \mathfrak{h}(\lambda).$$

The scattering matrix $S(\lambda)$ is unitary in $\mathfrak{h}(\lambda)$. The compactness of G ensures that $S(\lambda) - I_\lambda$ is compact for all $\lambda \in \delta$.

THEOREM 1.1. *Suppose that for some open interval $\delta \subset \mathbb{R}$, Hypothesis 1 holds true. Then for all $\lambda \in \delta$ formula (1.3) holds true.*

Next, we describe the trace class result. Instead of Hypothesis 1, we need the following stronger hypothesis:

HYPOTHESIS 2. The operator G is Hilbert-Schmidt. For all $\lambda \in \delta$, the derivatives $F'_0(\lambda)$ and $F'(\lambda)$ exist in the trace norm. The maps $\delta \ni \lambda \mapsto F'_0(\lambda)$ and $\delta \ni \lambda \mapsto F'(\lambda)$ are Hölder continuous (with some positive exponent) in the trace norm.

Again, it suffices to assume the existence and Hölder continuity of F'_0 and (1.5); then F' also exists and is Hölder continuous.

Under Hypothesis 2, the operator $S(\lambda) - I_\lambda$ is compact for all $\lambda \in \delta$. Thus, the spectrum of $S(\lambda)$ consists of eigenvalues on the unit circle which can only accumulate to 1. For $\lambda \in \delta$, let $e^{i\theta_n(\lambda)}$, $\theta_n(\lambda) \in (0, 2\pi)$, be the eigenvalues (enumerated with multiplicities taken into account) of $S(\lambda)$ distinct from 1. There may be finitely or infinitely many of these eigenvalues.

THEOREM 1.2. *Suppose that for an open interval $\delta \subset \mathbb{R}$, Hypothesis 2 holds true. Then for all $\lambda \in \delta$ the a.c. part of the operator $D(\lambda)$ is unitarily equivalent to a direct sum of operators of multiplication by x in $L^2([-a_n, a_n], dx)$, $a_n = \frac{1}{2}|e^{i\theta_n(\lambda)} - 1| = \sin(\theta_n(\lambda)/2)$.*

Using Theorems 1.1 and 1.2, one can also analyse the spectra of the operators (1.2) for certain classes of piecewise continuous functions φ ; we will not pursue this direction here.

It seems likely that the assertion of Theorem 1.2 holds true under Hypothesis 1. The trace class assumptions seem to be of a technical nature.

1.3. Krein's Example

In [10], M. G. Krein considers the operator H_0 in $L^2(0, \infty)$ with the integral kernel $H_0(x, y)$ given by

$$H_0(x, y) = \begin{cases} \sinh(x)e^{-y}, & x \leq y, \\ \sinh(y)e^{-x}, & x \geq y \end{cases}$$

and the operator H in the same Hilbert space with the integral kernel $H(x, y) = H_0(x, y) + e^{-x}e^{-y}$. Thus, $V = H - H_0$ is a rank one operator. In fact, H_0 and H are resolvents (with the spectral parameter -1) of the operator $-\frac{d^2}{dx^2}$ in $L^2(0, \infty)$ with the Dirichlet and Neumann boundary conditions at zero.

Krein shows that in this example $D(\lambda)$ is not a Hilbert-Schmidt operator for any $\lambda \in (0, 1)$. A more detailed analysis [9] shows that the spectrum of $D(\lambda)$ is simple, purely a.c. and coincides with $[-1, 1]$.

What can be said about the scattering matrix in this case? First note that the spectra of both H_0 and H are simple, purely a.c. and coincide with $[0, 1]$. Thus, the fibre spaces $\mathfrak{h}(\lambda)$ in (1.6) are one-dimensional and so the scattering matrix is simply a unimodular complex number. Krein calculates the spectral shift function $\xi(\lambda)$ for this pair of operators and shows that $\xi(\lambda) = 1/2$ on $[0, 1]$. Together with the Birman-Krein formula $\det S(\lambda) = e^{-2\pi i \xi(\lambda)}$ this shows that $S(\lambda) = -1$ for all $\lambda \in [0, 1]$. Thus, we have a complete agreement with Theorems 1.1 and 1.2. It is not difficult to check that Hypotheses 1 and 2 hold true with $\delta = (0, 1)$.

1.4. Example: Schrödinger operator

Let $H_0 = -\Delta$ in $\mathcal{H} = L^2(\mathbb{R}^d)$, $d = 1, 2, 3$, and $H = H_0 + V$, where V is the operator of multiplication by a function $V : \mathbb{R}^d \rightarrow \mathbb{R}$, which is assumed to satisfy

$$|V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1. \quad (1.7)$$

It is well known that under the assumption (1.7), the wave operators for the pair H_0 and H exist and are complete, and the scattering matrix $S(\lambda)$ is well defined and differs from the identity by a compact operator.

THEOREM 1.3. (i) Assume $\rho > 1$. Then for all $\lambda > 0$, formula (1.3) holds true.
(ii) Assume $\rho > d$. Then for all $\lambda > 0$, the conclusion of Theorem 1.2 holds true.

To the best of the author's knowledge, this result is new even for $d = 1$.

2. Proof of Theorem 1.3

1. Fix $a < 0$ such that $a < \inf \sigma(H)$. Consider the operators $h = (H - a)^{-1}$, $h_0 = (H_0 - a)^{-1}$. By the invariance principle for the scattering matrix (see [3] or [15]), we have

$$S(\lambda; H, H_0) = S(\mu; h, h_0), \quad \mu = \frac{1}{\lambda - a}, \quad \lambda > 0.$$

Also, denoting by $E_{h_0}(\mu)$ and $E_h(\mu)$ the spectral projections of h_0 and h associated with the interval $(-\infty, \mu)$, we have:

$$E(\lambda) - E_0(\lambda) = E_{h_0}(\mu) - E_h(\mu), \quad \mu = \frac{1}{\lambda - a}, \quad \lambda > 0.$$

Thus, Theorem 1.3 will follow from Theorems 1.1 and 1.2 if we show that the pair of operators h , h_0 satisfies Hypothesis 1 for $\rho > 1$ and Hypothesis 2 for $\rho > d$.

In order to check this, we need to fix an appropriate factorization of $h - h_0$. We shall use the factorization $h - h_0 = g^* v_0 g$, where

$$g = |V|^{1/2} h_0, \quad v_0 = -V_0 - V_0 |V|^{1/2} h |V|^{1/2} V_0, \quad V_0 = \text{sign}(V).$$

This factorization is merely an iterated resolvent identity written in different notation.

2. Assume $\rho > 1$. It is well known that $|V|^{1/2} h_0 \in \mathfrak{S}_\infty$. We use the notation

$$T_0(z) = |V|^{1/2} (H_0 - z)^{-1} |V|^{1/2}, \quad T(z) = |V|^{1/2} (H - z)^{-1} |V|^{1/2}, \quad \Im z > 0.$$

By the spectral theorem, we have

$$\frac{d}{d\mu} g E_{h_0}(\mu) g^* = (\lambda - a)^2 \frac{d}{d\lambda} |V|^{1/2} E_0(\lambda) |V|^{1/2} = (\lambda - a)^2 \frac{1}{\pi} \Im T_0(\lambda + i0). \quad (2.1)$$

The limit $T_0(\lambda + i0)$ exists and is continuous in $\lambda > 0$ in the operator norm. This fact is known as the limiting absorption principle; it stems from Sobolev's embedding theorems.

Next, we need to discuss the derivative $\frac{d}{d\mu} g E_h(\mu) g^*$. Before doing this, let us recall the following facts:

$$T(z) = T_0(z)(I + V_0 T_0(z))^{-1}, \quad \Im z > 0, \quad (2.2)$$

$$I + V_0 T_0(\lambda + i0) \text{ has a bounded inverse for all } \lambda > 0. \quad (2.3)$$

Formula (2.2) follows from the resolvent identity. Relation (2.3) goes back to Agmon [1] and uses the fact that (by Kato's theorem [8]) H has no positive eigenvalues. Also due to Agmon is the observation that one can put (2.2) and (2.3) together and prove that $T(\lambda + i0)$ is Hölder

continuous in $\lambda > 0$ in the operator norm. It follows that

$$\text{the derivative } \frac{d}{d\lambda}|V|^{1/2}E(\lambda)|V|^{1/2} = \frac{1}{\pi}\Im T(\lambda + i0) \text{ exists and is Hölder continuous.} \quad (2.4)$$

Let us return to the derivative $\frac{d}{d\mu}gE_h(\mu)g^*$. Using the resolvent identity, we get

$$\begin{aligned} gE_h(\mu)g^* &= |V|^{1/2}h_0E_h(\mu)h_0|V|^{1/2} \\ &= |V|^{1/2}hE_h(\mu)h|V|^{1/2} + |V|^{1/2}h_0VhE_h(\mu)hVh_0|V|^{1/2} \\ &\quad + |V|^{1/2}h_0VhE_h(\mu)h|V|^{1/2} + |V|^{1/2}hE_h(\mu)hVh_0|V|^{1/2}. \end{aligned} \quad (2.5)$$

Inspecting each term in the right hand side and using (2.4), we see that the derivative of the above expression exists and is Hölder continuous in μ in the operator norm. Thus, we have checked that h, h_0 satisfy Hypothesis 1.

3. Assume $\rho > d$. It is well known that $|V|^{1/2}h_0 \in \mathfrak{S}_2$; this follows from an inspection of the integral kernel of this operator.

Next, we claim that the derivative $\frac{d}{d\lambda}|V|^{1/2}E_0(\lambda)|V|^{1/2}$ exists and is Hölder continuous in the trace norm. This fact is probably well known to specialists; in any case, it follows from a simple computation involving factorization of the pre-limiting expressions into products of two Hilbert-Schmidt operators and estimating the Hilbert-Schmidt norm of each of these factors. The details of this computation can be found, e.g. in [12, Sections 6 and 7]. By (2.1), the derivative $\frac{d}{d\mu}gE_{h_0}(\mu)g^*$ also exists and is Hölder continuous in the trace norm.

Finally, consider the derivative $\frac{d}{d\mu}gE_h(\mu)g^*$. First, by using (2.5) we reduce the question to the existence and Hölder continuity of $\frac{d}{d\lambda}|V|^{1/2}E(\lambda)|V|^{1/2}$. The latter fact again follows from (2.2), (2.3) and the Hölder continuity of $\frac{d}{d\lambda}|V|^{1/2}E_0(\lambda)|V|^{1/2}$. Thus, we have checked that h, h_0 satisfy Hypothesis 2. This completes the proof.

3. Proof of Theorems 1.1 and 1.2

We use the notation

$$R_0(z) = (H_0 - zI)^{-1}, \quad R(z) = (H - zI)^{-1}, \quad T_0(z) = GR_0(z)G^*, \quad T(z) = GR(z)G^*.$$

For $\lambda \in \delta$, let us introduce an auxiliary operator in \mathcal{K} :

$$A(\lambda) = \pi^2(F'_0(\lambda))^{1/2}V_0F'(\lambda)V_0(F'_0(\lambda))^{1/2}. \quad (3.1)$$

Clearly, $A(\lambda)$ is compact, self-adjoint, and $A(\lambda) \geq 0$. This operator plays an important role in our construction. As we shall see, the spectrum of $A(\lambda)$ is related to the spectrum of the scattering matrix $S(\lambda)$. In order to describe this relation, let us introduce the following notation. For bounded normal operators X and Y in Hilbert spaces \mathcal{H}_X and \mathcal{H}_Y , we shall write

$$X \approx Y \text{ if } X|_{\mathcal{H}_X \ominus \text{Ker } X} \text{ is unitarily equivalent to } Y|_{\mathcal{H}_Y \ominus \text{Ker } Y}.$$

It is well known that $X^*X \approx XX^*$ for any bounded operator X ; we shall repeatedly use this fact.

LEMMA 3.1. *Suppose that Hypothesis 1 holds true. Then for all $\lambda \in \delta$,*

$$A(\lambda) \approx \frac{1}{4}(S(\lambda) - I_\lambda)^*(S(\lambda) - I_\lambda) = \frac{1}{2}(I_\lambda - \Re S(\lambda)). \quad (3.2)$$

In other words, the Lemma says that if $e^{i\theta_n}$ are the eigenvalues of $S(\lambda)$, then $(\sin(\theta_n/2))^2$ are the eigenvalues of $A(\lambda)$.

Proof. 1. First we recall the stationary representation for the scattering matrix. For $f \in \mathcal{H}_0^{(ac)}(\delta)$, let $\{f(\lambda)\}_{\lambda \in \delta}$, $f(\lambda) \in \mathfrak{h}(\lambda)$, be the representation of f in the direct integral (1.6). Then for all $\lambda \in \delta$, the operator $\mathcal{F}(\lambda) : \mathcal{K} \rightarrow \mathfrak{h}(\lambda)$, $f \mapsto (G^*f)(\lambda)$ is well defined, bounded, and

$$\mathcal{F}(\lambda)^* \mathcal{F}(\lambda) = F_0'(\lambda). \quad (3.3)$$

For a.e. $\lambda \in \delta$, the scattering matrix can be represented as

$$S(\lambda) = I_\lambda - 2\pi i \mathcal{F}(\lambda)(V_0 - V_0 T(\lambda + i0)V_0) \mathcal{F}(\lambda)^*. \quad (3.4)$$

2. Consider an auxiliary unitary operator $\tilde{S}(\lambda)$ in \mathcal{H} , defined by

$$\tilde{S}(\lambda) = I - 2\pi i (F_0'(\lambda))^{1/2} (V_0 - V_0 T(\lambda + i0)V_0) (F_0'(\lambda))^{1/2}. \quad (3.5)$$

By virtue of (3.3), we have

$$S(\lambda) - I_\lambda \approx \tilde{S}(\lambda) - I, \quad \lambda \in \delta.$$

This is well known; see e.g. [15, Lemma 7.7.1]. It follows that

$$(S(\lambda) - I_\lambda)^* (S(\lambda) - I_\lambda) \approx (\tilde{S}(\lambda) - I)^* (\tilde{S}(\lambda) - I). \quad (3.6)$$

3. For any $\varepsilon > 0$, employing the resolvent identity, we obtain

$$\begin{aligned} & (V_0 - V_0 T(\lambda - i\varepsilon)V_0)(\Im T_0(\lambda + i\varepsilon))(V_0 - V_0 T(\lambda + i\varepsilon)V_0) \\ &= V_0 G(I - R(\lambda - i\varepsilon)V)\varepsilon R_0(\lambda - i\varepsilon)R_0(\lambda + i\varepsilon)(I - VR(\lambda + i\varepsilon))G^*V_0 \\ &= V_0 G\varepsilon R(\lambda - i\varepsilon)R(\lambda + i\varepsilon)G^*V_0 = V_0(\Im T(\lambda + i\varepsilon))V_0. \end{aligned}$$

Taking $\varepsilon \rightarrow +0$ in the above identity and multiplying on both sides by $(F_0'(\lambda))^{1/2}$, we obtain

$$\frac{1}{4\pi} (\tilde{S}(\lambda) - I)^* (\tilde{S}(\lambda) - I) = (F_0'(\lambda))^{1/2} V_0 (\Im T(\lambda + i0))^{1/2} V_0 (F_0'(\lambda))^{1/2} = \frac{1}{\pi} A(\lambda).$$

Together with (3.6), this proves the required statement. \square

Without the loss of generality, let us assume that $\lambda = 0$ in the statements of Theorems 1.1 and 1.2. We use the notation $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$. The proofs of Theorems 1.1 and 1.2 will be deduced from the following Lemma, which might be of some interest in its own right.

LEMMA 3.2. (i) Assume Hypothesis 1 and let $0 \in \delta$. Then the essential spectra of the operators $E_0(\mathbb{R}_\pm)E(\mathbb{R}_\mp)E_0(\mathbb{R}_\pm)$ coincide with $[0, \|A(0)\|]$.

(ii) Assume Hypothesis 2 and let $0 \in \delta$. Let s_n be the non-zero eigenvalues of $A(0)$. Then the a.c. parts of the operators $E_0(\mathbb{R}_\pm)E(\mathbb{R}_\mp)E_0(\mathbb{R}_\pm)$ are unitarily equivalent to a direct sum of operators of multiplication by x in $L^2([0, s_n], dx)$.

Proof of Theorems 1.1 and 1.2.

1. First let us reduce our considerations to the case

$$E_0(\{0\}) = E(\{0\}) = 0. \quad (3.7)$$

Hypothesis 1 for $\lambda = 0$ implies that $GE_0(\{0\})G^* = 0$; therefore, $GE_0(\{0\}) = 0$ and so $VE_0(\{0\}) = 0$. It follows that the subspace $\text{Ran } E_0(\{0\})$ reduces both H and H_0 , and so $E_0(\{0\}) = E(\{0\})$. Denote $\tilde{\mathcal{H}} = \mathcal{H} \ominus \text{Ran } E_0(\{0\})$, $\tilde{H} = H|_{\tilde{\mathcal{H}}}$, $\tilde{H}_0 = H_0|_{\tilde{\mathcal{H}}}$, and let $\tilde{D}(0)$ be the difference (1.1) constructed for the operators \tilde{H}_0, \tilde{H} . Then we have $\tilde{D}(0) \approx D(0)$ and 0 is not an eigenvalue of \tilde{H} or of \tilde{H}_0 . Thus, without the loss of generality we can assume that (3.7) holds true from the start.

2. Let us denote $D = D(0)$ and

$$\begin{aligned}\mathcal{H}_+ &= \text{Ker}(D - I) = \text{Ran } E(\mathbb{R}_-) \cap \text{Ker } E_0(\mathbb{R}_-), \\ \mathcal{H}_- &= \text{Ker}(D + I) = \text{Ran } E_0(\mathbb{R}_-) \cap \text{Ker } E(\mathbb{R}_-), \\ \mathcal{H}_0 &= \mathcal{H} \ominus (\mathcal{H}_- \oplus \mathcal{H}_+).\end{aligned}$$

It is well known (see [4] or [2]) that $D|_{\mathcal{H}_0} \approx (-D)|_{\mathcal{H}_0}$. Therefore, the spectral analysis of D reduces to the spectral analysis of D^2 and to the spectral analysis of the dimensions of \mathcal{H}_+ and \mathcal{H}_- . Next, D^2 can be represented as

$$D^2 = E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-) + E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+), \quad (3.8)$$

and the right hand side provides a block-diagonal decomposition of D^2 with respect to the decomposition $\mathcal{H} = \text{Ran } E_0(\mathbb{R}_-) \oplus \text{Ran } E_0(\mathbb{R}_+)$. Thus, the spectral analysis of D^2 reduces to the spectral analysis of the two terms on the right hand side of (3.8).

3. Taking into account the decomposition (3.8), we see that Theorem 1.2 follows directly from Lemma 3.2(ii).

Similarly, Lemma 3.2(i) characterises $\sigma_{ess}(D)$ away from -1 and 1 . In order to complete the proof of Theorem 1.1, it remains to take care of the eigenvalues ± 1 of D . If $\|A(0)\| < 1$, then Lemma 3.2(i) ensures that the kernels

$$\text{Ker}(E_0(\mathbb{R}_\pm)E(\mathbb{R}_\mp)E_0(\mathbb{R}_\pm) - I) = \mathcal{H}_\pm$$

are finite dimensional, and so ± 1 do not contribute to the essential spectrum of D . On the other hand, if $\|A(0)\| = 1$, then by Lemma 3.2, $\sigma_{ess}(D) = [-1, 1]$ regardless of the dimensions of \mathcal{H}_\pm and so we have nothing to prove. \square

The key element in our proof of Lemma 3.2 is a representation of the product $E(\mathbb{R}_-)E_0(\mathbb{R}_+)$ in terms of some auxiliary operators Z, Z_0 which we proceed to define. These operators act from $L^2(\mathbb{R}_+, \mathcal{K})$ into \mathcal{H} ; here $L^2(\mathbb{R}_+, \mathcal{K})$ is the space of measurable functions $f : \mathbb{R}_+ \rightarrow \mathcal{K}$ such that

$$\int_0^\infty \|f(t)\|_{\mathcal{K}}^2 dt < \infty.$$

$L^1(\mathbb{R}_+, \mathcal{K})$ is defined similarly. On the dense subset $L^1(\mathbb{R}_+, \mathcal{K}) \cap L^2(\mathbb{R}, \mathcal{K})$, let us define the operators Z, Z_0 by

$$\begin{aligned}Z_0 f &= \int_0^\infty e^{-tH_0} E_0(\mathbb{R}_+) G^* f(t) dt, \\ Z f &= \int_0^\infty e^{tH} E(\mathbb{R}_-) G^* f(t) dt.\end{aligned}$$

We will see (in Lemma 4.3) that Z_0 and Z are bounded and

$$E(\mathbb{R}_-)E_0(\mathbb{R}_+) = -ZV_0Z_0^*. \quad (3.9)$$

From (3.9) we get the representation formula $E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+) = Z_0V_0Z^*ZV_0Z_0^*$, which will be important in our proof of Lemma 3.2. But first we need to develop some analysis related to the operators Z and Z_0 ; this is done in the beginning of the next section.

4. Hankel operators; Proof of Lemma 3.2

4.1. Hankel operators

We need to prepare some estimates for vector valued Hankel operators. These are straightforward generalisations of the well known technique of spectral theory of Hankel operators (see [11, 5, 6, 7]) to a vector valued case.

Suppose that for each $t \geq 0$, a bounded self-adjoint operator $K(t)$ in \mathcal{K} is given. Suppose that $K(t)$ is continuous in $t \geq 0$ in the operator norm. Define a Hankel type operator K in $L^2(\mathbb{R}_+, \mathcal{K})$ by

$$(Kf, g)_{L^2(\mathbb{R}_+, \mathcal{K})} = \int_0^\infty \int_0^\infty (K(t+s)f(s), g(t))_{\mathcal{K}} dt ds, \quad (4.1)$$

when $f, g \in L^2(\mathbb{R}_+, \mathcal{K}) \cap L^1(\mathbb{R}_+, \mathcal{K})$.

LEMMA 4.1.

(i) Suppose $\|K(t)\| \leq C_1/t$ for all $t > 0$. Then the operator K is bounded and $\|K\| \leq \pi C_1$.

(ii) Suppose $K(t)$ is compact for all t and $\|K(t)\|t \rightarrow 0$ as $t \rightarrow +0$ and as $t \rightarrow +\infty$. Then K is compact.

(iii) Suppose

$$K(t) = \int_0^\infty M(\lambda) e^{-\lambda t} d\lambda,$$

where $M(\lambda)$ is a measurable function of $\lambda \in (0, \infty)$ with values in the set of trace class operators in \mathcal{K} . Suppose that

$$C_2 := \int_0^\infty \|M(\lambda)\|_{\mathfrak{S}_1} \lambda^{-1} d\lambda < \infty;$$

then K is a trace class operator.

Proof. (i) and (ii) are straightforward generalisations of Proposition 1.1 from [7]. Indeed, since the Carlemann operator on $L^2(\mathbb{R}_+)$ with the kernel $(t+s)^{-1}$ is bounded with the norm π , we have

$$|(Kf, g)_{L^2}| \leq C_1 \int_0^\infty \int_0^\infty \frac{\|f(s)\|_{\mathcal{K}} \|g(t)\|_{\mathcal{K}}}{t+s} dt ds \leq \pi C_1 \|f\|_{L^2(\mathbb{R}_+, \mathcal{K})} \|g\|_{L^2(\mathbb{R}_+, \mathcal{K})},$$

which proves (i). To prove (ii), we need to approximate K by compact operators. Let $K_n(t) = K(t)\chi_{(1/n, n)}(t)$ and let K_n be the corresponding operator in $L^2(\mathbb{R}_+, \mathcal{K})$. It is not difficult to see that each K_n is compact. By (i), $\|K - K_n\| \rightarrow 0$ as $n \rightarrow \infty$.

(iii) For each λ , let us represent $M(\lambda)$ as a difference of its positive and negative parts: $M(\lambda) = M_+(\lambda) - M_-(\lambda)$, $M_\pm(\lambda) \geq 0$, $\|M(\lambda)\|_{\mathfrak{S}_1} = \text{Tr } M_+(\lambda) + \text{Tr } M_-(\lambda)$. Then K splits accordingly as $K = K_+ - K_-$. Let us factorize each of K_+ , K_- into a product of Hilbert-Schmidt operators as follows. Let

$$\begin{aligned} N_\pm &: L^2(\mathbb{R}_+, \mathcal{K}) \rightarrow L^2(\mathbb{R}_+, \mathcal{K}), \\ (N_\pm f)(\lambda) &= M_\pm(\lambda)^{1/2} \int_0^\infty e^{-\lambda t} f(t) dt. \end{aligned}$$

Then $K_\pm = N_\pm^* N_\pm$ and

$$\begin{aligned} \|N_\pm\|_{\mathfrak{S}_2}^2 &= \int_0^\infty d\lambda \int_0^\infty dt \|e^{-\lambda t} M_\pm(\lambda)^{1/2}\|_{\mathfrak{S}_2}^2 \\ &= \int_0^\infty d\lambda \|M_\pm(\lambda)\|_{\mathfrak{S}_1} \int_0^\infty e^{-2\lambda t} dt = \int_0^\infty \text{Tr}(M_\pm(\lambda)) (2\lambda)^{-1} d\lambda < \infty, \end{aligned}$$

which yields the required result. \square

Consider the self-adjoint operators Γ_0, Γ in $L^2(\mathbb{R}_+)$ which are given by the integral kernels

$$\Gamma_0(t, s) = \frac{e^{-t-s}}{t+s}, \quad \Gamma(t, s) = \frac{1 - e^{-t-s}}{t+s}.$$

It is well known that Γ_0 is bounded and has purely a.c. spectrum $[0, \pi]$ of multiplicity one; explicit diagonalisation of Γ_0 is available (see [14]). The following proposition is probably well known to specialists, but we were unable to find it in the literature.

LEMMA 4.2. *The operator Γ is unitarily equivalent to Γ_0 . Thus, Γ has a purely a.c. spectrum of multiplicity one which coincides with $[0, \pi]$.*

Proof. The proof is a combination of identities from [6]. Let $N : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ be the operator $(Nf)(t) = \int_0^\infty e^{-ts} f(s) ds$. We have $\Gamma = N\chi_{(0,1)}N$, $\Gamma_0 = N\chi_{(1,\infty)}N$. Next, let $U : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ be the unitary operator $(Uf)(x) = \frac{1}{x}f(1/x)$. Then $U^2 = I$, $U\chi_{(0,1)} = \chi_{(1,\infty)}U$ and $UN^2U = N^2$. Using the well known fact that $X^*X \approx XX^*$, we get

$$\begin{aligned} \Gamma &= N\chi_{(0,1)}N = (N\chi_{(0,1)}U)(U\chi_{(0,1)}N) = (NU\chi_{(1,\infty)})(\chi_{(1,\infty)}UN) \\ &\approx (\chi_{(1,\infty)}UN)(NU\chi_{(1,\infty)}) = (\chi_{(1,\infty)}N)(N\chi_{(1,\infty)}) \approx N\chi_{(1,\infty)}N = \Gamma_0. \end{aligned}$$

Thus, $\Gamma \approx \Gamma_0$. It remains to note that $\text{Ker } \Gamma = \text{Ker } \Gamma_0 = \{0\}$. \square

4.2. Proof of Lemma 3.2

Important ‘‘model’’ operators in our considerations are the integral Hankel operators in $L^2(\mathbb{R}_+, \mathcal{K})$ of the type (4.1) with the kernels given by

$$K(t) = F'(0) \frac{1 - e^{-t}}{t}, \quad K_0(t) = F'_0(0) \frac{1 - e^{-t}}{t}.$$

Identifying $L^2(\mathbb{R}_+, \mathcal{K})$ with $L^2(\mathbb{R}_+) \otimes \mathcal{K}$, we will denote these operators by $\Gamma \otimes F'(0)$ and $\Gamma \otimes F'_0(0)$.

LEMMA 4.3. (i) *Assume that*

$$\|F_0(\lambda) - F_0(0)\| = O(\lambda), \quad \|F(\lambda) - F(0)\| = O(\lambda), \quad \text{as } \lambda \rightarrow 0. \quad (4.2)$$

Then the operators Z and Z_0 are bounded.

(ii) *Assume Hypothesis 1 with $0 \in \delta$. Then the differences*

$$Z_0^*Z_0 - (\Gamma \otimes F'_0(0)) \quad \text{and} \quad Z^*Z - (\Gamma \otimes F'(0)) \quad (4.3)$$

are compact.

(iii) *Assume Hypothesis 2 with $0 \in \delta$. Then the differences (4.3) are trace class operators.*

Proof. We will prove the statements for Z_0 ; the proofs for Z are analogous.

(i) Let $f \in L^2(\mathbb{R}_+, \mathcal{K}) \cap L^1(\mathbb{R}_+, \mathcal{K})$; we have

$$\|Z_0 f\|^2 = \int_0^\infty \int_0^\infty (Ge^{-(t+s)H_0} E_0(\mathbb{R}_+) G^* f(t), g(s))_{\mathcal{K}} dt ds,$$

and so the above expression is a quadratic form of the operator of the type (4.1) with the kernel $K(t) = Ge^{-tH_0} E_0(\mathbb{R}_+) G^*$. By Lemma 4.1, it suffices to prove the bound $\|K(t)\| \leq C/t$, $t > 0$. Using our assumption (4.2), we have

$$\begin{aligned} \|K(t)\| &= \left\| t \int_0^\infty e^{-t\lambda} G E_0((0, \lambda)) G^* d\lambda \right\| \leq t \int_0^\infty e^{-t\lambda} \|F_0(\lambda) - F_0(0)\| d\lambda \\ &\leq Ct \int_0^\infty e^{-t\lambda} \lambda d\lambda = C/t. \end{aligned}$$

(ii) By the same reasoning, $Z_0^*Z_0 - (\Gamma \otimes F_0'(0))$ is an operator of the type (4.1) with

$$K(t) = Ge^{-tH_0}E_0(\mathbb{R}_+)G^* - F_0'(0) \int_0^1 e^{-t\lambda} d\lambda.$$

By Lemma 4.1, it suffices to prove that $\|K(t)\|t \rightarrow 0$ as $t \rightarrow 0$ and $t \rightarrow \infty$. For $t \rightarrow 0$ this is clearly true. Next, we have

$$K(t) = t \int_0^\infty e^{-t\lambda}(F_0(\lambda) - F_0(0))d\lambda - F_0'(0)t \int_0^\infty \min\{\lambda, 1\}e^{-t\lambda}d\lambda \quad (4.4)$$

and from $\|F_0(\lambda) - F_0(0) - \lambda F_0'(0)\| = o(\lambda)$, $\lambda \rightarrow 0$, we conclude that $\|K(t)\| = o(1/t)$ as $t \rightarrow \infty$.

(iii) Choose $\gamma > 0$ such that $[0, \gamma] \subset \delta$. As above, $Z_0^*Z_0 - (\Gamma \otimes F_0'(0))$ has the kernel (4.4). Let us write this kernel as $K(t) = K_1(t) + K_2(t)$, with

$$\begin{aligned} K_1(t) &= Ge^{-tH_0}E_0([0, \gamma])G^* - F_0'(0) \int_0^1 e^{-t\lambda} d\lambda \\ &= \int_0^\infty [F_0'(\lambda)\chi_{(0, \gamma)}(\lambda) - F_0'(0)\chi_{(0, 1)}(\lambda)]e^{-t\lambda}d\lambda, \end{aligned}$$

and $K_2(t) = Ge^{-tH_0}E_0((\gamma, \infty))G^*$, so $K_2 = (E_0((\gamma, \infty))Z_0)^*(E_0((\gamma, \infty))Z_0)$. By the Hölder continuity assumption,

$$\int_0^\infty \|F_0'(\lambda)\chi_{(0, \gamma)}(\lambda) - F_0'(0)\chi_{(0, 1)}(\lambda)\|_{\mathfrak{S}_1} \lambda^{-1} d\lambda < \infty,$$

and so by Lemma 4.1(iii), the Hankel operator with the kernel K_1 is trace class. Finally, it is easy to see that $E_0((\gamma, \infty))Z_0 \in \mathfrak{S}_2$, since

$$\|E_0((\gamma, \infty))Z_0\|_{\mathfrak{S}_2}^2 = \int_0^\infty \|e^{-tH_0}E_0((\gamma, \infty))G^*\|_{\mathfrak{S}_2}^2 dt \leq \int_0^\infty e^{-2\gamma t} \|G^*\|_{\mathfrak{S}_2}^2 dt < \infty,$$

and so the Hankel operator with the kernel K_2 also belongs to the trace class. This argument borrows its main idea from [5]. \square

LEMMA 4.4. *Assume (4.2) and $E_0(\{0\}) = E_0(\{0\}) = 0$. Then the identity (3.9) holds true.*

Proof. Let $\psi, \psi_0 \in \mathcal{H}$ be vectors such that $E((-\gamma, 0))\psi = E_0((0, \gamma))\psi_0 = 0$ for some $\gamma > 0$. Since the set of such vectors (over all $\gamma > 0$) is dense in \mathcal{H} , it suffices to prove that

$$(E_0(\mathbb{R}_+)\psi_0, E(\mathbb{R}_-)\psi) = -(V_0Z_0^*\psi_0, Z^*\psi)_{L^2(\mathbb{R}_+, \mathcal{K})} \quad (4.5)$$

for all such vectors ψ, ψ_0 . For ψ and ψ_0 of this class, $Z_0^*\psi_0$ and $Z^*\psi$ are given by

$$\begin{aligned} (Z_0^*\psi_0)(t) &= Ge^{-tH_0}E_0(\mathbb{R}_+)\psi_0, \\ (Z^*\psi)(t) &= Ge^{tH}E(\mathbb{R}_-)\psi, \end{aligned}$$

and so we have

$$\begin{aligned} (V_0Z_0^*\psi_0, Z^*\psi)_{L^2(\mathbb{R}_+, \mathcal{K})} &= \int_0^\infty (V_0Ge^{-tH_0}E_0(\mathbb{R}_+)\psi_0, Ge^{tH}E(\mathbb{R}_-)\psi)_{\mathcal{K}} dt \\ &= \int_0^\infty (Ve^{-tH_0}E_0(\mathbb{R}_+)\psi_0, e^{tH}E(\mathbb{R}_-)\psi) dt. \quad (4.6) \end{aligned}$$

Consider the function $L(t) = (e^{-tH_0}E_0(\mathbb{R}_+)\psi_0, e^{tH}E(\mathbb{R}_-)\psi)$. This function is continuous in $t \geq 0$ and we have

$$L(0) = (E_0(\mathbb{R}_+)\psi_0, E(\mathbb{R}_-)\psi), \quad L(+\infty) = 0, \quad L'(t) = (Ve^{-tH_0}E_0(\mathbb{R}_+)\psi_0, e^{tH}E(\mathbb{R}_-)\psi).$$

Combining this with (4.6), we get (4.5). \square

Proof of Lemma 3.2. We will prove the statement for $E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+)$; the proof for $E_0(\mathbb{R}_-)E(\mathbb{R}_+)E_0(\mathbb{R}_-)$ is analogous. By Lemma 4.4, we have

$$E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+) = Z_0V_0Z^*ZV_0Z_0^*.$$

Next, by Lemma 4.3(ii), for some compact operators X_0 and X we have

$$Z_0V_0Z^*ZV_0Z_0^* = Z_0V_0(\Gamma \otimes F'(0))V_0Z_0^* + X, \quad (4.7)$$

$$\begin{aligned} Z_0V_0(\Gamma \otimes F'(0))V_0Z_0^* &= \left\{ Z_0V_0(\Gamma^{1/2} \otimes (F'(0))^{1/2}) \right\} \left\{ (\Gamma^{1/2} \otimes (F'(0))^{1/2})V_0Z_0^* \right\} \\ &\approx \left\{ (\Gamma^{1/2} \otimes (F'(0))^{1/2})V_0Z_0^* \right\} \left\{ Z_0V_0(\Gamma^{1/2} \otimes (F'(0))^{1/2}) \right\} \\ &= (\Gamma^{1/2} \otimes (F'(0))^{1/2})V_0(\Gamma \otimes F'_0(0))V_0(\Gamma^{1/2} \otimes (F'(0))^{1/2}) + X_0 \\ &= \Gamma^2 \otimes (F'(0))^{1/2}V_0F'_0(0)V_0F'(0)^{1/2} + X_0. \quad (4.8) \end{aligned}$$

Finally, we note that

$$F'(0)^{1/2}V_0F'_0(0)V_0F'(0)^{1/2} \approx F'_0(0)^{1/2}V_0F'(0)V_0F'_0(0)^{1/2} = \pi^{-2}A(0).$$

Combining the above relations and using Weyl's theorem on the stability of the essential spectrum under compact perturbations, we get

$$\sigma_{ess}(E_0(\mathbb{R}_+)E(\mathbb{R}_-)E_0(\mathbb{R}_+)) \setminus \{0\} = \sigma_{ess}(\pi^{-2}\Gamma^2 \otimes A(0)) \setminus \{0\}.$$

By Lemma 4.2, the essential spectrum of $\pi^{-2}\Gamma^2 \otimes A(0)$ coincides with $[0, \|A(0)\|]$. This proves part (i) of the Lemma.

Next, assuming Hypothesis 2 and using part (iii) instead of part (ii) of Lemma 4.3, we arrive at (4.7), (4.8) with X_0 and X of the trace class. Thus, by the Kato-Rosenblum theorem on the stability of the a.c. spectrum under trace class perturbations, the a.c. part of $Z_0V_0Z^*ZV_0Z_0^*$ is unitarily equivalent to the a.c. part of $\pi^{-2}\Gamma^2 \otimes A(0)$. By Lemma 4.2, the latter operator is unitarily equivalent to a direct sum of operators of multiplication by x in $L^2([0, s_n], dx)$, where s_n are the eigenvalues of $A(0)$. \square

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