

# Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains

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## Abstract

We consider the spectrum of a two-dimensional Pauli operator with a compactly supported electric potential and a variable magnetic field with a positive mean value. The rate of accumulation of eigenvalues to zero is described in terms of the logarithmic capacity of the support of the electric potential. A connection between these eigenvalues and orthogonal polynomials in complex domains is established.

*Keywords:* Pauli operator, magnetic field, spectral asymptotics, logarithmic capacity, orthogonal polynomials

## 1 Introduction

**1. The unperturbed Pauli operator.** Let  $B = B(x)$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ , be a real valued function which has the physical meaning of the strength of a magnetic field in  $\mathbb{R}^2$ . A two-dimensional non-relativistic electron in the external magnetic field  $B$  can be described by the Pauli operator

$$h = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix} \quad \text{in } L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2).$$

The standard approach to the definition of the operators  $h^\pm$  in  $L^2(\mathbb{R}^2)$  involves introducing the magnetic vector potential  $A(x) = (A_1(x), A_2(x))$  such that  $B = \partial_{x_1}A_2 - \partial_{x_2}A_1$  and setting

$$h^\pm = (-i\nabla - A)^2 \mp B. \quad (1.1)$$

Instead, we adopt the approach advocated in [6], which consists of defining  $h^\pm$  in terms of a solution  $\Psi = \Psi(x)$  to the differential equation  $\Delta\Psi = B$ . Assume that  $B$  is such that a solution  $\Psi$  can be chosen subject to the condition

$$\Psi(x) = \frac{B_0}{4}|x|^2 + \Psi_1(x), \quad \bar{\Psi}_1 = \Psi_1 \in L^\infty(\mathbb{R}^2), \quad B_0 > 0. \quad (1.2)$$

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Important examples of magnetic fields  $B$  of this class are periodic fields with mean value  $B_0$  and constant magnetic fields  $B(x) = B_0$ .

Next, denote, as usual,  $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$  and  $\bar{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ . Consider the quadratic forms

$$\mathfrak{h}^+[u] = 4 \int_{\mathbb{R}^2} |\bar{\partial}(e^{\Psi(x)}u(x))|^2 e^{-2\Psi(x)} dx, \quad \mathfrak{h}^-[u] = 4 \int_{\mathbb{R}^2} |\partial(e^{-\Psi(x)}u(x))|^2 e^{2\Psi(x)} dx, \quad (1.3)$$

which are closed on the domains  $\text{Dom}(\mathfrak{h}^\pm) = \{u \in L^2(\mathbb{R}^2) \mid \mathfrak{h}^\pm[u] < \infty\}$ . Let us define  $h^\pm$  as the self-adjoint operators in  $L^2(\mathbb{R}^2)$ , corresponding to the quadratic forms  $\mathfrak{h}^\pm$ . For a wide class of magnetic fields this definition is equivalent to the standard definition (1.1) with  $A = (-\partial_{x_2}\Psi, \partial_{x_1}\Psi)$ ; see [6] for a detailed analysis of this issue. In fact, the magnetic field  $B$  or the magnetic vector potential  $A$  do not enter directly either the definition of Pauli operator or any of our considerations; instead, the ‘potential function’  $\Psi$  becomes the main functional parameter. Note that condition (1.2) is very close to the ‘admissibility’ condition used in [16].

We will denote by  $\mathfrak{h}_0^\pm$  and  $h_0^\pm$  the above defined forms and operators corresponding to the case of the constant magnetic field  $B(x) = B_0 > 0$ . Note that for all  $u \in \text{Dom}(\mathfrak{h}_0^-)$  one has

$$\mathfrak{h}_0^-[u] \geq 2B_0 \|u\|^2. \quad (1.4)$$

**2. Zero modes and the spectral gap.** It is well known that Pauli operator  $h$  has infinite dimensional kernel. More precisely, by the argument dating back to [1], we have  $\text{Ker } h^- = \{0\}$  and

$$\text{Ker } h^+ = \{u \in L^2(\mathbb{R}^2) \mid u(x) = f(x)e^{-\Psi(x)}, \bar{\partial}f = 0\}, \quad \dim \text{Ker } h^+ = \infty. \quad (1.5)$$

Next, the following well known supersymmetric argument (which has appeared in many forms in the literature; see e.g. [9] or [16]) establishes the existence of a spectral gap  $(0, m)$ ,  $m > 0$  of the operator  $h$ . Let  $a_0$  and  $a_0^*$  be the annihilation and creation operators in  $L^2(\mathbb{R}^2)$ , corresponding to the constant component  $B_0 > 0$  of the magnetic field  $B$ :

$$a_0 = -2ie^{-B_0|x|^2/4} \bar{\partial} e^{B_0|x|^2/4}, \quad a_0^* = -2ie^{B_0|x|^2/4} \partial e^{-B_0|x|^2/4}. \quad (1.6)$$

Then one can define

$$a = e^{-\Psi_1} a_0 e^{\Psi_1} \text{ on } \text{Dom}(a) = \{e^{-\Psi_1} u \mid u \in \text{Dom}(a_0)\} \text{ and } a^* = e^{\Psi_1} a_0^* e^{-\Psi_1}. \quad (1.7)$$

In terms of these operators, we have

$$h^+ = a^* a \text{ and } h^- = a a^* \quad (1.8)$$

and therefore  $\sigma(h^+) \setminus \{0\} = \sigma(h^-) \setminus \{0\}$ . Finally, comparing the form  $\mathfrak{h}^-$  with  $\mathfrak{h}_0^-$  and using (1.4), one obtains (see e.g. [3] or [16, Proposition 1.2])

$$\mathfrak{h}^-[u] \geq e^{2 \text{ess inf } \Psi_1} \mathfrak{h}_0^-[e^{-\Psi_1} u] \geq 2B_0 e^{2 \text{ess inf } \Psi_1} \|e^{-\Psi_1} u\|^2 \geq 2B_0 e^{-2 \text{osc } \Psi_1} \|u\|^2 \quad (1.9)$$

for all  $u \in \text{Dom}(\mathfrak{h}^-)$ , where  $\text{osc } \Psi_1 = \text{ess sup } \Psi_1 - \text{ess inf } \Psi_1$ . It follows that  $(0, m)$ ,  $m = 2B_0 e^{-2 \text{osc } \Psi_1} > 0$ , is a gap in the spectrum of  $h^+$  and of  $h$ . See [5] for a different point of view on the issue of existence of the spectral gap and [18] for recent progress on this topic.

**3. Perturbations of the Pauli operator and spectral asymptotics.** Let  $v \in L^p(\mathbb{R}^2)$ ,  $p > 1$ , be a non-negative compactly supported function, which has the physical meaning of the electric potential. The Pauli operator which describes a particle in the external magnetic field  $B$  and the electric field with the potential  $v$  is

$$h + vI = \begin{pmatrix} h^+ + v & 0 \\ 0 & h^- + v \end{pmatrix} \quad \text{in } L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2).$$

Along with  $h + vI$ , we will also consider the operator  $h - vI$ . In order to define  $h^\pm \pm v$  as a quadratic form sum, let us establish that  $v$  is  $h^+$ -form compact. By (1.7), (1.8) and boundedness of  $\Psi_1$ , we see that  $v$  is  $h^+$ -form compact if and only if  $ve^{-2\Psi_1}$  is  $h_0^+$ -form compact. As  $ve^{-2\Psi_1} \in L^p(\mathbb{R}^2)$ ,  $p > 1$ , we obtain that  $ve^{-2\Psi_1}$  is  $h_0^+$ -form compact (see [2]). A similar argument shows that  $v$  is  $h^-$ -form compact and so the operators  $h^\pm \pm v$  are also well defined.

The main object of interest in this paper is the rate of accumulation of the eigenvalues of  $h \pm vI$  to zero. By the above established relative compactness and by the estimate (1.9), the operators  $h^\pm \pm v$  can have only finitely many eigenvalues in a sufficiently small neighbourhood of zero. Thus, the question reduces to that of the rate of accumulation of the eigenvalues of  $h^\pm \pm v$  to zero.

Due to the assumption  $v \geq 0$ , the eigenvalues of  $h^+ + v$  can accumulate to 0 only from above, and the eigenvalues of  $h^+ - v$  can do so only from below. Let  $\lambda_1^- \leq \lambda_2^- \leq \dots$  be the negative eigenvalues of  $h^+ - v$ , and  $\lambda_1^+ \geq \lambda_2^+ \geq \dots$  be the eigenvalues of  $h^+ + v$  in the spectral gap  $(0, m)$ ; here and in the rest of the paper, we assume eigenvalues to be enumerated with multiplicities taken into account. Our aim is to describe the rate of convergence  $\lambda_n^\pm \rightarrow 0$  as  $n \rightarrow \infty$ . Roughly speaking, we prove the following asymptotics (precise statements are given in Section 2):

$$\log(\pm n! \lambda_n^\pm) = n \log(B_0/2) + 2n \log \text{Cap}(\text{supp } v) + o(n), \quad n \rightarrow \infty, \quad (1.10)$$

where  $\text{Cap}$  is the *logarithmic capacity* of a set. The notion of logarithmic capacity is introduced in the framework of potential theory; see e.g. [7, 11]. Recall that the logarithmic capacity of compact sets in  $\mathbb{R}^2$  has the following properties:

- (i) if  $\Omega_1 \subset \Omega_2$  then  $\text{Cap } \Omega_1 \leq \text{Cap } \Omega_2$ ;
- (ii)  $\text{Cap } \Omega$  coincides with the logarithmic capacity of the outer boundary of  $\Omega$  (=the boundary of the unbounded component of  $\mathbb{R}^2 \setminus \Omega$ );
- (iii) the logarithmic capacity of a disc of radius  $r$  is  $r$ ;
- (iv) if  $\Omega_2 = \{\alpha x \mid x \in \Omega_1\}$ ,  $\alpha > 0$ , then  $\text{Cap } \Omega_2 = \alpha \text{Cap } \Omega_1$ .

We establish (1.10) by means of the following simple chain of equivalent reformulations of the problem. Firstly, a perturbation theory argument reduces the problem to the spectral asymptotics of an auxiliary compact self-adjoint operator  $P_0 v P_0$ , where  $P_0$  is the spectral projection of  $h^+$ , corresponding to the eigenvalue 0. Next, we observe that the eigenvalues of

$P_0 v P_0$  coincide with the singular values of a certain embedding operator (see (4.2)). Using the approach of [13], we relate the singular numbers of this embedding operator to some sequence of orthogonal polynomials in the complex domain (see below). Finally, application of the results of [22] concerning the asymptotics of these orthogonal polynomials leads to (1.10).

Using the same technique, we are also able to treat two similar problems. First, we consider the Pauli operators  $h_0^\pm \pm v$  in the case of a constant magnetic field and describe the rate of accumulation of the eigenvalues to the higher Landau levels. Secondly, we consider the three-dimensional Pauli Hamiltonian with a constant magnetic field and a compactly supported electric potential and describe the rate of convergence of eigenvalues to 0. These results are presented in Section 2.

The rate of convergence of eigenvalues to zero for Pauli operators in dimensions two and three was investigated before in the case of constant magnetic field  $B(x) = B_0 > 0$  for various classes of potentials  $v$  with power or exponential decay at infinity; see [21, 19, 23, 14, 15, 10, 17]. We refer the reader to the discussion in [17]. The case of a constant magnetic field and compactly supported potentials  $v$  was considered in [17] and [12]. The case of a two-dimensional operator with variable magnetic field and potentials  $v$  with power or exponential decay and also with compactly supported potentials was treated in [16]. The results of [17, 12, 16] for the case of compactly supported potentials read as

$$\log(\pm\lambda_n^\pm) = -n \log n + O(n), \quad n \rightarrow \infty. \quad (1.11)$$

As far as we are aware, a connection between the spectral asymptotics of magnetic operators and logarithmic capacity or orthogonal polynomials has not been made before.

Some physical intuition concerning these problems with constant magnetic field can be gained from [8].

**4. Orthogonal and Chebyshev polynomials.** We identify  $\mathbb{R}^2$  and  $\mathbb{C}$  in a standard way:  $z = x_1 + ix_2$  for  $(x_1, x_2) \in \mathbb{R}^2$ , denote by  $dm(z)$  the Lebesgue measure in  $\mathbb{C}$  and consider  $v$  as a function of  $z$ . It appears that the sequence of polynomials in  $z$ , orthogonal with respect to the measure  $v(z)dm(z)$ , is related to the asymptotics of  $\lambda_n^\pm$ . Here we recall necessary facts from the theory of Chebyshev and orthogonal polynomials in complex domains; see e.g. [7] and [22] for the details.

For any  $n = 0, 1, 2, \dots$ , let  $\mathcal{P}_n$  be the set of all monic polynomials in  $z$  of degree  $n$ :

$$\mathcal{P}_n = \{z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0 \mid a_0, \dots, a_{n-1} \in \mathbb{C}\}. \quad (1.12)$$

Let  $\Omega \subset \mathbb{C}$  be a compact set. For a fixed  $n$ , consider the problem of minimization of the norm  $\|t\|_{C(\Omega)} \equiv \sup_{z \in \Omega} |t(z)|$  on the set  $t \in \mathcal{P}_n$ . It is clear that the minimum is positive and attained at some polynomial  $t_n \in \mathcal{P}_n$ . The polynomial  $t_n$  is called the *n'th Chebyshev polynomial* for the set  $\Omega$ . (One can prove that such a polynomial is unique, but we will not need this fact). It is well known that all zeros of  $t_n$  lie in the closed convex hull of  $\Omega$ . The *n'th root asymptotics of  $t_n$*  is given by

$$\lim_{n \rightarrow \infty} \|t_n\|^{1/n} = \text{Cap } \Omega. \quad (1.13)$$

Next, let  $v \in L^1(\mathbb{C}, dm)$  be a non-negative compactly supported function. Denote

$$M_n(v) = \inf_{p \in \mathcal{P}_n} \int_{\mathbb{C}} |p(z)|^2 v(z) dm(z). \quad (1.14)$$

It is easy to prove that the infimum in (1.14) is attained at the polynomials  $\{p_n\}_{n=0}^{\infty}$ ,  $p_n \in \mathcal{P}_n$ , which can be obtained by applying the Gram–Schmidt orthogonalisation process in  $L^2(\mathbb{C}, v(z)dm(z))$  to the sequence  $1, z, z^2, \dots$ . All zeros of  $p_n$  lie in the closed convex hull of  $\text{supp } v$ .

Regarding the  $n$ 'th root asymptotics of  $p_n$ , the following facts are known (see [22]). Denote

$$\rho_+(v) = \limsup_{n \rightarrow \infty} M_n(v)^{1/n}, \quad \rho_-(v) = \liminf_{n \rightarrow \infty} M_n(v)^{1/n}. \quad (1.15)$$

In general, it can happen that  $\rho_-(v) < \rho_+(v)$  (see the proof of Theorem 1.1.9 in [22]). One has the estimates

$$\rho_+(v) \leq (\text{Cap } \text{supp } v)^2, \quad (1.16)$$

$$\rho_-(v) \geq (\text{Cap } \Omega_-(v))^2, \quad \Omega_-(v) = \{z \in \mathbb{C} \mid \limsup_{r \rightarrow +0} \frac{\log \int_{|z-\zeta| \leq r} v(\zeta) dm(\zeta)}{\log r} < \infty\}. \quad (1.17)$$

The inequality (1.16) is a part of Corollary 1.1.7 of [22]. The inequality (1.17), although not stated explicitly in [22], follows directly from the proof of Theorem 4.2.1 therein.

**Remark 1.1.** Let  $\Omega \subset \mathbb{C}$  be a compact set with a Lipschitz boundary, and let  $v \in L^1(\mathbb{C}, dm)$  be such that  $v(z) \geq c > 0$  for all  $z \in \Omega$  and  $v(z) = 0$  for all  $z \in \mathbb{C} \setminus \Omega$ . Then we easily find that  $\Omega_-(v) = \Omega = \text{supp } v$  and therefore  $\rho_+(v) = \rho_-(v) = (\text{Cap } \Omega)^2$ .

## 2 Main results

**1. Two-dimensional Pauli operators with variable magnetic field.** Let  $h^+$ , as in the Introduction, be the Pauli operator defined via (1.3) with  $\Psi$  subject to (1.2). Let  $v$  and  $\lambda_n^{\pm}$  be as in the Introduction.

**Theorem 2.1.** *Let  $0 \leq v \in L^p(\mathbb{R}^2)$ ,  $p > 1$ , be a compactly supported potential and let  $M_n(v)$  be as defined in (1.14). Then there exists  $k \in \mathbb{N}$  such that*

$$(B_0/2)M_{n+k}(v)^{1/n}(1+o(1)) \leq (n!\lambda_n^+)^{1/n} \leq (B_0/2)M_{n-1}(v)^{1/n}(1+o(1)), \quad (2.1)$$

$$(B_0/2)M_{n-1}(v)^{1/n}(1+o(1)) \leq (-n!\lambda_n^-)^{1/n} \leq (B_0/2)M_{n-k}(v)^{1/n}(1+o(1)), \quad (2.2)$$

as  $n \rightarrow \infty$ . In particular,

$$\limsup_{n \rightarrow \infty} (\pm n!\lambda_n^{\pm})^{1/n} = B_0 \rho_{\pm}(v)/2, \quad \liminf_{n \rightarrow \infty} (\pm n!\lambda_n^{\pm})^{1/n} = B_0 \rho_{\pm}(v)/2,$$

where  $\rho_{\pm}(v)$  are defined by (1.15). If  $v$  is of the class described in Remark 1.1, then the asymptotics (1.10) holds true.

**Remark 2.2.** Let  $\mu$  be a compactly supported finite measure in  $\mathbb{R}^2$  such that the quadratic form  $\int_{\mathbb{C}} |u(x)|^2 d\mu(x)$  is compact with respect to the quadratic form  $\mathfrak{h}^+$ . Then one can define the self-adjoint operators corresponding to the quadratic forms

$$\mathfrak{h}^+[u] \pm \int_{\mathbb{R}^2} |u(x)|^2 d\mu(x).$$

All our considerations remain valid for such operators. For example, the case of a measure  $\mu$ , supported by a curve, can be interesting.

**2. Two-dimensional Pauli operators with constant magnetic field.** Let  $B(x) = B_0 > 0$ ; consider the corresponding operator  $h_0^+$ . As it is well known, the spectrum of  $h_0^+$  consists of the eigenvalues  $\{2qB_0\}_{q=0}^{\infty}$  of infinite multiplicities; these eigenvalues are known as Landau levels. Consider the problem of accumulation of eigenvalues of  $h_0^+ \pm v$  to a fixed higher Landau level  $2qB_0$ ,  $q \geq 1$ . Let  $\lambda_{q,1}^- \leq \lambda_{q,2}^- \leq \dots$  be the eigenvalues of  $h_0^+ - v$  in the interval  $(2(q-1)B_0, 2qB_0)$ , and let  $\lambda_{q,1}^+ \geq \lambda_{q,2}^+ \geq \dots$  be the eigenvalues of  $h_0^+ + v$  in  $(2qB_0, 2(q+1)B_0)$ .

**Theorem 2.3.** Let  $\Omega \subset \mathbb{R}^2$  be a compact set with Lipschitz boundary and let  $v \in L^p(\mathbb{R}^2)$ ,  $p > 1$ , be such that  $v(x) \geq c > 0$  for  $x \in \Omega$  and  $v(x) = 0$  for  $x \in \mathbb{R}^2 \setminus \Omega$ . Then for the corresponding eigenvalues  $\lambda_{q,n}^{\pm}$  we have:

$$\lim_{n \rightarrow \infty} (\pm n! (\lambda_{q,n}^{\pm} - 2qB_0))^{1/n} = \frac{B_0}{2} (\text{Cap } \Omega)^2.$$

The rate of convergence of  $\lambda_{q,n}^{\pm} \rightarrow 2qB_0$  as  $n \rightarrow \infty$ ,  $q \geq 1$ , was studied before in [17, 12], where the asymptotics

$$\log(\pm (\lambda_{q,n}^{\pm} - 2qB_0)) = -n \log n + O(n), \quad n \rightarrow \infty$$

was obtained. Note that if the potential  $v$  depends only on  $|x|$ , then the result of Theorem 2.3 can be obtained by a direct calculation using separation of variables, see e.g. [17, Proposition 3.2].

**3. Three-dimensional Pauli operator with a constant magnetic field.** Let

$$H = (-i\nabla - \mathbf{A}(\mathbf{x}))^2 - B_0 \text{ in } L^2(\mathbb{R}^3, d\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, x_3),$$

where  $\mathbf{A}(\mathbf{x}) = (-\frac{1}{2}B_0x_2, \frac{1}{2}B_0x_1, 0)$ . It is well known that the spectrum of  $H$  is absolutely continuous and coincides with the interval  $[0, \infty)$ . The background information concerning the spectral theory of  $H$  and its perturbations can be found in [2].

Let  $V \in L^{3/2}(\mathbb{R}^3)$  be a non-negative compactly supported potential. The operator of multiplication by  $V$  in  $L^2(\mathbb{R}^3)$  is  $H$ -form compact (cf. [2]). Thus, one can define the self-adjoint operator  $H - V$  via the corresponding quadratic form; the essential spectrum of  $H - V$  is also  $[0, \infty)$ . Let  $\Lambda_1 \leq \Lambda_2 \leq \dots$  be the negative eigenvalues of  $H - V$ ; we have  $\Lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ . Below we describe the asymptotic behaviour of  $\Lambda_n$  as  $n \rightarrow \infty$  in terms of the auxiliary weight function

$$w(x_1, x_2) = \int_{-\infty}^{\infty} V(x_1, x_2, x_3) dx_3, \quad \text{a.e. } (x_1, x_2) \in \mathbb{R}^2. \quad (2.3)$$

As above, we consider  $w$  as a function of  $z = x_1 + ix_2$ .

**Theorem 2.4.** *Let  $0 \leq V \in L^{3/2}(\mathbb{R}^3)$  be a compactly supported potential and  $w$  be defined by (2.3). Then there exists  $k \in \mathbb{N}$  such that*

$$(B_0/2)^2 M_{n+k}(w)^{2/n}(1+o(1)) \leq (-(n!)^2 \Lambda_n)^{1/n} \leq (B_0/2)^2 M_{n-k}(w)^{2/n}(1+o(1)), \quad (2.4)$$

as  $n \rightarrow \infty$ . In particular,

$$\limsup_{n \rightarrow \infty} (-(n!)^2 \Lambda_n)^{1/n} = (B_0 \rho_+(w)/2)^2, \quad \liminf_{n \rightarrow \infty} (-(n!)^2 \Lambda_n)^{1/n} = (B_0 \rho_-(w)/2)^2,$$

where  $\rho_{\pm}(w)$  are defined by (1.15).

The rate of accumulation  $\Lambda_n \rightarrow 0$  for potentials  $V$  with power or exponential decay was considered before in [21, 19, 23, 14, 15, 10, 17]. For compactly supported potentials, this problem was considered in [17, 12], where the asymptotics

$$\log(-\Lambda_n) = -2n \log n + O(n), \quad n \rightarrow \infty$$

was obtained.

**Remark 2.5.** Theorem 2.4 remains valid under the following assumptions on  $V$ : (i)  $V \geq 0$ ,  $V$  is  $H$ -form compact; (ii)  $\int_{\mathbb{R}^3} V(\mathbf{x})(1+|x_3|^2)d\mathbf{x} < \infty$ ; (iii) the function  $w$ , defined by (2.3), is compactly supported.

### 3 Proof of Theorems 2.1, 2.3 and 2.4

*Proof of Theorem 2.1:* Let  $\mathcal{H}_0 \subset L^2(\mathbb{R}^2)$  be the kernel of  $h^+$ , and let  $P_0$  be the corresponding eigenprojection,  $\text{Ran } P_0 = \mathcal{H}_0$ . Consider the compact self-adjoint operator  $P_0 v P_0$ . The key ingredient in the proof is the following

**Lemma 3.1.** *Let  $v \in L^1(\mathbb{R}^2)$  be a non-negative compactly supported function and let  $s_1 \geq s_2 \geq \dots > 0$  be the eigenvalues of  $P_0 v P_0$ . Then*

$$((n+1)! s_{n+1})^{1/n} = (B_0/2) M_n(v)^{1/n} (1+o(1)), \quad n \rightarrow \infty, \quad (3.1)$$

where  $M_n(v)$  are defined by (1.14).

The proof is given in Section 4. Now it remains to employ a perturbation theory argument (see [16, Proposition 3.1] or [17, Proposition 4.1]) based on the Birman-Schwinger principle and on Weyl inequalities for eigenvalues of a sum of compact operators. This argument shows that there exists  $k \in \mathbb{N}$  such that for all sufficiently large  $n \in \mathbb{N}$  one has

$$s_n \leq -\lambda_n^- \leq 2s_{n-k}, \quad \frac{1}{2}s_{n+k} \leq \lambda_n^+ \leq s_n.$$

Combining these inequalities with Lemma 3.1, we obtain the required result. ■

*Proof of Theorem 2.3:* For any  $q \geq 0$ , denote  $\mathcal{H}_q = \text{Ker}(h^+ - 2qB_0)$  and let  $P_q$  be the eigenprojection of  $h_0^+$  corresponding to the eigenvalue  $2qB_0$ . Consider the compact self-adjoint operator  $P_q v P_q$ , and let  $s_1^{(q)} \geq s_2^{(q)} \geq \dots$  be the eigenvalues of this operator. As in the proof of Theorem 2.1, using a perturbation theory argument based on the Birman-Schwinger principle and on Weyl inequalities (see [17, Proposition 4.1]), one shows that there exists  $k \in \mathbb{N}$  such that for all sufficiently large  $n \in \mathbb{N}$ ,

$$\frac{1}{2}s_{n+k}^{(q)} \leq \pm(\lambda_{q,n}^\pm - 2qB_0) \leq 2s_{n-k}^{(q)}. \quad (3.2)$$

Now the proof of Theorem 2.3 reduces to

**Lemma 3.2.** *Let  $\Omega \subset \mathbb{R}^2$  be a compact set with Lipschitz boundary and let  $v \in L^1(\mathbb{R}^2)$  be such that  $v(x) \geq c > 0$  for  $x \in \Omega$  and  $v(x) = 0$  for  $x \in \mathbb{R}^2 \setminus \Omega$ . Fix  $q \in \mathbb{N}$  and let  $s_1^{(q)} \geq s_2^{(q)} \geq \dots$  be the eigenvalues of  $P_q v P_q$ . Then one has*

$$\lim_{n \rightarrow \infty} (n! s_n^{(q)})^{1/n} = (B_0/2)(\text{Cap } \Omega)^2. \quad (3.3)$$

The proof of Lemma 3.2 is given in Section 5. From Lemma 3.2 and the estimate (3.2), we immediately obtain the required result. ■

*Proof of Theorem 2.4:* The proof repeats almost word for word the construction of [19]. According to the Birman-Schwinger principle, for  $E > 0$  we have:

$$\#\{n \mid \Lambda_n < -E\} = n_+(1; \sqrt{V}(H_0 + E)^{-1}\sqrt{V}). \quad (3.4)$$

The operator  $\sqrt{V}(H_0 + E)^{-1}\sqrt{V}$  can be represented as

$$\sqrt{V}(H_0 + E)^{-1}\sqrt{V} = \frac{1}{2\sqrt{E}}K_1 + K_2 + K_3.$$

Here  $K_1, K_2$  are the operators in  $L^2(\mathbb{R}^3)$  with the integral kernels

$$\begin{aligned} K_1(\mathbf{x}, \mathbf{y}) &= \sqrt{V(\mathbf{x})}P_0(x_\perp, y_\perp)\sqrt{V(\mathbf{y})}, \\ K_2(\mathbf{x}, \mathbf{y}) &= \sqrt{V(\mathbf{x})}P_0(x_\perp, y_\perp)\frac{e^{-\sqrt{E}|x_3-y_3|} - 1}{2\sqrt{E}}\sqrt{V(\mathbf{y})}, \end{aligned}$$

where the notation  $x_\perp = (x_1, x_2)$ ,  $y_\perp = (y_1, y_2)$  is used, and  $P_0(x_\perp, y_\perp)$  is the integral kernel of the operator  $P_0$  in  $L^2(\mathbb{R}^2)$ . Finally,  $K_3$  is the operator

$$K_3 = \sqrt{V}Q_0(H_0 + E)^{-1}\sqrt{V},$$

where  $Q_0 = (I - P_0) \otimes I$  in the decomposition  $L^2(\mathbb{R}^3, dx_1 dx_2 dx_3) = L^2(\mathbb{R}^2, dx_1 dx_2) \otimes L^2(\mathbb{R}, dx_3)$ .

The operators  $K_2$  and  $K_3$  have limits (in the operator norm) as  $E \rightarrow +0$ ; these limits are compact self-adjoint operators. Thus, by the Weyl's inequalities for eigenvalues (see e.g. [4]), we have for  $E \rightarrow +0$ :

$$n_+(1; \sqrt{V}(H_0 + E)^{-1}\sqrt{V}) \leq n_+(\frac{1}{2}; \frac{1}{2\sqrt{E}}K_1) + n_+(\frac{1}{2}; K_2 + K_3) \leq n_+(\sqrt{E}; K_1) + O(1), \quad (3.5)$$

$$n_+(1; \sqrt{V}(H_0 + E)^{-1}\sqrt{V}) \geq n_+(\frac{3}{2}; \frac{1}{2\sqrt{E}}K_1) - n_+(\frac{1}{2}; -K_2 - K_3) \geq n_+(3\sqrt{E}; K_1) - O(1). \quad (3.6)$$

Finally, again as in [19], let us prove that the non-zero eigenvalues of  $K_1$  coincide with those of  $P_0wP_0$ , where  $w$  is defined by (2.3). It suffices to prove this statement for continuous  $V$  with compact support; the general case  $V \in L^{3/2}$  then follows by approximation argument. Let  $N_1 : L^2(\mathbb{R}^3, dx_1 dx_2 dx_3) \rightarrow L^2(\mathbb{R}^2, dx_1 dx_2)$  and  $N_2 : L^2(\mathbb{R}^2, dx_1 dx_2) \rightarrow L^2(\mathbb{R}^3, dx_1 dx_2 dx_3)$  be the following operators:

$$(N_1u)(x_1, x_2) = \int_{-\infty}^{\infty} V^{1/2}(x_1, x_2, x_3)u(x_1, x_2, x_3)dx_3, \\ (N_2u)(x_1, x_2, x_3) = V^{1/2}(x_1, x_2, x_3)u(x_1, x_2).$$

Then  $K_1 = N_2P_0N_1 = (N_2P_0)(P_0N_1)$  and  $P_0wP_0 = (P_0N_1)(N_2P_0)$ . It follows that the non-zero eigenvalues of  $K_1$  coincide with  $\{s_n\}$ , the non-zero eigenvalues of  $P_0wP_0$ , and so  $n_+(\sqrt{E}; K_1) = \#\{n \mid (s_n)^2 > E\}$ . From here and (3.4), (3.5), (3.6) it follows that for some  $k \in \mathbb{N}$  and all sufficiently large  $n \in \mathbb{N}$ , one has

$$\frac{1}{9}(s_{n+k})^2 \leq \Lambda_n \leq (s_{n-k})^2. \quad (3.7)$$

Combining this with Lemma 3.1, we get the statement of Theorem 2.4. ■

## 4 Proof of Lemma 3.1

First let us consider the case of a constant magnetic field  $B(x) = B_0 > 0$ . Let  $F^2$  be the Hilbert space of all entire functions  $f$  such that

$$\|f\|_{F^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-B_0|z|^2/2} dm(z) < \infty. \quad (4.1)$$

In the case  $B_0 = 2$ , the space  $F^2$  is usually called Fock space or Segal-Bargmann space. By (1.5), we have an isometry between  $\mathcal{H}_0 = \text{Ker } h^+ \subset L^2(\mathbb{C}, dm)$  and  $F^2$ , given by  $u(z) = e^{-B_0|z|^2/4}f(z)$ ,  $u \in \mathcal{H}_0$ ,  $f \in F^2$ . Thus, the quadratic form of the operator  $P_0vP_0|_{\mathcal{H}_0}$  is unitarily equivalent to the quadratic form

$$\int_{\mathbb{C}} |f(z)|^2 v(z) e^{-B_0|z|^2/2} dm(z), \quad f \in F^2.$$

It follows that the non-zero eigenvalues  $s_n$  of  $P_0 v P_0$  coincide with the singular values  $\mu_n$  of the embedding operator

$$F^2 \subset L^2(\mathbb{C}, v(z) e^{-B_0 |z|^2/2} dm(z)). \quad (4.2)$$

The case of a variable magnetic field can be also reduced to the embedding (4.2). Indeed, if  $s_n$  correspond to the case of a variable magnetic field, then, using the boundedness of  $\Psi_1$ , one obtains (see [16, Proposition 3.2]):

$$\mu_n e^{-2 \operatorname{osc} \Psi_1} \leq s_n \leq \mu_n e^{2 \operatorname{osc} \Psi_1}, \quad n \in \mathbb{N}.$$

Thus, it remains to prove the asymptotic formula

$$((n+1)! \mu_{n+1})^{1/n} = (B_0/2) M_n(v)^{1/n} (1 + o(1)), \quad n \rightarrow \infty \quad (4.3)$$

for the singular values  $\mu_n$  of the embedding (4.2). We shall assume  $B_0 = 2$ ; the general case can be reduced to this one by a linear change of coordinates.

The embedding  $F^2 \subset C(\Omega)$ , where  $\Omega$  is a compact set in  $\mathbb{C}$ , was studied in [13]. Below we repeat the arguments of [13] (with trivial modifications) to obtain the required asymptotics.

By the minimax principle, we have the following variational characterisation of  $\mu_n$ :

$$\mu_{n+1} = \inf_{L_n^+ \subset F^2} \sup_{f \in L_n^+ \setminus \{0\}} \frac{\int_{\mathbb{C}} |f(z)|^2 v(z) e^{-|z|^2} dm(z)}{\|f\|_{F^2}^2}, \quad \operatorname{codim} L_n^+ = n, \quad (4.4)$$

$$\mu_{n+1} = \sup_{L_n^- \subset F^2} \inf_{f \in L_n^- \setminus \{0\}} \frac{\int_{\mathbb{C}} |f(z)|^2 v(z) e^{-|z|^2} dm(z)}{\|f\|_{F^2}^2}, \quad \dim L_n^- = n + 1. \quad (4.5)$$

### 1. Upper bound on $\mu_{n+1}$ .

1. For the subspaces  $L_n^+$  from (4.4), we will take

$$L_n^+ = \{f \in F^2 \mid f(z) = p_n(z)g(z), \text{ } g \text{ is entire function}\},$$

where  $p_n$  is the sequence of monic polynomials orthogonal with respect to the measure  $v(z)dm(z)$ .

In order to estimate the ratio in (4.4) from above, let us prove the following auxiliary statement. Denote  $R_0 = \max_{z \in \operatorname{supp} v} |z|$ . We claim that for any  $\varepsilon \in (0, \frac{1}{3})$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  and any  $f = p_n g \in L_n^+$ , we have

$$\sup_{|z| \leq R_0} |g(z)|^2 \leq (1 - \varepsilon)^{-2n} \frac{1}{(n+1)!} \|p_n g\|_{F^2}^2. \quad (4.6)$$

Indeed, we have

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{p_n(\zeta)(\zeta - z)} d\zeta, \quad r > R_0,$$

and therefore

$$\sup_{|z| \leq R_0} |g(z)|^2 \leq \frac{r}{2\pi} \sup_{|z| \leq R_0} \int_{|\zeta|=r} \frac{|f(\zeta)|^2}{|p_n(\zeta)|^2 |\zeta - z|^2} d|\zeta|$$

for any  $r > R_0$ . Denote  $R = R_0/\varepsilon$ . Since all zeros of  $p_n$  lie in the closed convex hull of  $\text{supp } v$ , we obtain:

$$|p_n(\zeta)| |\zeta - z| \geq ((1 - \varepsilon)r)^{n+1}, \quad |z| \leq R_0, \quad |\zeta| = r \geq R.$$

Thus, we get

$$\sup_{|z| \leq R_0} |g(z)|^2 \leq \frac{r^{-2n-1}}{2\pi(1 - \varepsilon)^{2n+2}} \int_{|\zeta|=r} |f(\zeta)|^2 d|\zeta|, \quad r \geq R.$$

Integrating the last inequality over  $r$  from  $R$  to  $\infty$  with the weight  $e^{-r^2} r^{2n+1}$ , and using the fact that

$$\int_R^\infty e^{-r^2} r^{2n+1} dr = \frac{1}{2}(n+1)! - \int_0^R e^{-r^2} r^{2n+1} dr \geq \frac{1}{2\pi}(1 - \varepsilon)^{-2}(n+1)!$$

for all sufficiently large  $n$ , we obtain (4.6).

2. From (4.6) we obtain for any  $f = p_n g \in L_n^+$ :

$$\int_{\mathbb{C}} |f(z)|^2 v(z) e^{-|z|^2} dm(z) \leq M_n(v) \|g\|_{C(\text{supp } v)}^2 \leq M_n(v) (1 - \varepsilon)^{-2n} \frac{1}{(n+1)!} \|f\|_{F^2}^2.$$

Together with (4.4), the last estimate yields

$$((n+1)! \mu_{n+1})^{1/n} \leq (1 - \varepsilon)^{-2} M_n(v)^{1/n} \quad (4.7)$$

for all sufficiently large  $n$ .

**2. Lower bound for  $\mu_{n+1}$ .** Let us use formula (4.5) and take  $L_n^-$  to be the set of all polynomials in  $z$  of degree  $\leq n$ . As in the proof of the upper bound, we denote  $R_0 = \max_{z \in \text{supp } v} |z|$ , fix  $\varepsilon > 0$  and set  $R = R_0/\varepsilon$ . We shall use the following norm in  $F^2$ :

$$\|f\|_{F^2}^2 = \int_{|z| \geq R} |f(z)|^2 e^{-|z|^2} dm(z). \quad (4.8)$$

This norm is equivalent to the earlier introduced norm (4.1). Indeed, the inequality  $\|f\|_{F^2} \leq \|f\|_{F^2}$  is trivial; the inequality  $\|f\|_{F^2} \leq C(R) \|f\|_{F^2}$  is easy to obtain by application of the Cauchy integral formula.

Let  $q_n \in L_n^- \setminus \{0\}$  be the polynomial which minimizes the ratio

$$\frac{\int_{\mathbb{C}} |q_n(z)|^2 v(z) e^{-|z|^2} dm(z)}{\|q_n\|_{F^2}^2} \quad (4.9)$$

among all polynomials in  $L_n^- \setminus \{0\}$ . Next, without the loss of generality, we may assume that  $q_n$  is monic. Denote  $k = \deg q_n \leq n$ .

The following standard argument shows that all zeros of  $q_n$  are confined to the disk  $\{z \mid |z| \leq R_0\}$ . Suppose that one of the zeros  $z_0$  is outside the disk; then replace  $q_n(z)$  by  $q_n(z)|z_0|(z - R_0^2/\bar{z}_0)/(R_0(z - z_0))$ . One has

$$\frac{|z_0||z - R_0^2/\bar{z}_0|}{R_0|z - z_0|} \leq 1 \text{ for } |z| \leq R_0 \text{ and } \frac{|z_0||z - R_0^2/\bar{z}_0|}{R_0|z - z_0|} \geq 1 \text{ for } |z| \geq R_0,$$

so this change decreases the ratio (4.9) — contradiction. Thus, we get the estimate

$$|q_n(z)| \leq (1 + \varepsilon)^k |z|^k, \quad |z| \geq R, \quad k = \deg q_n.$$

It follows that

$$\|q_n\|_{F^2}^2 = \int_{|z| \geq R} |q_n(z)|^2 e^{-|z|^2} dm(z) \leq \int_{|z| \geq R} |z|^{2k} (1 + \varepsilon)^{2k} e^{-|z|^2} dm(z) \leq (1 + \varepsilon)^{2k} \pi (k + 1)!.$$

On the other hand, for the numerator of (4.9), we have

$$\int_{\mathbb{C}} |q_n(z)|^2 v(z) e^{-|z|^2} dm(z) \geq e^{-R_0^2} \int_{\mathbb{C}} |q_n(z)|^2 v(z) dm(z) \geq e^{-R_0^2} M_k(v).$$

Combining the above estimates, we obtain:

$$\mu_{n+1} \geq \inf_{f \in L_n^- \setminus \{0\}} \frac{\int_{\mathbb{C}} |f(z)|^2 v(z) e^{-|z|^2} dm(z)}{C(R) \|f\|_{F^2}^2} \geq \min_{0 \leq k \leq n} \frac{M_k(v)}{C_1(R) (1 + \varepsilon)^{2k} (k + 1)!}. \quad (4.10)$$

As  $zp_k(z) \in \mathcal{P}_{k+1}$ , from the definition (1.14) of  $M_k(v)$  we get a trivial estimate  $M_{k+1}(v) \leq R_0^2 M_k(v)$ . This estimate shows that for a sufficiently large  $n$ , the minimum in (4.10) is attained at  $k = n$ . Therefore,

$$((n + 1)! \mu_{n+1})^{1/n} \geq \left( \frac{1}{C_1(R)} \right)^{1/n} \frac{M_n(v)^{1/n}}{(1 + \varepsilon)^2} \geq (1 + \varepsilon)^{-3} M_n(v)^{1/n}$$

for all sufficiently large  $n$ . The latter estimate together with (4.7) completes the proof of the Lemma. ■

## 5 Proof of Lemma 3.2

First recall some well known facts concerning the spectral decomposition of the operator  $h_0^+$ . As above, we use the notation  $\mathcal{H}_q = \text{Ker}(h^+ - 2qB_0)$  and  $P_q$  is the orthogonal projection in  $L^2(\mathbb{C}, dm)$  onto the subspace  $\mathcal{H}_q$ . The operator  $h_0^+$  can be represented in terms of the annihilation and creation operators (1.6) as  $h_0^+ = a_0^* a_0$ . The operators  $a_0, a_0^*$  obey the commutation relation  $[a_0, a_0^*] = 2B_0$ , wherefrom we get the identity  $a_0^q (a_0^*)^q u = (2B_0)^q q! u$  for all  $u \in \mathcal{H}_0$  and  $q \in \mathbb{N}$ . It follows that

$$(2B_0)^{-q/2} (q!)^{-1/2} (a_0^*)^q : \mathcal{H}_0 \rightarrow \mathcal{H}_q \text{ is an isometry onto } \mathcal{H}_q. \quad (5.1)$$

Recalling the explicit isomorphism between  $\mathcal{H}_0$  and the space  $F^2$  (see the previous section), we see that the change  $u = (2B_0)^{-q/2} (q!)^{-1/2} (a_0^*)^q (e^{-B_0|z|^2/4} f(z))$  gives a unitary equivalence between the operator  $P_q v P_q$  and the operator in  $F^2$  defined by the quadratic form

$$(2B_0)^{-q} (q!)^{-1} \int_{\mathbb{C}} |(a_0^*)^q (e^{-B_0|z|^2/4} f(z))|^2 v(z) dm(z).$$

We will consider the case  $B_0 = 2$ ; the general case can be reduced to this one by a linear change of variables. With this simplification, the above quadratic form becomes

$$(q!)^{-1} \int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 v(z) e^{-|z|^2} dm(z), \quad f \in F^2. \quad (5.2)$$

Let us prove the asymptotics (3.3) for the eigenvalues  $\{s_n^{(q)}\}_{n=1}^{\infty}$  corresponding to the form (5.2).

**1. Upper bound for  $s_n^{(q)}$ .** Let  $\Omega_\delta = \{z \mid \text{dist}(z, \Omega) \leq \delta\}$ . By the Cauchy integral formula, we have

$$\sup_{\Omega} |f^{(j)}| \leq \frac{j!}{\delta^j} \sup_{\Omega_\delta} |f|, \quad f \in F^2, \quad j \in \mathbb{N}.$$

Thus, we have the following bound for the form (5.2):

$$(q!)^{-1} \int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 v(z) e^{-|z|^2} dm(z) \leq C \|f\|_{C(\Omega_\delta)}^2,$$

where  $C$  depends on  $\Omega$ ,  $\delta$ ,  $v$ ,  $q$ . Let us define

$$L_n^+ = \{f \in F^2 \mid f(z) = t_n(z)g(z), g \text{ entire}\},$$

where  $t_n$  is the  $n$ 'th Chebyshev polynomial for the set  $\Omega_\delta$ . Note that the proof of (4.6) uses only the fact that all zeros of  $p_n$  lie in the closed convex hull of  $\text{supp } v$ . Therefore, the same estimate remains valid with the change  $p_n \mapsto t_n$ . Thus, we get

$$\begin{aligned} (q!)^{-1} \int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 v(z) e^{-|z|^2} dm(z) &\leq C \|f\|_{C(\Omega_\delta)}^2 \leq C \|t_n\|_{C(\Omega_\delta)}^2 \|g\|_{C(\Omega_\delta)}^2 \\ &\leq C(1 - \varepsilon)^{-2n} \frac{1}{(n+1)!} \|t_n\|_{C(\Omega_\delta)}^2 \|f\|_{F^2}^2, \quad f \in L_n^+, \end{aligned}$$

for all sufficiently large  $n$ . This yields

$$\limsup_{n \rightarrow \infty} (n! s_n^{(q)})^{1/n} \leq (1 - \varepsilon)^{-2} \lim_{n \rightarrow \infty} \|t_n\|_{C(\Omega_\delta)}^{2/n} = (1 - \varepsilon)^{-2} (\text{Cap } \Omega_\delta)^2.$$

It remains to note that  $\varepsilon$  and  $\delta$  can be chosen arbitrary small and that  $\lim_{\delta \rightarrow +0} \text{Cap } \Omega_\delta = \text{Cap } \Omega$  for any compact set  $\Omega$  (see e.g. [7]).

## 2. Lower bound for $s_n^{(q)}$ .

1. Due to the compactness of the embedding of the Sobolev space  $W_2^1(\Omega) \subset L^2(\Omega)$ , for any  $\gamma > 0$  there exists a subspace in  $W_2^1(\Omega)$  of a finite codimension such that for all elements  $u$  in this subspace we have  $\|u\|_{L^2(\Omega)} \leq \gamma \|\nabla u\|_{L^2(\Omega)}$ . It follows that for any  $\gamma > 0$  there exists a subspace in  $F^2$  of a finite codimension such that for any element of this subspace we have  $\|f\|_{L^2(\Omega)} \leq \gamma \|f'\|_{L^2(\Omega)}$ . Arguing by induction, we see that for any  $\gamma > 0$  there exists a subspace  $N = N(\gamma, q) \subset F^2$  of a finite codimension  $l$  such that

$$\|\partial^{k-1} f\|_{L^2(\Omega)} \leq \gamma \|\partial^k f\|_{L^2(\Omega)}, \quad \forall f \in N(\gamma, q), \quad \forall k = 1, 2, \dots, q. \quad (5.3)$$

2. We need to estimate the form (5.2) from below. Using the assumption  $v(z) \geq c > 0$ ,  $z \in \Omega$ , on the first step and the triangle inequality on the second step, we obtain

$$\begin{aligned} \int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 v(z) e^{-|z|^2} dm(z) &\geq C \|(\partial - \bar{z})^q f\|_{L^2(\Omega)}^2 \\ &\geq C \left( \|\partial^q f\|_{L^2(\Omega)} - \sum_{k=1}^q \binom{q}{k} \|(-\bar{z})^k \partial^{q-k} f\|_{L^2(\Omega)} \right)^2 \\ &\geq C \left( \|\partial^q f\|_{L^2(\Omega)} - \sum_{k=1}^q \binom{q}{k} R_0^k \|\partial^{q-k} f\|_{L^2(\Omega)} \right)^2, \end{aligned} \quad (5.4)$$

where  $R_0 = \max_{z \in \Omega} |z|$ . From this estimate it is clear that by choosing  $\gamma$  sufficiently small we can ensure that for all  $f$  in the corresponding subspace  $N(\gamma, q)$  (see (5.3)) we have

$$\int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 v(z) e^{-|z|^2} dm(z) \geq \frac{C}{2} \|\partial^q f\|_{L^2(\Omega)}^2 \geq \frac{C}{2} \gamma^q \|f\|_{L^2(\Omega)}^2.$$

3. Now, as in the proof of Lemma 3.1, let  $L_n^-$  be the set of all polynomials in  $z$  of degree  $\leq n$ . Consider the subspace  $\tilde{L}_n^- = L_n^- \cap N(\gamma, q)$ ; clearly,  $\dim \tilde{L}_n^- \geq n + 1 - l$ , where  $l = \text{codim } N(\gamma, q)$ . Next, again as in the proof of Lemma 3.1 (see (4.10) and the following argument), we obtain for all sufficiently large  $n$ :

$$s_{n+1-l}^{(q)} \geq \frac{\int_{\mathbb{C}} |(\partial - \bar{z})^q f(z)|^2 e^{-|z|^2} v(z) dm(z)}{\|f\|_{F^2}^2} \geq C' \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{F^2}^2} \geq C'' \frac{M_n(\chi_\Omega)}{(1 + \varepsilon)^{2n} (n + 1)!}, \quad f \in \tilde{L}_n^-,$$

where  $\chi_\Omega$  denotes the characteristic function of the set  $\Omega$  in  $\mathbb{C}$ . As stated in Remark 1.1,  $\lim_{n \rightarrow \infty} M_n(\chi_\Omega)^{1/n} = (\text{Cap } \Omega)^2$ . Thus,

$$\liminf_{n \rightarrow \infty} (n! s_n^{(q)})^{1/n} \geq \frac{(\text{Cap } \Omega)^2}{(1 + \varepsilon)^2}$$

for any  $\varepsilon > 0$ . ■

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