The spectral shift function and the invariance principle

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The new representation formula for the spectral shift function due to F. Gesztesy and K. A. Makarov is considered. This formula is extended to the case of relatively trace class perturbations. The proof is based on the analysis of a certain new unitary invariant for a pair of self-adjoint operators.

Key Words: Spectral shift function, scattering matrix, Birman-Krein formula, invariance principle, spectral flow

1. INTRODUCTION

1. First we briefly recall the definition of the spectral shift function (SSF). For the details and references to the literature, see [7, 26].

Let H_0 and H be self-adjoint operators in a Hilbert space $\mathcal{H}$, and let their difference belong to the trace class:

$$H - H_0 \in \mathfrak{S}_1. \tag{1.1}$$

Then there exists a unique function $\xi(\cdot; H, H_0) \in L_1(\mathbb{R})$, such that the following trace formula holds [15]:

$$\operatorname{Tr}(\varphi(H) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda)\xi(\lambda; H, H_0)d\lambda, \quad \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{1.2}$$

The function ξ is called the SSF for the pair H_0, H.

Let $\Delta_{H/H_0}(z) = \det((H-zI)(H_0-zI)^{-1})$, $\operatorname{Im} z > 0$, be the perturbation determinant of the pair H_0, H. The following Krein's formula expresses the SSF in terms of Δ_{H/H_0}:

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{y\to+0} \arg \Delta_{H/H_0}(\lambda + iy), \quad \text{a.e. } \lambda \in \mathbb{R}, \tag{1.3}$$

where the branch of the argument is fixed by the condition

$$\lim_{y\to+\infty} \arg \Delta_{H/H_0}(\lambda + iy) = 0. \tag{1.4}$$

The Birman-Krein formula [6] relates the SSF to the scattering matrix $\mathcal{S}(\lambda; H, H_0)$ for the pair H_0, H (for the definition of the scattering matrix, see, e.g., [26]):

$$\det \mathcal{S}(\lambda; H, H_0) = \exp(-2\pi i \xi(\lambda; H, H_0)), \tag{1.5}$$

for a.e. λ on the absolutely continuous spectrum of H_0.

2. In [11], a new representation for the SSF has been found. In order to write down this representation, let us present the perturbation $V := H - H_0$ in the factorized form $V = G^* J G$, where G is a Hilbert-Schmidt operator, and $J = J^* = J^{-1} = \operatorname{sign} V$. Further, denote

$$\begin{aligned} A(\lambda + i0) &:= \lim_{y\to+0} \operatorname{Re}\big(G(H_0 - (\lambda + iy)I)^{-1} G^*\big), \\ B(\lambda + i0) &:= \lim_{y\to+0} \operatorname{Im}\big(G(H_0 - (\lambda + iy)I)^{-1} G^*\big). \end{aligned} \tag{1.6}$$

Note that the limits in (1.6) exist for a.e. $\lambda \in \mathbb{R}$ in the operator norm (and even in the norm of the Schatten-von Neumann ideal $\mathfrak{S}_p$ for any $p > 1$ — see [5, 18, 19]).

The representation of [11, Theorem 5.4] reads as follows:

$$\begin{aligned} &\xi(\lambda; H, H_0) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} \operatorname{index}\big(E_{J+A(\lambda+i0)+tB(\lambda+i0)}((-\infty, 0)), E_J((-\infty, 0))\big), \\ &\qquad\qquad \text{a.e. } \lambda \in \mathbb{R}. \end{aligned} \tag{1.7}$$

Here $E_M(\cdot)$ stands for the spectral projection of a self-adjoint operator M, and $\operatorname{index}(\cdot, \cdot)$ denotes the index of a Fredholm pair of projections (see (2.3) below). In the special case of perturbations of a definite sign (where $J = \pm I$) the formula (1.7) was originally found in [20]. In its turn, [20] used as a starting point the paper [25], where the case $J = \pm I$ and $\lambda \in \mathbb{R} \setminus (\sigma(H) \cup \sigma(H_0))$ was considered.

3. In applications, the assumption (1.1) becomes too restrictive. Instead of (1.1), it is usually possible to check that

$$f(H) - f(H_0) \in \mathfrak{S}_1, \tag{1.8}$$

where $f : \sigma(H_0) \cup \sigma(H) \to \mathbb{R}$ is a locally monotone (i.e., monotone on each component of $\sigma(H_0) \cup \sigma(H)$) smooth enough function.

Under the assumption (1.8), the SSF for the pair $f(H_0)$, $f(H)$ exists and the corresponding trace formula is valid. The change of variables $\lambda \mapsto f(\lambda)$ leads to the trace formula (1.2) for the pair H_0, H with

$$\xi(\lambda; H, H_0) = (\operatorname{sign} f'(\lambda))\, \xi(f(\lambda); f(H), f(H_0)). \tag{1.9}$$

Usually formula (1.9) is treated as the definition of the SSF $\xi(\cdot; H, H_0)$ under the assumption (1.8). Further details can be found in [26, §8.11]. For the function f, one often takes $f(\lambda) = (\lambda - \lambda_0)^{-m}$ or $f(\lambda) = e^{-a\lambda}$. In what follows, we mainly consider (1.9) locally, i.e., for a fixed value of λ; in this case we will for simplicity assume that $f'(\lambda) \geq 0$ (otherwise one can replace f by $-f$). Formula (1.9) is sometimes called the invariance principle for the SSF by analogy with the invariance principle for the scattering matrix [4].

4. For the case of perturbations V of a definite sign and semibounded from below operators H_0, H, formula (1.7) has been extended (in [20, Theorem 1.2]) to the case when the inclusion (1.8) (but not necessarily (1.1)) holds true with $f(\lambda) = (\lambda - \lambda_0)^{-m}$. This extension has proved to be useful in applications to differential operators (see [21]).

The aim of this paper is to prove a similar result, but (i) without the assumption on the sign of the perturbation (ii) without assuming that H_0 and H are semibounded from below (iii) for a broader class of functions f.

Below we briefly describe our main result; for a precise statement, see Theorem 8.1.

Let H_0 be a self-adjoint operator and suppose that the perturbation V of H_0 has the form $V = G^* J G$, where the operator G is such that $G(|H_0| + I)^{-1/2}$ is compact, and the operator $J = J^*$ is bounded and has a bounded inverse.[1] Under these assumptions, one can define the perturbed operator $H = H_0 + G^* J G$. If H_0 is semibounded from below, the sum $H_0 + G^* J G$ is understood in the quadratic form sense. If H_0 is not semibounded from below, one can still define the operator H using the resolvent identity. This construction goes back to [13] and is discussed in detail in [26, §1.9, 1.10]; we recall its basic features in §2.2 below.

Next, we fix an open interval $\delta \subset \mathbb{R}$ and assume that the operator $G E_{H_0}(\delta)$ belongs to the Hilbert–Schmidt class $\mathfrak{S}_2$. The above assumptions ensure (see [5]) that for a.e. $\lambda \in \mathbb{R}$, the limits $A(\lambda + i0)$, $B(\lambda + i0)$ (see (1.6) or, for a rigorous definition, (2.6)) exist in the operator norm and $B(\lambda + i0)$ belongs to the trace class. This implies that the r.h.s. of (1.7) (and of its generalization (1.10) below) is well defined.

[1] In contradistinction to [11], we do not assume that $J^2 = I$. This does not increase the generality (one can always replace G by $|J|^{1/2} G$ and J by $\operatorname{sign} J$), but may be convenient in applications — see [24].

Further, we accept the following assumption on the function f (this assumption depends on the spectral parameter λ).

Assumption 1.1. *Let $\Omega \subset \mathbb{R}$ be a Borel set, and let $f : \Omega \to \mathbb{R}$ satisfy the following two conditions at the point λ:*
(i) λ is an interior point of Ω, f is continuous and differentiable at λ, and $f'(\lambda) > 0$;
(ii) for any $\delta > 0$, one has $\inf\{|f(x) - f(\lambda)| \mid x \in \Omega, \quad |x - \lambda| > \delta\} > 0$.

We suppose that $\sigma(H_0) \cup \sigma(H) \subset \Omega$, the inclusion (1.8) holds and the Assumption 1.1 holds for all $\lambda \in \delta$. Thus, the SSF for the pair $f(H_0)$, $f(H)$ is well defined. Under these assumptions, we prove that for a.e. $\lambda \in \delta$ one has

$$\xi(f(\lambda); f(H), f(H_0)) \\ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} \operatorname{index}\big(E_{J^{-1}+A(\lambda+i0)+tB(\lambda+i0)}((-\infty,0)), E_{J^{-1}}((-\infty,0))\big). \tag{1.10}$$

In many applications, imposing the appropriate requirements on the coefficients of the differential operators H_0, H, one can easily verify all the above assumptions on H_0, H. In fact, while this paper was in the stage of preparation, formula (1.10) has been already applied in [24] to the computation of the asymptotics of the SSF of the Dirac operator.

5. The proof is based on the analysis of a certain new (to the best of our knowledge) unitary invariant for a pair of self-adjoint operators H_0, H. This invariant is an integer valued function, which depends on two variables $\theta \in (0, 2\pi)$ and $\lambda \in \mathbb{R}$. We denote this invariant by $\mu(\theta; \lambda, H, H_0)$. We postpone the definition of μ till §4; below we only list some of the properties of μ (without giving precise statements) and explain how formula (1.10) can be deduced from these properties.

(i) The function μ is defined *outside the trace class scheme*. The definition of μ requires certain assumptions on the operators H_0, H, but these assumptions are rather in the spirit of the 'smooth' scattering theory. We state and discuss these assumptions in §4.

The function $\mu(\theta; \lambda, H, H_0)$ is defined as a *spectral flow* of a certain family of unitary operators. The notion of spectral flow of a family of unitary operators is discussed in §3.

(ii) When λ is on the absolutely continuous spectrum of H_0, the function $\mu(\theta)$ up to an integer constant coincides with the eigenvalue counting function for the spectrum of the scattering matrix $\mathcal{S}(\lambda; H, H_0)$ (see §9.1):

$$\mu(\theta_1) - \mu(\theta_2) = \sum_{\theta \in [\theta_1, \theta_2)} \dim \operatorname{Ker}(\mathcal{S}(\lambda; H, H_0) - e^{i\theta} I), \quad 0 < \theta_1 < \theta_2 < 2\pi. \tag{1.11}$$

(iii) When λ is outside the essential spectrum of H_0, the function μ does not depend on θ. For such λ, it can be determined from the eigenvalue counting function of H_0 and H (see §9.2).

Thus, we see that μ, as well as the SSF, in a compact form contains information about the perturbation of both continuous and discrete spectrum. The following property shows that μ actually contains more information than the SSF.

(iv) If (1.1) holds, then $\mu(\theta; \lambda, H, H_0)$ is well defined for a.e. $\lambda \in \mathbb{R}$ and the SSF is given by (see §6):

$$\xi(\lambda; H, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu(\theta; \lambda, H, H_0) d\theta. \tag{1.12}$$

Thus, ξ can be recovered from μ.

(v) The function μ obeys the invariance principle (see §7):

$$\mu(\theta; \lambda, H, H_0) = \mu(\theta; f(\lambda), f(H), f(H_0)). \tag{1.13}$$

(vi) Suppose that the perturbation $V = H - H_0$ can be written down as $V = G^* J G$, where the operator G is such that $G(|H_0| + I)^{-1/2}$ is compact, and $J = J^*$ is bounded and has a bounded inverse. If the limits (1.6) exist in the operator norm, then the following formula for μ is valid (see §5):

$$\mu(\theta) = \operatorname{index}\big(E_{J^{-1}}((-\infty, 0)), E_{J^{-1} + A(\lambda + i0) + \cot(\theta/2) B(\lambda + i0)}((-\infty, 0))\big). \tag{1.14}$$

Thus, the function μ is an 'intermediate' object between SSF and the scattering matrix. It uses only the information on the *spectrum* of the scattering matrix, disregarding its eigenvectors. On the other hand, it contains more information than the spectrum of the scattering matrix. Roughly speaking, this additional information reduces to an integer constant at every point λ. Outside the essential spectrum this constant merely equals $-\xi(\lambda; H, H_0)$. On the absolutely continuous spectrum, observe that the Birman-Krein formula (1.5) determines the SSF up to an integer constant; the 'additional information' contained in μ fixes this constant in accordance with the normalisation condition (1.4).

Note that, taking into account (1.11), the equality (1.12) modulo $\mathbb{Z}$ is merely the Birman-Krein formula (1.5), and the relation (1.13) modulo $\mathbb{Z}$ is a trivial consequence of the invariance principle for the scattering matrix. It is the adequate choice of an integer constant in the definition of μ, that makes it possible to establish formulae (1.12)–(1.13) in the full scale.

Combining (1.12) and (1.14) and performing the change of variable $t = \cot(\theta/2)$ in the resulting integral, we obtain (1.7) (if $J^{-1} = J$); this can be considered as an alternative proof of (1.7). Combining (1.12), (1.13), (1.14), we obtain (1.10).

In fact, the properties (ii) and (iii) above are not used in the proof of (1.10); we have mentioned them here only in order to explain the idea behind the definition of μ.

6. Let us describe the structure of the paper. In §2, we introduce some notation and recall the definition of the sum $H_0 + G^*JG$ (without the assumption that H_0 is semibounded from below). In §3 we discuss the notion of the spectral flow for unitary operators. In §4 we define the function μ. In §5, 6, 7, we prove formulae (1.14), (1.12), (1.13), respectively. In §8 we state and prove the main result of the paper on the representation (1.10). In §9, we prove formula (1.11) and explain the relation of the function μ to the eigenvalue counting functions of the operators H_0, H away from their essential spectrum; this material is not used in the proof of the main result of the paper.

In each section, the statement and discussion of all the results are given first and the proofs are postponed till the end of the section.

7. In different parts of the paper, we use two different points of view on the pair of operators H_0, H (in accord with the nature of the question under consideration). The first point of view is that the 'basic' operators are the unperturbed operator H_0 and the perturbation G^*JG; the perturbed operator H is defined as the sum $H = H_0 + G^*JG$. This point of view is aimed at applications.

According to the second point of view, the operators H_0 and H are defined independently one of another and have equal roles; in this case we do not use the factorization of the perturbation $H - H_0$.

2. NOTATION AND PRELIMINARIES

2.1. Notation

1. Below $\mathcal{H}$, $\mathcal{K}$ are separable Hilbert spaces; I is the identity operator. For a closable linear operator $T : \mathcal{H} \to \mathcal{K}$, by $\operatorname{Dom} T$ we denote its domain and by $\overline{T}$ — the closure of T. For a self-adjoint operator A in a Hilbert space, the symbols $\sigma(A)$, $\sigma_{ess}(A)$, $\rho(A)$ denote its spectrum, essential spectrum and resolvent set and $E_A(\delta)$ is the spectral projection associated to a Borel set $\delta \subset \mathbb{R}$. We also denote by $\Xi(A)$ the Ξ operator associated with A (see [10, 11]): $\Xi(A) := E_A((-\infty, 0))$.

By $\mathcal{B}(\mathcal{H},\mathcal{K})$ we denote the Banach space of all bounded operators acting from $\mathcal{H}$ to $\mathcal{K}$; $\mathfrak{S}_\infty(\mathcal{H},\mathcal{K}) \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ is the space of all compact operators and $\mathfrak{S}_p(\mathcal{H},\mathcal{K})$, $p \geq 1$, is the standard Schatten–von Neumann class. We write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$, $\mathfrak{S}_p(\mathcal{H}) := \mathfrak{S}_p(\mathcal{H},\mathcal{H})$; the norm in the classes $\mathcal{B}$, $\mathfrak{S}_p$ is denoted by $\|\cdot\|$, $\|\cdot\|_{\mathfrak{S}_p}$ and the limits — by n-lim, $\mathfrak{S}_\mathrm{p}$-lim, respectively.

We shall often use the well-known fact that

$$A \in \mathfrak{S}_p, \quad M_n \xrightarrow{s} 0 \quad \Rightarrow \quad \|M_n A\|_{\mathfrak{S}_p} \to 0, \quad p \in [1,\infty]; \tag{2.1}$$

here $\xrightarrow{s}$ denotes strong convergence. If, in addition, $M_n^* \xrightarrow{s} 0$, then also $\|AM_n\|_{\mathfrak{S}_p} \to 0$. In particular, (2.1) implies that

$$A_n \in \mathfrak{S}_p, \quad \|A_n - A\|_{\mathfrak{S}_p} \to 0, \quad M_n \xrightarrow{s} M \quad \Rightarrow \quad \|M_n A_n - MA\|_{\mathfrak{S}_p} \to 0. \tag{2.2}$$

Formulas and statements with double indices ($\pm$ and $\mp$) should be read as pairs of statements, in one of which all the indices take upper values and in another — the lower ones. A constant which first appears in formula $(i.j)$ is denoted by $C_{i.j}$. We denote $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. The open ball in a metric space with the centre x and radius r is denoted by $B(x;r)$.

2. A pair P, Q of orthogonal projections in $\mathcal{H}$ is called Fredholm if

$$\{+1,-1\} \cap \sigma_{ess}(P-Q) = \emptyset.$$

In particular, if $P - Q$ is compact, then the pair P, Q is Fredholm. The index of a Fredholm pair is determined by the formula

$$\operatorname{index}(P,Q) := \dim(\operatorname{Ker}(P-Q-I)) - \dim(\operatorname{Ker}(P-Q+I)). \tag{2.3}$$

Clearly,

$$\operatorname{index}(P,Q) = -\operatorname{index}(Q,P).$$

If either $(P-Q)$ or $(Q-R)$ is compact and both P, Q and Q, R are Fredholm pairs, then the pair P, R is also Fredholm and the following chain rule is valid:

$$\operatorname{index}(P,R) = \operatorname{index}(P,Q) + \operatorname{index}(Q,R). \tag{2.4}$$

See, e.g., [2] for the details.

2.2. Operator $H(H_0, G, J)$

Let $\mathcal{H}$ be a 'basic' and $\mathcal{K}$ an 'auxiliary' Hilbert space. Fix a self-adjoint operator H_0 in $\mathcal{H}$ and let $G : \mathcal{H} \to \mathcal{K}$ and J in $\mathcal{K}$ be such operators that

$$\begin{gathered}\operatorname{Dom}(|H_0| + I)^{1/2} \subset \operatorname{Dom} G, \quad G(|H_0| + I)^{-1/2} \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{K}), \\ J = J^* \in \mathcal{B}(\mathcal{K}), \quad 0 \in \rho(J).\end{gathered} \tag{2.5}$$

Below we define a self-adjoint operator H, which corresponds to the formal sum $H_0 + G^*JG$. Sometimes we shall explicitly indicate the dependence of H on H_0, G, J by writing $H(H_0, G, J)$. The construction below goes back to [13] and is discussed in detail in [26, §1.9, 1.10].

For $z \in \rho(H_0)$ define the following operators of the class $\mathfrak{S}_\infty(\mathcal{K})$:

$$\begin{aligned}&T(z) = T(z; H_0, G) = (G(|H_0| + I)^{-1/2}) \frac{|H_0| + I}{H_0 - zI} (G(|H_0| + I)^{-1/2})^*, \\ &A(z) = A(z; H_0, G) = \operatorname{Re} T(z), \qquad B(z) = B(z; H_0, G) = \operatorname{Im} T(z).\end{aligned} \tag{2.6}$$

It is easy to check (see, e.g., [26, Lemma 1.10.5]) that

$$0 \in \rho(I + JT(z)) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7}$$

Under the assumptions (2.5), there exists a unique self-adjoint operator $H = H(H_0, G, J)$ (see [26, §1.9, 1.10]), such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ its resolvent satisfies the equation

$$\begin{aligned}&(H - zI)^{-1} - (H_0 - zI)^{-1} \\ &\qquad = -(G(H_0 - \overline{z}I)^{-1})^*(I + JT(z))^{-1}(JG(H_0 - zI)^{-1}).\end{aligned} \tag{2.8}$$

The inverse operator $(I + JT(z))^{-1}$ in the r.h.s. of (2.8) exists by (2.7). Note that (2.7) implies

$$0 \in \rho(J^{-1} + T(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.9}$$

and (2.8) can be recast as

$$\begin{aligned}&(H - zI)^{-1} - (H_0 - zI)^{-1} \\ &\qquad = -(G(H_0 - \overline{z}I)^{-1})^*(J^{-1} + T(z))^{-1}(G(H_0 - zI)^{-1}).\end{aligned} \tag{2.10}$$

If H_0 is semibounded from below, then H coincides with the sum $H_0 + G^*JG$ in the quadratic form sense. More precisely, if $h_0[\cdot, \cdot]$ is the sesquilinear form of H_0 with the domain $d[h_0](= \operatorname{Dom}(|H_0| + I)^{1/2})$, then the sesquilinear form $h[\cdot,\cdot]$ of H is defined on the domain $d[h] = d[h_0]$ by the relation

$$h[f,g] = h_0[f,g] + (JGf, Gg), \quad f,g \in d[h_0].$$

If the operator G^*JG is well defined and H_0-bounded with a relative bound < 1, then $H = H_0 + G^*JG$ in the sense of the Kato–Rellich theorem.

Finally, by (2.10), the difference of the resolvents of H and H_0 is compact, and therefore the essential spectra of H_0 and H coincide.

3. THE SPECTRAL FLOW FOR UNITARY OPERATORS

3.1. Introduction

Let $A(t)$, $t \in [0,1]$, be a family of self-adjoint Fredholm operators. If $A(t)$ is continuous in t in some appropriate sense, one can define the *spectral flow* of A, $\mathrm{sf}(A)$. A 'naive' definition of the spectral flow is the following:

$$\begin{aligned}\mathrm{sf}(A) = &\langle \text{the number of eigenvalues of } A(t) \text{ that cross 0 rightwards}\rangle \\ &- \langle \text{the number of eigenvalues of } A(t) \text{ that cross 0 leftwards}\rangle\end{aligned}$$

as t grows monotonically from 0 to 1. The spectral flow was introduced in [1, §7] as the intersection number of the graph $\cup_{t\in[0,1]}\sigma(A(t))$ of the spectrum of $A(t)$ with the line $\lambda = -\varepsilon$, where ε is a sufficiently small positive number (one can take $\varepsilon = 0$ if both $A(0)$ and $A(1)$ are invertible). The spectral flow is an important homotopy invariant of the family $A(t)$ — see, e.g., recent treatments in [22] and [9] and references therein.

In this paper, we will need the notion of the spectral flow for *unitary*, rather than self-adjoint, operators. Namely, let us fix a Hilbert space $\mathcal{H}$ and a parameter $p \in [1,\infty]$. Let $Y_p = Y_p(\mathcal{H})$ be the set of all unitary operators W in $\mathcal{H}$ such that $W - I \in \mathfrak{S}_p(\mathcal{H})$. Clearly, Y_p is a metric space with the metric $d(W_1,W_2) = \|W_1 - W_2\|_{\mathfrak{S}_p}$, $p < \infty$ and $d(W_1,W_2) = \|W_1 - W_2\|$, $p = \infty$. Consider a mapping $U : [0,1] \to Y_p$. We do not suppose that U is continuous; instead, we assume that the spectrum $\sigma(U(t))$ depends continuously on t in a certain precise sense to be defined below. In this section we define the spectral flow of the family $U(t)$ through the points $z \in \mathbb{T} \setminus \{1\}$. A 'naive' definition of the spectral flow is the following:

$$\begin{aligned}\mathrm{sf}(z;U) = &\langle \text{the number of eigenvalues of } U(t) \text{ that cross } z \text{ anti-clockwise}\rangle \\ &- \langle \text{the number of eigenvalues of } U(t) \text{ that cross } z \text{ clockwise}\rangle\end{aligned} \tag{3.1}$$

as t grows monotonically from 0 to 1.

In our subsequent construction, we will have to deal with $\mathrm{sf}(z;U)$ as the function of the spectral parameter $z \in \mathbb{T} \setminus \{1\}$. For example, we will have to consider the integral

$$\int_0^{2\pi} \mathrm{sf}(e^{i\theta};U)d\theta$$

for the families $U : [0,1] \to Y_1$. Therefore, the behaviour of $\mathrm{sf}(e^{i\theta};U)$ as an element of the functional spaces on $(0,2\pi)$ (such as $L_1(0,2\pi)$) is essential for us.

Because of this, we find it convenient to give our own definition of the spectral flow (see Definition 3.1 below), rather than to use the standard definition. Our definition is adapted to the specific purposes of this paper and consistently takes into account the dependence of $\mathrm{sf}(z;U)$ on the spectral parameter z.

In §3.5 we will show that our definition coincides with the naive definition (3.1) (whenever the latter makes sense) and therefore is consistent with the standard definition of the spectral flow. However, we do not use this fact and work entirely in terms of our definition.

For the proofs of the main result of this paper we shall need only the cases $p = 1$, $p = \infty$. Nevertheless, we find it instructive to give a universal treatment of all the cases $p \in [1,\infty]$, since this does not require any considerable modification of the proofs.

3.2. Covering spaces

For the reader's convenience, we recall the definition of covering spaces and their basic properties. The details can be found in any textbook in algebraic topology; see, e.g., [17, Chapter 5].

Let X and $\widetilde{X}$ be topological spaces. We suppose that X and $\widetilde{X}$ are *arcwise connected* (i.e., any two points can be joint by a path) and *locally arcwise connected* (i.e., any point has a basic family of arcwise connected neighbourhoods). A continuous mapping $\pi : \widetilde{X} \to X$ is called a *covering,* if every point $x \in X$ has an arcwise connected open neighbourhood U with the following property. The restriction of π onto each arc component V of $\pi^{-1}(U)$ is a homeomorphism between V and U.

The important property of covering spaces is that paths and their homotopies can be lifted from X to $\widetilde{X}$. More precisely:

Proposition 3.1. *Let $\widetilde{x} \in \widetilde{X}$, $x = \pi(\widetilde{x})$. For any path $\gamma : [0,1] \to X$ with the initial point $\gamma(0) = x$, there exists a unique path (a lift of γ) $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ such that $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = \widetilde{x}$.*

The idea of the proof is to express the path γ as a sequence of a finite number of 'short' paths, each of which is contained in an elementary neighbourhood, and then lift each of these paths. For the details (and the proof of the uniqueness part), see, e.g., [17, Chapter 5, §3].

PROPOSITION 3.2. *Let $\widetilde{\gamma}_0, \widetilde{\gamma}_1 : [0,1] \to \widetilde{X}$ be paths in $\widetilde{X}$ which have the same initial point: $\widetilde{\gamma}_0(0) = \widetilde{\gamma}_1(0)$. If $\pi \circ \widetilde{\gamma}_0$ is homotopic to $\pi \circ \widetilde{\gamma}_1$, then $\widetilde{\gamma}_0$ is homotopic to $\widetilde{\gamma}_1$; in particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.*

The idea of the proof is essentially the same as that of Proposition 3.1. Let $F : [0,1] \times [0,1] \to X$ be a homotopy between $\pi \circ \widetilde{\gamma}_0$ and $\pi \circ \widetilde{\gamma}_1$:

$$F(t,0) = \pi(\widetilde{\gamma}_0(t)), \qquad F(t,1) = \pi(\widetilde{\gamma}_1(t)),$$
$$F(0,s) = \pi(\widetilde{\gamma}_0(0)), \qquad F(1,s) = \pi(\widetilde{\gamma}_0(1)).$$

Then the square $[0,1] \times [0,1]$ can be subdivided into 'small' rectangles such that F maps each rectangle into an elementary neighbourhood. After that, F can be lifted to $\widetilde{X}$ locally on each rectangle. The result of this lifting gives a homotopy between $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$. For the details, see, e.g., [17, Chapter 5, Lemma 3.3].

3.3. The covering $\pi_p : \widetilde{X_p} \to X_p$

1. First we define the function space $\widetilde{X}_p$ which the function $\mathrm{sf}(\cdot; U)$ will belong to. Let $\widetilde{X}_\infty$ be the set of all functions $f : \mathbb{T} \setminus \{1\} \to \mathbb{Z}$ such that the function $(0, 2\pi) \ni \theta \mapsto f(e^{i\theta})$ is left continuous and non-increasing. Clearly, the points $z \in \mathbb{T} \setminus \{1\}$ where $f \in \widetilde{X}_\infty$ is discontinuous, can accumulate only to 1. For any $f \in \widetilde{X}_\infty$, let us introduce the function $\nu(\cdot; f) : \mathbb{Z} \to [0, 2\pi]$ by

$$\nu(n; f) := \sup(\{0\} \cup \{\theta \in (0, 2\pi) \mid f(e^{i\theta}) > n\}). \tag{3.2}$$

Clearly, $\nu(\cdot; f)$ is non-increasing and

$$\lim_{n \to +\infty} \nu(n; f) = 0, \qquad \lim_{n \to -\infty} \nu(n; f) = 2\pi.$$

Note that f can be recovered from $\nu(\cdot; f)$ by the formula

$$f(e^{i\theta}) := \inf\{n \in \mathbb{Z} \mid \nu(n; f) < \theta\}. \tag{3.3}$$

For $p \in [1, \infty)$, let $\widetilde{X}_p \subset \widetilde{X}_\infty$ be the set of functions f such that

$$\sum_{n \geq 0} (\nu(n; f))^p + \sum_{n < 0} (2\pi - \nu(n; f))^p < \infty.$$

For any $p \in [1, \infty]$ and any $f, g \in \widetilde{X}_p$, define

$$\widetilde{\rho}_p(f, g) := \|\nu(\cdot; f) - \nu(\cdot; g)\|_{l_p(\mathbb{Z})}.$$

Note that

$$\widetilde{\rho}_1(f,g) = \int_0^{2\pi} |f(e^{i\theta}) - g(e^{i\theta})| d\theta.$$

PROPOSITION 3.3. *The function $\widetilde{\rho}_p$ is a metric on $\widetilde{X}_p$. With respect to this metric, $\widetilde{X}_p$ is arcwise connected and locally arcwise connected.*

2. Consider the following equivalence relation on $\widetilde{X}_p$:

$$f \sim g \Longleftrightarrow \exists n \in \mathbb{Z} : \forall z \in \mathbb{T} \setminus \{1\}, \quad f(z) = g(z) + n.$$

Let X_p be the quotient space $\widetilde{X}_p/\sim$, and let $\pi_p : \widetilde{X}_p \to X_p$ be the corresponding projection. For $f, g \in X_p$ define

$$\rho_p(f,g) = \inf\{\widetilde{\rho}_p(\widetilde{f}, \widetilde{g}) \mid \pi_p(\widetilde{f}) = f, \pi_p(\widetilde{g}) = g\}.$$

PROPOSITION 3.4. *The function ρ_p is a metric on X_p. With respect to this metric, X_p is arcwise connected and locally arcwise connected.*

Obviously, the mapping $\pi_p : \widetilde{X}_p \to X_p$ is continuous.

PROPOSITION 3.5. *The mapping $\pi_p : \widetilde{X}_p \to X_p$ is a covering.*

Remark. Clearly, an element $f \in X_p$ is uniquely determined by specifying the set of discontinuities $z_n \in \mathbb{T} \setminus \{1\}$ of an element $\widetilde{f} \in \pi_p^{-1}(f)$ together with the heights $m(z_n)$ of the jumps of $\widetilde{f}$ at the points z_n. Thus, the space X_p can be identified with the set of the spectra of all unitary operators $W \in Y_p$; under this identification, z_n become eigenvalues with the multiplicities $m(z_n)$.

Notation. Let $\gamma : [0,1] \to \widetilde{X}_p$ be any mapping. Then γ depends on two variables, $t \in [0,1]$ and $z \in \mathbb{T} \setminus \{1\}$. If we need to indicate the dependence of γ on both variables z and t, we write $\gamma(z;t)$. If γ is considered as an element of the function space $\widetilde{X}_p$ (for a fixed t), we write $\gamma(t)$.

3. It is obvious that the following diagram is commutative for any $1 \le q < r \le \infty$:

$$\begin{array}{ccc} \widetilde{X}_q & \xrightarrow{in_{\widetilde{X}_q \to \widetilde{X}_r}} & \widetilde{X}_r \\ {\scriptstyle \pi_q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_r} \\ X_q & \xrightarrow{in_{X_q \to X_r}} & X_r \end{array} \tag{3.4}$$

Here $in_{\widetilde{X}_q \to \widetilde{X}_r}$ and $in_{X_q \to X_r}$ are the natural embeddings.

3.4. The mapping $\eta_p : Y_p \to X_p$

1. Below we use the following natural notation for the arcs of the unit circle in the complex plane:

$$(e^{i\theta_1}, e^{i\theta_2}) = \{e^{i\theta} \mid \theta_1 < \theta < \theta_2\}, \quad \theta_1 < \theta_2,$$

with the obvious modifications for $[e^{i\theta_1}, e^{i\theta_2}]$, $(e^{i\theta_1}, e^{i\theta_2}]$, $[e^{i\theta_1}, e^{i\theta_2})$.

Let $W \in Y_p$ and $\theta_1, \theta_2 \in (0, 2\pi)$. Define

$$N(e^{i\theta_1}, e^{i\theta_2}; W) = \begin{cases} \operatorname{rank} E_W([e^{i\theta_1}, e^{i\theta_2})), & \theta_1 < \theta_2, \\ 0, & \theta_1 = \theta_2, \\ -\operatorname{rank} E_W([e^{i\theta_2}, e^{i\theta_1})), & \theta_2 < \theta_1. \end{cases} \tag{3.5}$$

It is easy to see that for any $z_0 \in \mathbb{T} \setminus \{1\}$ the function $\mathbb{T} \setminus \{1\} \ni z \mapsto N(z, z_0; W) \in \mathbb{Z}$ belongs to the space $\widetilde{X}_p$.

Proposition 3.6. *Fix $z_0 \in \mathbb{T} \setminus \{1\}$. The mapping*

$$Y_p \ni W \mapsto N(\cdot, z_0; W) \in \widetilde{X}_p$$

is continuous at the 'points' W such that $z_0 \in \mathbb{T} \setminus \sigma(W)$.

2. Let us define the mapping η_p:

$$\eta_p : \quad Y_p \ni W \mapsto \eta_p(W) := \pi_p(N(\cdot, z_0; W)) \in X_p, \quad z_0 \in \mathbb{T} \setminus \sigma(W). \tag{3.6}$$

Clearly, this definition does not depend on z_0, since the change of z_0 results in adding an integer constant to $N(\cdot, z_0; W)$. By Proposition 3.6, the mapping η_p is continuous.

3. Note that the following diagram is commutative for any $1 \le q < r \le \infty$:

$$\begin{array}{ccc} Y_q & \xrightarrow{in_{Y_q \to Y_r}} & Y_r \\ \eta_q \big\downarrow & & \big\downarrow \eta_r \\ X_q & \xrightarrow{in_{X_q \to X_r}} & X_r \end{array} \tag{3.7}$$

Here $in_{X_q \to X_r}$ and $in_{Y_q \to Y_r}$ are the natural embeddings.

3.5. The spectral flow

1. Now we are ready to define the spectral flow of a family $U : [0, 1] \to Y_p$. But first we have to take into account one complication of a formal nature. In our construction below (see §4.1) we have to deal with the families, defined on an open, rather than closed, interval $(0, 1)$. At the same time, it appears that the composition $\eta_p \circ U$ can be extended by continuity to the endpoints 0 and 1. Thus, first we need the notation for such an extension. Suppose that a mapping $\gamma : (0, 1) \to X_p$ is continuous and the limits $\lim_{t\to+0} \gamma(t)$, $\lim_{t\to1-} \gamma(t)$ exist. Then we write that *the extension of* γ *exists* and denote by

$$\mathrm{ext}(\gamma)$$

the mapping γ, extended by continuity to the whole interval $[0, 1]$.

Definition 3.1. Let $U : (0, 1) \to Y_p$ be such a mapping that the extension $\gamma := \mathrm{ext}(\eta_p \circ U)$ exists. Let $\widetilde{\gamma}$ be a lift of γ into $\widetilde{X}_p$. Then we define

$$\mathrm{sf}(z; U) := \widetilde{\gamma}(z; 1) - \widetilde{\gamma}(z; 0). \tag{3.8}$$

Definition 3.1 does not depend on the choice of the lift $\widetilde{\gamma}$. Indeed, let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ. Then the function $\widetilde{\gamma}_2(0) - \widetilde{\gamma}_1(0)$ is an integer constant; let us denote this constant by n. By the uniqueness of the lift of a path with a fixed initial point, one has $\widetilde{\gamma}_2(t) \equiv \widetilde{\gamma}_1(t) + n$ and therefore $\widetilde{\gamma}_2(1) - \widetilde{\gamma}_2(0) = \widetilde{\gamma}_1(1) - \widetilde{\gamma}_1(0)$.

Definition 3.1 does not depend on p *in the following sense.* Let $1 \le q < r \le \infty$ and let $U_q : (0, 1) \to Y_q$ be such a mapping that the extension $\gamma_q = \mathrm{ext}(\eta_q \circ U_q)$ exists. Let $\widetilde{\gamma}_q$ be the lift of γ_q and $\widetilde{\gamma}_q(1) - \widetilde{\gamma}_q(0)$ be the spectral flow of U_q. Consider the mapping $U_r := in_{Y_q \to Y_r} \circ U_q : (0, 1) \to Y_r$. It follows from (3.7) that the extension $\gamma_r = \mathrm{ext}(\eta_r \circ U_r)$ exists and $\gamma_r = in_{X_q \to X_r} \circ \gamma_q$. Consider the lift $\widetilde{\gamma}_r$ of γ_r. Taking into account (3.4), one sees that $in_{\widetilde{X}_q \to \widetilde{X}_r} \circ \widetilde{\gamma}_q$ is also a lift of γ_r. From here it follows that

$$in_{\widetilde{X}_q \to \widetilde{X}_r}(\widetilde{\gamma}_q(1)) - in_{\widetilde{X}_q \to \widetilde{X}_r}(\widetilde{\gamma}_q(0)) = \widetilde{\gamma}_r(1) - \widetilde{\gamma}_r(0).$$

2. Thus defined, the spectral flow is homotopy invariant:

Proposition 3.7. *Let $U_1, U_2 : (0,1) \to Y_p$ be two mappings such that the extensions $\gamma_1 = \mathrm{ext}(\eta_p \circ U_1)$ and $\gamma_2 = \mathrm{ext}(\eta_p \circ U_2)$ exist and are homotopic (in particular, this implies that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$). Then*

$$\mathrm{sf}(z; U_1) = \mathrm{sf}(z; U_2), \quad z \in \mathbb{T} \setminus \{1\}. \tag{3.9}$$

Proof. A direct application of Proposition 3.2. ∎

Note that our proof of the invariance principle (1.13) depends heavily on the homotopy invariance of the spectral flow.

3. In this paper we do not explicitly use the fact that Definition 3.1 agrees with the 'naive' definition (3.1), whenever the latter makes sense. However, let us give a sketch of proof of this fact. Here for the sake of simplicity of notation we assume that our mappings U are already defined on the whole of $[0,1]$ and thus need not be extended.

First suppose that for a mapping $U : [0,1] \to Y_p$ (such that $\eta_p \circ U$ is continuous), there exists $z_0 \in \mathbb{T} \setminus \{1\}$ such that $z_0 \in \rho(U(t))$ for all $t \in [0,1]$. One easily checks that in this case, according to Definition 3.1,

$$\mathrm{sf}(z; U) = N(z, z_0; U(1)) - N(z, z_0; U(0)).$$

Clearly, this agrees with (3.1).

Further, for an arbitrary mapping $U : [0,1] \to Y_p$ (such that $\eta_p \circ U$ is continuous), one can always find a finite cover of $[0,1]$ by the intervals δ_n, $n = 1, \dots, N$, with the property that for any n there exists $z_n \in \mathbb{T} \setminus \{1\}$, $z_n \in \rho(U(t))$ for any $t \in \delta_n$. In this case, one can write

$$\mathrm{sf}(z; U) = \sum_{n=1}^{N} \big(N(z, z_n; U(t_n)) - N(z, z_n; U(t_{n-1})) \big) \tag{3.10}$$

for a set of points $0 = t_0 < t_1 < \cdots < t_N = 1$, $t_n \in \delta_n \cap \delta_{n+1}$ for $n = 1, \dots, N-1$. Formula (3.10) also agrees with (3.1).

3.6. Proof of Propositions 3.3—3.6

1. *Proof of Proposition 3.3:* 1. Let us prove that $\widetilde{\rho}_p$ is a metric. Clearly, $\widetilde{\rho}_p(f,g) = \widetilde{\rho}_p(g,f)$ and $\widetilde{\rho}_p(f,g) \geq 0$. Suppose that $f \not\equiv g$; by (3.3), it follows that $\nu(\cdot; f) \not\equiv \nu(\cdot; g)$ and therefore $\widetilde{\rho}_p(f,g) \neq 0$.

The triangle inequality for $\widetilde{\rho}_p$ is evident.

2. We shall prove that any ball in $\widetilde{X}_p$ is arcwise connected; clearly, this will imply that $\widetilde{X}_p$ is arcwise connected and locally arcwise connected.

For every $f_0, f_1 \in \widetilde{X}_p$, let

$$\nu_\alpha(n) = \alpha\nu(n; f_1) + (1-\alpha)\nu(n; f_0), \quad \alpha \in [0,1], \quad n \in \mathbb{Z}.$$

The formula (3.3) recovers the family f_α of the functions such that $\nu(n; f_\alpha) = \nu_\alpha(n)$. Clearly, the path $[0,1] \ni \alpha \mapsto f_\alpha \in \widetilde{X}_p$ connects f_0 and f_1; moreover, $\widetilde{\rho}_p(f_0, f_\alpha) \leq \widetilde{\rho}_p(f_0, f_1)$. Thus, every ball in $\widetilde{X}_p$ is arcwise connected. ▌

2. *Auxiliary facts*

1. Note that

$$\widetilde{\rho}_p(f+n, g+n) = \widetilde{\rho}_p(f,g) \text{ for any constant } n \in \mathbb{Z}. \tag{3.11}$$

2. Clearly, for any $f \in \widetilde{X}_p$ one has

$$\inf_{n\in\mathbb{Z}\setminus\{0\}} \widetilde{\rho}_p(f+n, f) = \widetilde{\rho}_p(f+1, f) > 0. \tag{3.12}$$

3. Let us prove that

$$\forall f, g \in \widetilde{X}_p \quad \exists n \in \mathbb{Z} : \quad \inf_{m\in\mathbb{Z}} \widetilde{\rho}_p(f+m, g) = \widetilde{\rho}_p(f+n, g). \tag{3.13}$$

In other words, the infimum in (3.13) is always attained.

First let $p \neq \infty$. Then, clearly,

$$\lim_{|m|\to\infty} \widetilde{\rho}_p(f+m, g) = \infty,$$

which proves (3.13). Next, let $p = \infty$. Then

$$\lim_{|m|\to\infty} \widetilde{\rho}_\infty(f+m, g) = 2\pi,$$

whereas $\widetilde{\rho}_\infty(f+m, g) \leq 2\pi$ for any m. This proves (3.13) for $p = \infty$.

3. *Proof of Proposition 3.4:* 1. Let us prove that ρ_p is a metric. Clearly, $\rho_p(f,g) = \rho_p(g,f)$ and $\rho_p(f,g) \geq 0$. Suppose that $\rho_p(f,g) = 0$; let us check that $f = g$. Fix $\widetilde{f} \in \pi_p^{-1}(f)$, $\widetilde{g} \in \pi_p^{-1}(g)$. By (3.13), the relation $\rho_p(f,g) = 0$ implies that $\widetilde{\rho}_p(\widetilde{f}+n, \widetilde{g}) = 0$ for some $n \in \mathbb{Z}$ and thus $\widetilde{f}+n = \widetilde{g}$ and therefore $f = g$.

The triangle inequality for ρ_p follows directly from the triangle inequality for $\widetilde{\rho}_p$.

2. Obviously, $\pi_p(\widetilde{X}_p) = X_p$. Since $\widetilde{X}_p$ is arcwise connected, it follows that X_p is also arcwise connected.

3. Let us prove that X_p is locally arcwise connected. To this end, we prove that every ball in X_p is arcwise connected. Fix $f \in X_p$, $\widetilde{f} \in \pi_p^{-1}(f)$ and $r > 0$ and consider the open ball $B(f;r)$ with the centre f and radius r. Below we prove that π_p maps the ball $B(\widetilde{f};r)$ onto $B(f;r)$. Since $B(\widetilde{f};r)$ is arcwise connected (see the proof of Proposition 3.3), this will imply that $B(f;r)$ is also arcwise connected.

The inclusion $\pi_p(B(\widetilde{f};r)) \subset B(f;r)$ is evident. Let us prove that $B(f;r) \subset \pi_p(B(\widetilde{f};r))$. If $g \in B(f;r)$ and $\widetilde{g} \in \pi_p^{-1}(g)$, then $\inf_{m\in\mathbb{Z}} \widetilde{\rho}_p(\widetilde{f}+m,\widetilde{g}) < r$, which, by (3.13), implies that $\widetilde{\rho}_p(\widetilde{f}+m,\widetilde{g}) < r$ for some $m \in \mathbb{Z}$. Thus, $\widetilde{\rho}_p(\widetilde{f},\widetilde{g}-m) < r$ and therefore $\widetilde{g}-m \in B(\widetilde{f};r)$ and $g = \pi_p(\widetilde{g}-m) \in \pi_p(B(\widetilde{f};r))$. ▮

4. *Proof of Proposition 3.5:* Fix $f \in X_p$, $\widetilde{f} \in \pi_p^{-1}(f)$ and $\varepsilon < \widetilde{\rho}_p(\widetilde{f}+1,\widetilde{f})/3$. Let us prove that the ball $B(f;\varepsilon)$ is an elementary neighbourhood. We shall prove that $\pi_p^{-1}(B(f;\varepsilon)) = \cup_{n\in\mathbb{Z}} B(\widetilde{f}+n;\varepsilon)$, where the balls $B(\widetilde{f}+n;\varepsilon)$ are mutually disjoint, arcwise connected and the restriction $\pi_p \mid B(\widetilde{f}+n;\varepsilon)$ is a homeomorphism between $B(\widetilde{f}+n;\varepsilon)$ and $B(f;\varepsilon)$.

Let us first check that the balls $B(\widetilde{f}+n;\varepsilon)$ are mutually disjoint. Indeed, let $\widetilde{g} \in B(\widetilde{f}+n;\varepsilon) \cap B(\widetilde{f}+m;\varepsilon)$. Then $\widetilde{\rho}_p(\widetilde{f}+n,\widetilde{f}+m) \le \widetilde{\rho}_p(\widetilde{f}+n,\widetilde{g}) + \widetilde{\rho}_p(\widetilde{g},\widetilde{f}+m) < 2\varepsilon$. By (3.12) and the choice of ε, the last inequality implies $m = n$.

In the course of the proof of Proposition 3.4, we have checked that $\pi_p(B(\widetilde{f}+n;\varepsilon)) = B(f;\varepsilon)$ for any $n \in \mathbb{Z}$. The same reasoning also shows that $\pi_p^{-1}(B(f;\varepsilon)) = \cup_{n\in\mathbb{Z}} B(\widetilde{f}+n;\varepsilon)$.

Let us prove that the restriction $\pi_p \mid B(\widetilde{f}+n;\varepsilon)$ is injective. Let $\pi_p(\widetilde{g}) = \pi_p(\widetilde{h})$ for $\widetilde{g},\widetilde{h} \in B(\widetilde{f}+n;\varepsilon)$. Then $\widetilde{g} = \widetilde{h}+m$ for some $m \in \mathbb{Z}$. Using (3.11), one has:

$$\begin{aligned} \widetilde{g} \in B(\widetilde{f}+n;\varepsilon) \quad &\Rightarrow \quad \widetilde{\rho}_p(\widetilde{f}+n,\widetilde{g}) < \varepsilon \quad \Rightarrow \quad \widetilde{\rho}_p(\widetilde{f}+n-m,\widetilde{h}) < \varepsilon \\ &\Rightarrow \quad \widetilde{h} \in B(\widetilde{f}+n-m;\varepsilon) \Rightarrow \quad m = 0 \quad \Rightarrow \quad \widetilde{g} = \widetilde{h}. \end{aligned}$$

4. Finally, let us check that $(\pi_p \mid B(\widetilde{f}+n;\varepsilon))^{-1}$ is continuous. Let $\widetilde{g},\widetilde{h} \in B(\widetilde{f}+n;\varepsilon)$, $g = \pi_p(\widetilde{g})$, $h = \pi_p(\widetilde{h})$. Below we show that if $\rho_p(g,h) < \varepsilon$, then $\widetilde{\rho}_p(\widetilde{g},\widetilde{h}) = \rho_p(g,h)$. Indeed, by (3.13), one has $\rho_p(g,h) = \widetilde{\rho}_p(\widetilde{g}+m,\widetilde{h})$ for some $m \in \mathbb{Z}$. Let us show that $m = 0$. Using (3.11), one has

$$\begin{aligned} \widetilde{\rho}_p(\widetilde{f}+m,\widetilde{f}) = \widetilde{\rho}_p(\widetilde{f}+n+m,\widetilde{f}+n) &\le \widetilde{\rho}_p(\widetilde{f}+n+m,\widetilde{g}+m) + \widetilde{\rho}_p(\widetilde{g}+m,\widetilde{h}) \\ &+ \widetilde{\rho}_p(\widetilde{h},\widetilde{f}+n) < 3\varepsilon, \end{aligned}$$

which, by (3.12) and the choice of ε, implies $m = 0$. ▮

5. The proof of Proposition 3.6 is based on the following

LEMMA 3.1. *For any $\varepsilon \in (0, 2\pi)$ there exists $C_{3.14}(\varepsilon) > 0$ such that for any $z_0 \in \mathbb{T} \setminus \{1\}$ and any operators W_1, $W_2 \in Y_p$ with the property*

$$[z_0 e^{-i\varepsilon}, z_0 e^{i\varepsilon}] \cap \sigma(W_j) = \emptyset, \quad j = 1, 2,$$

the following estimate holds:

$$\widetilde{\rho}_p(N(\cdot, z_0; W_1), N(\cdot, z_0; W_2)) \leq C_{3.14}(\varepsilon) \|W_1 - W_2\|_{\mathfrak{S}_p}. \tag{3.14}$$

Proof 1. Let us first prove the following auxiliary statement. For an operator $A = A^* \in \mathfrak{S}_p$, let $\{\lambda_n^{(+)}(A)\}_{n\in\mathbb{N}}$ be the sequence of its non-negative eigenvalues listed in decreasing order counting multiplicities, and let $\lambda_n^{(-)}(A) := \lambda_n^{(+)}(-A)$. Denote $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. Let $\Lambda(A) \in l_p(\mathbb{Z}_0)$ be the sequence

$$\Lambda_n(A) = \begin{cases} \lambda_n^{(+)}(A), & n > 0; \\ \lambda_{-n}^{(-)}(A), & n < 0. \end{cases}$$

Let us prove that for any self-adjoint operators $A_1, A_2 \in \mathfrak{S}_p$,

$$\|\Lambda(A_1) - \Lambda(A_2)\|_{l_p(\mathbb{Z}_0)} \leq \|A_1 - A_2\|_{\mathfrak{S}_p}. \tag{3.15}$$

For $p = \infty$, the above relation follows directly from the variational characterization of the eigenvalues. For $p = 1$, it can be proven by using some simple tricks with trace. Anyway, we proceed straight to the general case, which is a consequence of a slight modification of Lidski's theorem [16] (see also [12, Chapter 2, §6.5]). First note that it is sufficient to prove (3.15) for finite rank operators A_1, A_2. In the finite rank case, Lidski's theorem says that

$$\lambda_n(A_1) - \lambda_n(A_2) = \sum_m \sigma_{nm} \lambda_m(A_1 - A_2), \tag{3.16}$$

where $\{\lambda_n(A)\}$ is the sequence of all (positive and negative) eigenvalues of A, listed in the order of decreasing of the absolute value $|\lambda_n(A)|$, and σ_{nm} is a matrix satisfying

$$\sum_n |\sigma_{nm}| \leq 1, \quad \sum_m |\sigma_{nm}| \leq 1. \tag{3.17}$$

The relations (3.16), (3.17) imply (cf. [12]) that

$$\sum_n |\lambda_n(A_1) - \lambda_n(A_2)|^p \leq \sum_n |\lambda_n(A_1 - A_2)|^p = \|A_1 - A_2\|_{\mathfrak{S}_p}^p, \quad p \in [1, \infty),$$

which differs from the desired inequality (3.15) only by the method of numbering the eigenvalues. Following the proof of Lidski's theorem, it is not difficult to see that it holds also in the case when the positive and negative eigenvalues are numbered separately; more precisely, one has

$$\begin{gathered}\lambda_n^{(\pm)}(A_1)-\lambda_n^{(\pm)}(A_2)=\sum_m \sigma_{nm}^{(\pm)}\lambda_m(A_1-A_2),\\ \sum_n \left|\sigma_{nm}^{(+)}\right|+\left|\sigma_{nm}^{(-)}\right|\le 1,\quad \sum_m \left|\sigma_{nm}^{(\pm)}\right|\le 1.\end{gathered}\tag{3.18}$$

In the same way as above, (3.18) implies (3.15).

2. Below we will need the following fact. For any $\varphi\in C^\infty(\mathbb{T})$ and any two unitary operators W_1, W_2 such that $W_1-W_2\in\mathfrak{S}_p$, one has

$$\|\varphi(W_1)-\varphi(W_2)\|_{\mathfrak{S}_p}\le C_{3.19}(\varphi)\|W_1-W_2\|_{\mathfrak{S}_p}.\tag{3.19}$$

In order to prove (3.19) (see, e.g., [7, §5.4] for the details and discussion), one first writes a representation

$$\varphi(z)=\sum_{n\in\mathbb{Z}}c_n z^n,\quad \sum_{n\in\mathbb{Z}}|n||c_n|<\infty,$$

which is valid for all smooth enough φ. Next, it is easy to check that

$$\|W_1^n-W_2^n\|_{\mathfrak{S}_p}\le n\|W_1-W_2\|_{\mathfrak{S}_p}.$$

Therefore, (3.19) holds with $C_{3.19}(\varphi)=\sum_{n\in\mathbb{Z}}|n||c_n|$.

3. Now we are ready to prove the estimate (3.14). Let $\varphi_\varepsilon\in C^\infty(\mathbb{T})$ be such a function that $\varphi_\varepsilon(e^{i\theta})=\theta$ for all $\theta\in[-2\pi+\varepsilon,-\varepsilon]$. Denote $\varphi_{\varepsilon,z_0}(z):=\varphi_\varepsilon(z/z_0)+\arg z_0$, where $\arg z_0\in(0,2\pi)$. It is straightforward to see that for $j=1,2$ and $n=1,2,\dots$, one has

$$\begin{gathered}\nu(n-1;N(\cdot,z_0;W_j))=\lambda_n^{(+)}(\varphi_{\varepsilon,z_0}(W_j)),\\ \nu(-n;N(\cdot,z_0;W_j))=2\pi-\lambda_n^{(-)}(\varphi_{\varepsilon,z_0}(W_j)),\end{gathered}\tag{3.20}$$

and therefore

$$\widetilde{\rho}_p(N(\cdot,z_0;W_1),N(\cdot,z_0;W_2))=\|\Lambda(\varphi_{\varepsilon,z_0}(W_1))-\Lambda(\varphi_{\varepsilon,z_0}(W_2))\|_{l_p(\mathbb{Z}_0)}.\tag{3.21}$$

The relations (3.21), (3.15) and (3.19) together imply (3.14) with the constant

$$C_{3.14}(\varepsilon)=\sup_{z_0\in\mathbb{T}\setminus\{1\}}C_{3.19}(\varphi_{\varepsilon,z_0}).\quad\blacksquare$$

Proof of Proposition 3.6 Fix W_0 such that $z_0 \in \mathbb{T} \setminus \sigma(W_0)$ and $\varepsilon > 0$ such that $[z_0 e^{-i\varepsilon}, z_0 e^{i\varepsilon}] \cap \sigma(W_0) = \emptyset$. Then for any $W \in Y_p$ such that $\|W - W_0\| < \varepsilon/2$, one has $[z_0 e^{-i\varepsilon/2}, z_0 e^{i\varepsilon/2}] \cap \sigma(W) = \emptyset$. Thus, we can apply Lemma 3.1, which yields

$$\widetilde{\rho}_p(N(\cdot, z_0; W), N(\cdot, z_0; W_0)) \leq C_{3.14}(\varepsilon/2) \|W - W_0\|_{\mathfrak{S}_p}.$$

Clearly, this implies the continuity of the mapping in hand at the 'point' W_0. ▮

3.7. Lemma on convergence in X_p

In the proof of Theorem 7.1 below we shall need the following

LEMMA 3.2. *Let W_n and W_n' be sequences of operators in Y_p such that $\lim_{n\to\infty} \|W_n - W_n'\|_{\mathfrak{S}_p} = 0$. Then the limit $\mathrm{X_p}\text{-}\lim_{n\to\infty} \eta_p(W_n)$ exists if and only if the limit $\mathrm{X_p}\text{-}\lim_{n\to\infty} \eta_p(W_n')$ exists. If these limits exist, they coincide.*

Proof. 1. For any $f \in X_\infty$, let us introduce the notation

$$\sigma(f) := \{\exp(i\nu(n; \widetilde{f})) \mid n \in \mathbb{Z}\} \cup \{1\}, \quad \widetilde{f} \in \pi_\infty^{-1}(f)$$

(recall that $\nu(n; \widetilde{f})$ is defined by (3.2)). Clearly, this definition does not depend on the choice of an element $\widetilde{f} \in \pi_\infty^{-1}(f)$. It is also clear that in this notation,

$$\sigma(W) = \sigma(\eta_\infty(W)), \quad W \in Y_\infty.$$

2. Suppose that the limit $f := \mathrm{X_p}\text{-}\lim_{n\to\infty} \eta_p(W_n)$ exists. Below we prove that the limit $\mathrm{X_p}\text{-}\lim_{n\to\infty} \eta_p(W_n')$ also exists and is equal to f. Fix $z_0 \in \mathbb{T} \setminus \sigma(f)$ and $\varepsilon > 0$ such that $[z_0 e^{-i\varepsilon}, z_0 e^{i\varepsilon}] \cap \sigma(f) = \emptyset$. If n is large enough so that $\rho_\infty(f, \eta_\infty(W_n)) < \varepsilon/3$, we get

$$[z_0 e^{-i2\varepsilon/3}, z_0 e^{i2\varepsilon/3}] \cap \sigma(\eta_\infty(W_n)) = \emptyset.$$

Further, if n is large enough so that $\rho_\infty(f, \eta_\infty(W_n)) < \varepsilon/3$ and $\|W_n - W_n'\| < \varepsilon/3$, we get

$$[z_0 e^{-i\varepsilon/3}, z_0 e^{i\varepsilon/3}] \cap \sigma(\eta_\infty(W_n')) = \emptyset.$$

For such n we can apply Lemma 3.1, which yields

$$\rho_p(\eta_p(W_n), \eta_p(W_n')) \leq C_{3.14}(\varepsilon/3) \|W_n - W_n'\|_{\mathfrak{S}_p} \to 0 \quad \text{as } n \to \infty.$$

Thus, $\lim_{n\to\infty} \rho_p(\eta_p(W_n'), f) = 0$. ▮

4. THE FUNCTION μ: DEFINITION

4.1. Definition

Let H_0 and H be self-adjoint operators in a Hilbert space $\mathcal{H}$. For any $z \in \rho(H_0) \cap \rho(H)$ define a unitary operator in $\mathcal{H}$ by

$$\begin{aligned} M(z; H, H_0) &:= \frac{H - \overline{z}I}{H - zI} \frac{H_0 - zI}{H_0 - \overline{z}I} \\ &= (I + (z - \overline{z})(H - zI)^{-1})(I + (\overline{z} - z)(H_0 - \overline{z}I)^{-1}). \end{aligned} \tag{4.1}$$

Next, in what follows we fix $p \in [1, \infty]$. We introduce

Assumption 4.2. *(i) For any $z \in \rho(H_0) \cap \rho(H)$ one has*

$$(H - zI)^{-1} - (H_0 - zI)^{-1} \in \mathfrak{S}_p. \tag{4.2}$$

(ii) For any $\lambda \in \mathbb{R}$ one has

$$\lim_{y \to +\infty} y \left\| (H - (\lambda + iy)I)^{-1} - (H_0 - (\lambda + iy)I)^{-1} \right\|_{\mathfrak{S}_p} = 0. \tag{4.3}$$

By the identity

$$M(z) - I = (z - \overline{z})((H - zI)^{-1} - (H_0 - zI)^{-1}) \frac{H_0 - zI}{H_0 - \overline{z}I}, \tag{4.4}$$

the inclusion (4.2) is equivalent to

$$M(z; H, H_0) - I \in \mathfrak{S}_p(\mathcal{H}), \tag{4.5}$$

and the relation (4.3) is equivalent to

$$\lim_{y \to +\infty} \|M(\lambda + iy; H, H_0) - I\|_{\mathfrak{S}_p} = 0. \tag{4.6}$$

Proposition 4.1. *(i) If (4.2) holds for one value of z, then it holds for all $z \in \rho(H_0) \cap \rho(H)$.*
(ii) If (4.3) holds for one value of λ, then it holds for all $\lambda \in \mathbb{R}$.
(iii) Assumption 4.2(i) implies that the mapping

$$\mathbb{C} \setminus \mathbb{R} \ni z \mapsto M(z; H, H_0) - I \in \mathfrak{S}_p(\mathcal{H})$$

is continuous.

Further, we need one more assumption. Recall that the class X_p and the mapping η_p have been defined in §3.3, 3.4. Fix $\lambda \in \mathbb{R}$.

Assumption 4.3. *The limit*

$$\mathrm{X_p}\text{-}\lim_{y\to+0} \eta_p(M(\lambda+iy;H,H_0)) \tag{4.7}$$

exists.

Under the Assumptions 4.2 and 4.3, consider the mapping

$$U:(0,1)\ni t\mapsto M(\lambda+i(1-t)t^{-1};H,H_0)\in Y_p. \tag{4.8}$$

Clearly, the mapping U satisfies the hypothesis of Definition 3.1 and therefore $\mathrm{sf}(z;U)$ is well defined.

Definition 4.1. Suppose that for a pair of selfadjoint operators H_0, H and for $\lambda\in\mathbb{R}$, the Assumptions 4.2, 4.3 hold true. Let U be the mapping (4.8); then we define

$$\mu(\theta;\lambda,H,H_0):=\mathrm{sf}(e^{i\theta};U),\quad \theta\in(0,2\pi). \tag{4.9}$$

4.2. Sufficient conditions

Let $\mathcal{H}$ be a 'basic' and $\mathcal{K}$ an 'auxiliary' Hilbert spaces and let operators $H_0, G, J, H=H(H_0,G,J)$ be as described in §2.2. Below we give sufficient conditions (in terms of H_0, G, J), which ensure that the Assumptions 4.2 and 4.3 hold true for the pair H_0, H. In addition to (2.5), assume that

$$G(|H_0|+I)^{-1/2}\in\mathfrak{S}_{2p}(\mathcal{H},\mathcal{K}) \tag{4.10}$$

for some $p\in[1,\infty]$.

Proposition 4.2. *Assume* (2.5), (4.10). *Then, for the pair of operators H_0, H, Assumption 4.2 holds true.*

In particular, if $H-H_0\in\mathfrak{S}_p$, then Assumption 4.2 holds true.

Proposition 4.3. *Assume* (2.5), (4.10) *and define the operators* (2.6). *Suppose that for some $\lambda\in\mathbb{R}$*
(i) *the limit* $\mathrm{s}\text{-}\lim_{y\to+0}(J^{-1}+T(\lambda+iy))^{-1}$ *exists;*
(ii) *the limit* $\mathfrak{S}_\mathrm{p}\text{-}\lim_{y\to+0} B(\lambda+iy)=:B(\lambda+i0)$ *exists.*
Then, for the pair H_0, H, Assumption 4.3 holds at the point λ.

PROPOSITION 4.4. *Assume* (2.5), (4.10) *and suppose that for an open interval* $\delta \subset \mathbb{R}$ *one has*

$$GE_{H_0}(\delta) \in \mathfrak{S}_2(\mathcal{H}, \mathcal{K}). \tag{4.11}$$

Then for a.e. $\lambda \in \delta$

(i) *the limits*

$$\mathfrak{S}_{\mathrm{q}}\text{-}\lim_{y \to +0} T(\lambda + iy), \quad \mathfrak{S}_{\mathrm{p}}\text{-}\lim_{y \to +0} B(\lambda + iy) \tag{4.12}$$

exist, where $q = p$ *if* $p > 1$ *and* q *is any number greater than* 1, *if* $p = 1$;

(ii) *one has* $0 \in \rho(J^{-1} + T(\lambda + i0))$.

Thus, the hypotheses (i), (ii) of Proposition 4.3 hold true and the pair H_0, H *satisfies Assumption 4.3.*

In particular, if $H - H_0 \in \mathfrak{S}_1$, then Assumption 4.2 holds true for $p = 1$ and a.e. $\lambda \in \mathbb{R}$.

4.3. Operator $S(z)$

In order to prove Propositions 4.2–4.4, below we introduce an auxiliary operator $S(z)$. Let $\mathcal{H}$ be a 'basic' and $\mathcal{K}$ an 'auxiliary' Hilbert spaces. Let the operators H_0, G, J be as described in §2.2; assume (2.5) and (4.10) for some $p \in [1, \infty]$ and let $H = H(H_0, G, J)$. For any $z \in \mathbb{C} \setminus \mathbb{R}$ define

$$S(z) = S(z; H_0, G, J) := I - 2iB^{1/2}(z)(J^{-1} + T(z))^{-1}B^{1/2}(z). \tag{4.13}$$

The inverse operator in the r.h.s. of (4.13) exists by (2.9). A straightforward calculation shows that $S(z)$ is unitary in $\mathcal{K}$. Clearly, $S(z) - I \in \mathfrak{S}_p$. The operator $S(z)$ can also be presented as

$$\begin{aligned} S(z) &= I - 2iB^{1/2}(z)(I + JT(z))^{-1}JB^{1/2}(z) = \\ &= I - 2iB^{1/2}(z)J(I + T(z)J)^{-1}B^{1/2}(z). \end{aligned}$$

The definition of the operator $S(z)$ copies the stationary representation for the scattering matrix (see (9.1)). For this reason, the operators of this type are well studied (see, e.g., [8] and references therein).

LEMMA 4.1. *Assume* (2.5) *and* (4.10). *Then the mapping*

$$\mathbb{C} \setminus \mathbb{R} \ni z \mapsto S(z) - I \in \mathfrak{S}_p(\mathcal{K}) \tag{4.14}$$

is continuous and

$$\|S(z) - I\|_{\mathfrak{S}_p} \to 0 \text{ as } \operatorname{Im} z \to +\infty. \tag{4.15}$$

Proof. 1. Let us first check that

$$\text{the mapping} \quad \rho(H_0) \ni z \mapsto T(z) \in \mathfrak{S}_p \quad \text{is continuous} \tag{4.16}$$

and

$$\|T(z)\|_{\mathfrak{S}_p} \to 0 \quad \text{as } \operatorname{Im} z \to +\infty. \tag{4.17}$$

In order to do this, observe that the mapping

$$\rho(H_0) \ni z \mapsto \frac{|H_0| + I}{H_0 - zI} \in \mathcal{B}(\mathcal{H}) \tag{4.18}$$

is continuous (in the operator norm) and

$$\frac{|H_0| + I}{H_0 - zI} \xrightarrow{s} 0 \text{ as } \operatorname{Im} z \to +\infty. \tag{4.19}$$

Now recall the definition (2.6) of $T(z)$. By (2.1), the relation (4.16) follows from (4.10) and the continuity of (4.18). Similarly, (4.17) follows from (4.10) and (4.19).

2. Clearly, the relations (4.16) and (2.9) imply that

$$\text{the mapping} \quad \mathbb{C} \setminus \mathbb{R} \ni z \mapsto (J^{-1} + T(z))^{-1} \in \mathcal{B}(\mathcal{H}) \quad \text{is continuous.} \tag{4.20}$$

3. By (2.2), the relations (4.16) and (4.20) imply the continuity of the mapping (4.14). The relation (4.17) implies (4.15). ▮

THEOREM 4.1. *Assume* (2.5) *and let* $H = H(H_0, G, J)$. *For any* $z \in \mathbb{C} \setminus \mathbb{R}$ *the operator* $M(z; H, H_0) - I$ *is compact and*

$$\eta_\infty(M(z; H, H_0)) = \eta_\infty(S(z; H_0, G, J)). \tag{4.21}$$

Proof. 1. By (4.4) and (2.10), one has

$$\begin{aligned} M(z) = I - (z - \overline{z})(G(H_0 - \overline{z}I)^{-1})^* \\ \times (J^{-1} + T(z))^{-1}(G(H_0 - zI)^{-1})(I - (z - \overline{z})(H_0 - \overline{z}I)^{-1}). \end{aligned} \tag{4.22}$$

It follows that $M(z) - I \in \mathfrak{S}_\infty$.

2. For $R > 0$, denote $P^{(R)} = E_{H_0}((-R,R))$, $G^{(R)} = GP^{(R)}$, $H_0^{(R)} = H_0 P^{(R)}$. Note that $G^{(R)} \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{K})$ and $H_0^{(R)} \in \mathcal{B}(\mathcal{H})$. Further, let $H^{(R)} = H_0^{(R)} + (G^{(R)})^* J G^{(R)} (\in \mathcal{B}(\mathcal{H}))$. By (2.1), the relation $P^{(R)} = (P^{(R)})^* \xrightarrow{s} I$ implies that

$$\|G^{(R)}(|H_0|+I)^{-1/2} - G(|H_0|+I)^{-1/2}\| \to 0 \text{ as } R \to +\infty,$$

and thus

$$\|T(z; H_0^{(R)}, G^{(R)}) - T(z; H_0, G)\| \to 0 \text{ as } R \to +\infty.$$

By the definition (4.13) of $S(z)$ it follows that

$$\|S(z; H_0^{(R)}, G^{(R)}, J) - S(z; H_0, G, J)\| \to 0 \text{ as } R \to +\infty$$

and by (4.22) it follows that

$$\|M(z; H^{(R)}, H_0^{(R)}) - M(z; H, H_0)\| \to 0 \text{ as } R \to +\infty.$$

Therefore, since the mapping $\eta_\infty : Y_\infty \to X_\infty$ is continuous, it is sufficient to prove that

$$\eta_\infty(M(z; H^{(R)}, H_0^{(R)}) = \eta_\infty(S(z; H_0^{(R)}, G^{(R)}, J)) \tag{4.23}$$

for any $R > 0$. For the sake of brevity, below we suppress the index R in the notation and suppose that $H_0 \in \mathcal{B}(\mathcal{H})$ and $G \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{K})$. We also denote $V := G^* J G$.

3. Recall that for any two bounded operators A, B and any $\lambda \neq 0$ one has

$$\dim \operatorname{Ker}(AB - \lambda I) = \dim \operatorname{Ker}(BA - \lambda I). \tag{4.24}$$

By (4.24), for any $\lambda \in \mathbb{T} \setminus \{1\}$, one has

$$\begin{aligned}
\dim \operatorname{Ker}(M(z) - \lambda I) &= \dim \operatorname{Ker}\left(\frac{H - \overline{z} I}{H - zI}\frac{H_0 - zI}{H_0 - \overline{z} I} - \lambda I\right) \\
&= \dim \operatorname{Ker}\big((H - \overline{z} I)(H_0 - \overline{z} I)^{-1}((H - zI)(H_0 - zI)^{-1})^{-1} - \lambda I\big) \\
&= \dim \operatorname{Ker}\big((I + V(H_0 - \overline{z} I)^{-1})(I + V(H_0 - zI)^{-1})^{-1} - \lambda I\big) \\
&= \dim \operatorname{Ker}\big(I - 2iV\operatorname{Im}((H_0 - zI)^{-1})(I + V(H_0 - zI)^{-1})^{-1} - \lambda I\big) \\
&= \dim \operatorname{Ker}\big(I - 2iG\operatorname{Im}((H_0 - zI)^{-1})(I + V(H_0 - zI)^{-1})^{-1}G^* J - \lambda I\big).
\end{aligned}$$

A direct computation shows that

$$(I + V(H_0 - zI)^{-1})^{-1} G^* J = G^*(J^{-1} + T(z))^{-1}.$$

Thus,

$$\begin{aligned}
&\dim \operatorname{Ker}(M(z) - \lambda I) \\
&\quad = \dim \operatorname{Ker}\big(I - 2iG\operatorname{Im}((H_0 - zI)^{-1})G^*(J^{-1} + T(z))^{-1} - \lambda I\big) \\
&\quad = \dim \operatorname{Ker}\big(I - 2iB(z)(J^{-1} + T(z))^{-1} - \lambda I\big) \\
&\quad = \dim \operatorname{Ker}\Big(I - 2iB^{1/2}(z)(J^{-1} + T(z))^{-1} B^{1/2}(z) - \lambda I\Big) \\
&\quad = \dim \operatorname{Ker}(S(z) - \lambda I),
\end{aligned}$$

which implies (4.21). ▌

4.4. Proofs of Propositions 4.1, 4.2–4.4

Proof of Proposition 4.1 (i) follows from the identity

$$\begin{aligned}
&(H - zI)^{-1} - (H_0 - zI)^{-1} \\
&\qquad = \frac{H - z_0 I}{H - zI}((H - z_0 I)^{-1} - (H_0 - z_0 I)^{-1})\frac{H_0 - z_0 I}{H_0 - zI}. \quad (4.25)
\end{aligned}$$

(ii) Suppose that (4.3) holds for $\lambda = \lambda_0$. In (4.25), take $z = \lambda + iy$, $z_0 = \lambda_0 + iy$. Now the desired assertion follows from the fact that

$$\sup_{y>1} \left\| \frac{H - (\lambda_0 + iy)I}{H - (\lambda + iy)I} \right\| < \infty, \quad \sup_{y>1} \left\| \frac{H_0 - (\lambda_0 + iy)I}{H_0 - (\lambda + iy)I} \right\| < \infty.$$

(iii) Let us use (4.22) and check that the r.h.s. of this identity depends continuously on z in the $\mathfrak{S}_p$ norm. Similarly to the proof of Lemma 4.1, factorizing

$$G(H_0 - zI)^{-1} = [G(|H_0| + I)^{-1/2}][(|H_0| + I)^{1/2}(H_0 - zI)^{-1}],$$

and using (2.1), we check that the operator $G(H_0 - zI)^{-1}$ depends continuously on z in $\mathfrak{S}_{2p}$ norm. Taking into account (4.20) and the fact that the operator $(I - (z - \overline{z})(H_0 - \overline{z}I)^{-1})$ depends continuously on z in the operator norm, we get the desired assertion. ▌

Proof of Proposition 4.2 Let us use (2.10). Since $(J^{-1} + T(z))^{-1}$ is bounded and $G(H_0 - zI)^{-1} \in \mathfrak{S}_{2p}$, we get the inclusion (4.2). The relation (4.3) is equivalent to (4.6); the latter follows from Theorem 4.1 and (4.15). ▌

Proof of Proposition 4.3 By Theorem 4.1, it is sufficient to prove that the limit

$$\mathfrak{S}_{\mathrm{p}}\text{-}\lim_{y\to+0}(S(\lambda+iy;H_0,G,J)-I)$$

exists. By (2.2), the existence of the above limit follows directly from the definition of operator S and the hypothesis of the proposition. ▮

Proof of Proposition 4.4

1. For any $\delta'\subset\mathbb{R}$, denote

$$T_{\delta'}(z)=T(z;H_0,GE_{H_0}(\delta')).$$

Denoting $\Delta=\mathbb{R}\setminus\delta$, we see that

$$T(z)=T_\delta(z)+T_\Delta(z).$$

It is one of the classical results of the trace class scattering theory (see [5, 18, 19]) that the inclusion (4.11) implies that for a.e. $\lambda\in\mathbb{R}$ the limit $T_\delta(\lambda+i0)$ exists in $\mathfrak{S}_r(\mathcal{K})$ (for any $r>1$) and the limit $\lim_{y\to+0}\operatorname{Im}T_\delta(\lambda+iy)$ exists in $\mathfrak{S}_1(\mathcal{K})$. On the other hand, the function $T_\Delta(z)\in\mathfrak{S}_p(\mathcal{K})$ is analytic in $\mathbb{C}\setminus\Delta$ and $\operatorname{Im}T_\Delta(\lambda)=0$ for all $\lambda\in\delta$. Thus, for a.e. $\lambda\in\delta$ the limits (4.12) exist.

2. It remains to check that the limit $\text{n-}\lim_{y\to+0}(J^{-1}+T(\lambda+iy))^{-1}$ exists for a.e. $\lambda\in\delta$. In order to do this, write

$$(J^{-1}+T(z))^{-1}=(J^{-1}+T_\Delta(z))^{-1}(I+F(z))^{-1},$$
$$F(z)=T_\delta(z)(J^{-1}+T_\Delta(z))^{-1}.$$

Let us check that for a.e. $\lambda\in\mathbb{R}$ the limits

$$\text{n-}\lim_{y\to+0}(J^{-1}+T_\Delta(\lambda+iy))^{-1}\quad\text{and}\quad\text{n-}\lim_{y\to+0}(I+F(\lambda+iy))^{-1}\tag{4.26}$$

exist.

3. By the Fredholm analytic alternative, the set

$$\mathcal{N}=\{\lambda\in\delta\mid 0\in\sigma(J^{-1}+T_\Delta(\lambda))\}$$

is discrete in δ (i.e., the points of $\mathcal{N}$ can possibly accumulate only to the endpoints of the interval δ). Thus, the limit $\text{n-}\lim_{y\to+0}(J^{-1}+T_\Delta(\lambda+iy))^{-1}$ exists for all $\lambda\in\delta\setminus\mathcal{N}$.

4. The function $F(z)\in\mathfrak{S}_1(\mathcal{K})$ is analytic in $\mathbb{C}_+$ and for a.e. $\lambda\in\delta$ has limit values $F(\lambda+i0)$ in $\mathfrak{S}_q(\mathcal{K})$ (for any $q>1$). Thus, using Theorem 1.8.5 from [26], we obtain that the limit $\text{n-}\lim_{y\to+0}(I+F(\lambda+iy))^{-1}$ exists for a.e. $\lambda\in\delta$. ▮

5. FORMULA FOR μ

5.1. Statement of result

Let the operators H_0, G, J be as described in §2.2, assume (2.5) and let $H = H(H_0, G, J)$. For a self-adjoint operator A, we denote $\Xi(A) := E_A((-\infty, 0))$; see [10] for the reasoning behind this notation.

THEOREM 5.1. *Suppose that, for some $\lambda \in \mathbb{R}$, the limit* $\text{n-lim}_{y\to+0} T(\lambda + i\varepsilon)$ *exists and $0 \in \rho(J^{-1} + T(\lambda + i0))$. Then for all $\theta \in (0, 2\pi)$ the pair of projections $\Xi(J^{-1})$, $\Xi(J^{-1} + A(\lambda + i0) + \cot(\theta/2)B(\lambda + i0))$ is Fredholm and*

$$\mu(\theta; \lambda, H, H_0) = \operatorname{index}\big(\Xi(J^{-1}), \Xi(J^{-1} + A(\lambda + i0) + \cot(\theta/2)B(\lambda + i0))\big). \tag{5.1}$$

If $J = \pm I$, then (5.1) takes the form

$$\mu(\theta; \lambda, H, H_0) = -\operatorname{rank} E_{A(\lambda+i0)+\cot(\theta/2)B(\lambda+i0)}((-\infty, -1)), \qquad J = I,$$
$$\mu(\theta; \lambda, H, H_0) = \operatorname{rank} E_{A(\lambda+i0)+\cot(\theta/2)B(\lambda+i0)}([1, \infty)), \qquad J = -I.$$

Note that, in particular, this implies the following monotonicity rule for the function μ:

$$\pm J \geq 0 \quad \Rightarrow \quad \mp\mu(\theta; \lambda, H, H_0) \geq 0.$$

Related statements are well known in the spectral analysis of the scattering matrix — see [8] and references therein.

The relation (5.1) also implies the following estimates for μ:

$$\pm\mu(\theta; \lambda, H, H_0) \leq \operatorname{rank} \Xi(\pm J).$$

In particular, if the perturbation G^*JG has rank $n < \infty$, then the absolute value of μ does not exceed n.

5.2. The spectrum of $S(z)$

Consider the following operators A, B, J:

$$\begin{aligned} &A = A^* \in \mathfrak{S}_\infty(\mathcal{K}), \quad 0 \leq B \in \mathfrak{S}_\infty(\mathcal{K}), \quad J = J^* \in \mathcal{B}(\mathcal{K}), \\ &0 \in \rho(J), \quad 0 \in \rho(J^{-1} + A + iB). \end{aligned} \tag{5.2}$$

Under these assumptions, define a unitary operator in $\mathcal{K}$ by

$$S = I - 2iB^{1/2}(J^{-1} + A + iB)^{-1}B^{1/2}. \tag{5.3}$$

The proof of (5.1) is based on the following simple characterization of the spectrum of S.

LEMMA 5.1. *Assume* (5.2) *and let* S *be defined by* (5.3)*. Then for any* $\theta \in (0, 2\pi)$ *one has*

$$\dim \operatorname{Ker}(S - e^{i\theta} I) = \dim \operatorname{Ker}(J^{-1} + A + \cot(\theta/2)B). \tag{5.4}$$

Proof One has (using (4.24)):

$$\begin{aligned}\dim \operatorname{Ker}(S - e^{i\theta} I) &= \dim \operatorname{Ker}(I - 2iB(J^{-1} + A + iB)^{-1} - e^{i\theta} I) \\ &= \dim \operatorname{Ker}((J^{-1} + A - iB)(J^{-1} + A + iB)^{-1} - e^{i\theta} I) \\ &= \dim \operatorname{Ker}(J^{-1} + A - iB - e^{i\theta}(J^{-1} + A + iB)) \\ &= \dim \operatorname{Ker}(J^{-1} + A + \cot(\theta/2)B). \quad \blacksquare\end{aligned}$$

We shall need the following auxiliary statement, which is a very slight modification of one of the results of [11].

LEMMA 5.2. *Let* $M = M^* \in \mathcal{B}(\mathcal{K})$*,* $0 \leq B \in \mathfrak{S}_\infty(\mathcal{K})$ *and* $0 \in \rho(M + \tau B)$ *for some* $\tau \in \mathbb{R}$*. Then* $\Xi(M), \Xi(M + B)$ *is a Fredholm pair of projections and*

$$\operatorname{index}(\Xi(M), \Xi(M + B)) = \sum_{s \in (0,1]} \dim \operatorname{Ker}(M + sB). \tag{5.5}$$

Proof. 1. In [11, Corollary 4.8], the desired assertion has been proven under the additional assumption $B \in \mathfrak{S}_1(\mathcal{K})$. Below we show that this assumption can be lifted.

2. First note that the condition $0 \in \rho(M + \tau B)$ implies that $0 \notin \sigma_{ess}(M)$. Further, it is easy to see that

$$\Xi(M) - \Xi(M + B) \in \mathfrak{S}_\infty(\mathcal{K}).$$

This can be proven by representing the above projections by Riesz integrals and using the resolvent identity (cf. [11, Lemmas 3.5, 3.8]). The above inclusion implies that $\Xi(M)$, $\Xi(M + B)$ is a Fredholm pair.

3. First assume that $0 \in \rho(M)$ and $0 \in \rho(M + B)$. Let $0 \leq B_n \in \mathfrak{S}_1(\mathcal{K})$, $\|B_n - B\| \to 0$ as $n \to \infty$. For all large enough n we will have $0 \in \rho(M + \tau B_n)$. By [11, Corollary 4.8], for such n one has

$$\operatorname{index}(\Xi(M), \Xi(M + B_n)) = \sum_{s \in (0,1]} \dim \operatorname{Ker}(M + sB_n). \tag{5.6}$$

Our aim is to pass to the limit in (5.6).

4. By [11, Theorem 3.12], the l.h.s. of (5.6) tends to the l.h.s. of (5.5) as $n \to \infty$. Further, by the Birman–Schwinger principle in a gap (see, e.g., [3]), one has

$$\sum_{s\in(0,1]} \dim\operatorname{Ker}(M+sB) = \operatorname{rank} E_{B^{1/2}M^{-1}B^{1/2}}((-\infty,-1]).$$

Since $\|B_n^{1/2}M^{-1}B_n^{1/2} - B^{1/2}M^{-1}B^{1/2}\| \to 0$, we see that the r.h.s of (5.6) tends to the r.h.s. of (5.5).

5. In order to get rid of the assumptions $0 \in \rho(M)$, $0 \in \rho(M+B)$, we observe that for all small enough $\varepsilon > 0$ one has $0 \in \rho(M+\varepsilon B)$, $0 \in \rho(M+B+\varepsilon B)$ and thus

$$\operatorname{index}(\Xi(M+\varepsilon B), \Xi(M+B+\varepsilon B)) = \sum_{s\in(\varepsilon,1+\varepsilon]} \dim\operatorname{Ker}(M+sB).$$

Taking $\varepsilon \to +0$ in the above formula, we get (5.5). ▮

Lemma 5.3. *Assume* (5.2) *and let* S *be defined by* (5.3)*. Then for the function* $N(\cdot,\cdot;S)$*, defined by* (3.5)*, one has for any* $\theta_1,\theta_2 \in (0,2\pi)$*:*

$$\begin{aligned} N(e^{i\theta_1}, e^{i\theta_2}; S) &= \operatorname{index}\big(\Xi(J^{-1}+A+\cot(\theta_2/2)B), \Xi(J^{-1}+A+\cot(\theta_1/2)B)\big) \\ &= \operatorname{index}\big(\Xi(J^{-1}), \Xi(J^{-1}+A+\cot(\theta_1/2)B)\big) \\ &\quad + \operatorname{index}\big(\Xi(J^{-1}+A+\cot(\theta_2/2)B), \Xi(J^{-1})\big); \end{aligned} \tag{5.7}$$

all the three pairs of projections in the r.h.s. are Fredholm.

Proof. 1. First of all we note that

$$\Xi(J^{-1}+A+\cot(\theta_j/2)B) - \Xi(J^{-1}) \in \mathfrak{S}_\infty(\mathcal{K}), \quad j=1,2. \tag{5.8}$$

As in the previous lemma, this can be proven by representing $\Xi(J^{-1} + A+\cot(\theta_j/2)B)$ and $\Xi(J^{-1})$ by the Riesz integrals and using the resolvent identity (cf. [11, Lemmas 3.5, 3.8]). The inclusion (5.8) implies that all the three pairs of projections in the r.h.s. of (5.7) are Fredholm.

2. It is sufficient to prove (5.7) for $\theta_1 < \theta_2$. Indeed, the case $\theta_1 > \theta_2$ follows from the above mentioned one by changing the roles of θ_1 and θ_2; for $\theta_1 = \theta_2$ the relation (5.7) trivially holds.

In the case $\theta_1 < \theta_2$, using Lemmas 5.1 and 5.2, one has:

$$
\begin{aligned}
\mathrm{rank} E_S([e^{i\theta_1}, e^{i\theta_2})) &= \sum_{\theta\in[\theta_1,\theta_2)} \dim \mathrm{Ker}(S - e^{i\theta} I) \\
&= \sum_{\theta\in[\theta_1,\theta_2)} \dim \mathrm{Ker}(J^{-1} + A + \cot(\theta/2)B) \\
&= \sum_{\theta\in(\cot(\theta_2/2),\cot(\theta_1/2)]} \dim \mathrm{Ker}(J^{-1} + A + tB) \\
&= \mathrm{index}(\Xi(J^{-1} + A + \cot(\theta_2/2)B), \Xi(J^{-1} + A + \cot(\theta_1/2)B)).
\end{aligned}
$$

Note that Lemma 5.2 is applicable, since, by the analytic Fredholm alternative, the assumption $0 \in \rho(J^{-1} + A + iB)$ (see (5.2)) implies that $0 \in \rho(J^{-1} + A + \tau B)$ for all $\tau \in \mathbb{R}$ but for a discrete set of points.

3. Thus, we have proven the first equality in (5.7). The second one follows by the chain rule (2.4). Note that the inclusion (5.8) ensures the applicability of the chain rule. ▮

5.3. Proof of Theorem 5.1

1. First we need a simple result which shows that the r.h.s. of (5.1) depends continuously on $A(\lambda+i0)$ and $B(\lambda+i0)$. This statement is closely related to [20, Lemma 2.5] and [11, Theorem 3.12].

Lemma 5.4. *Assume* (5.2) *and let, in addition,* $B \in \mathfrak{S}_p$, $p \in [1,\infty]$. *Let* $A_k = A_k^* \in \mathfrak{S}_\infty(\mathcal{K})$, $0 \le B_k \in \mathfrak{S}_p(\mathcal{K})$, $J_k = J_k^* \in \mathcal{B}(\mathcal{K})$, $k \in \mathbb{N}$ *be such operators that* $0 \in \rho(J_k)$, $0 \in \rho(J_k^{-1} + A_k + iB_k)$, $\lim_{k\to\infty} \|A_k - A\| = 0$, $\lim_{k\to\infty} \|B_k - B\|_{\mathfrak{S}_p} = 0$, $\lim_{k\to\infty} \|J_k - J\| = 0$. *Define the functions*

$$
\begin{aligned}
f &: \mathbb{T}\setminus\{1\} \ni e^{i\theta} \mapsto f(e^{i\theta}) = \mathrm{index}(\Xi(J^{-1}), \Xi(J^{-1} + A + \cot(\theta/2)B)) \in \mathbb{Z}, \\
f_k &: \mathbb{T}\setminus\{1\} \ni e^{i\theta} \mapsto f_k(e^{i\theta}) = \mathrm{index}(\Xi(J_k^{-1}), \Xi(J_k^{-1} + A_k + \cot(\theta/2)B_k)) \in \mathbb{Z}.
\end{aligned}
$$

Then $f, f_k \in \widetilde{X}_p$ *and*

$$
\widetilde{\rho}_p(f_k, f) \to 0 \quad \text{as } k \to \infty. \tag{5.9}
$$

Proof. 1. Define the operator S by (5.3) and let

$$
S_k = I - 2iB_k^{1/2}(J_k^{-1} + A_k + iB_k)^{-1}B_k^{1/2}.
$$

As in Proposition 4.3, we see that $\|S_k - S\|_{\mathfrak{S}_p} \to 0$ as $k \to \infty$.

Fix $\theta_0 \in (0, 2\pi)$ such that $e^{i\theta_0} \in \rho(S)$. By Proposition 3.6,

$$\widetilde{\rho}_p(N(\cdot, e^{i\theta_0}; S_k), N(\cdot, e^{i\theta_0}; S)) \to 0 \quad \text{as } k \to \infty. \tag{5.10}$$

2. By Lemma 5.3,

$$N(e^{i\theta}, e^{i\theta_0}; S) = f(e^{i\theta}) + C(\theta_0),$$
$$N(e^{i\theta}, e^{i\theta_0}; S_k) = f_k(e^{i\theta}) + C_k(\theta_0)$$

with

$$C(\theta_0) = \operatorname{index}(\Xi(J^{-1} + A + \cot(\theta_0/2)B), \Xi(J^{-1})),$$
$$C_k(\theta_0) = \operatorname{index}(\Xi(J_k^{-1} + A_k + \cot(\theta_0/2)B_k), \Xi(J_k^{-1})).$$

Since $e^{i\theta_0} \in \rho(S)$, by Lemma 5.1 one has $0 \in \rho(J^{-1} + A + \cot(\theta_0/2)B)$. By [11, Theorem 3.12], it follows that $\lim_{k\to\infty} C_k(\theta_0) = C(\theta_0)$. Since $C_k(\theta_0)$ and $C(\theta_0)$ are integer valued, one has $C_k(\theta_0) = C(\theta_0)$ for all large enough k. Thus, by (3.11), the relation (5.10) implies (5.9). ▌

2. *Proof of Theorem 5.1:* 1. First of all, we note that for all $\theta \in (0, 2\pi)$

$$\Xi(J^{-1}) - \Xi(J^{-1} + A(\lambda + i0) + \cot(\theta/2)B(\lambda + i0)) \in \mathfrak{S}_\infty(\mathcal{K})$$

(cf. (5.8)) and thus the pair of projections in the r.h.s. of (5.1) is Fredholm.

2. Let U be the mapping (4.8) (for $p = \infty$) and $\gamma = \operatorname{ext}(\eta_\infty \circ U)$ (recall that η_∞ has been introduced in §3.4, and ext — in §3.5). Below we explicitly construct the lift of γ. Let us define the mapping $\widetilde{\gamma} : [0, 1] \to \widetilde{X}_\infty$ by

$$\begin{aligned}
&\widetilde{\gamma}(e^{i\theta}; 0) = 0;\\
&\widetilde{\gamma}(e^{i\theta}; t) = \operatorname{index}\bigl(\Xi(J^{-1}), \Xi(J^{-1} + A(z) + \cot(\theta/2)B(z))\bigr),\\
&\qquad\qquad z = \lambda + i(1-t)t^{-1}, \quad t \in (0, 1);\\
&\widetilde{\gamma}(e^{i\theta}; 1) = \operatorname{index}\bigl(\Xi(J^{-1}), \Xi(J^{-1} + A(\lambda + i0) + \cot(\theta/2)B(\lambda + i0))\bigr).
\end{aligned}$$

Below we show that:
(i) $\widetilde{\gamma}$ is continuous;
(ii) $\pi_\infty \circ \widetilde{\gamma} = \gamma$.

The statements (i), (ii) mean that $\widetilde{\gamma}$ is the lift of γ with $\widetilde{\gamma}(0) = 0$. Since the r.h.s. of (5.1) coincides with $\widetilde{\gamma}(e^{i\theta}; 1)$, this implies the statement of the theorem.

3. By Lemma 5.4, the continuity of $\widetilde{\gamma}$ for $t \in (0, 1)$ follows from the norm continuity of $A(z)$, $B(z)$ (see (4.16)) in z. Similarly, the continuity of $\widetilde{\gamma}$ at $t = 0$ follows from (4.17) and the continuity at $t = 1$ is evident.

The relation $\pi_\infty \circ \widetilde{\gamma} = \gamma$ follows from Theorem 4.1 and Lemma 5.3. ▮

6. THE FUNCTION μ AND THE PERTURBATION DETERMINANT

6.1. Statement of result

Let the operators H_0, G, J be as described in §2.2. Assume (2.5) and (4.10) with $p = 1$ and let $H = H(H_0, G, J)$. As in [26, §8.1.4], we introduce the 'modified perturbation determinant'

$$D_{H/H_0}(z) = \det(I + JT(z)), \quad z \in \rho(H_0). \tag{6.1}$$

If the operator $V = G^*JG$ is well defined and $V(H_0 - zI)^{-1} \in \mathfrak{S}_1(\mathcal{H})$, then $D_{H/H_0}(z)$ coincides with the usual perturbation determinant $\Delta_{H/H_0}(z)$. By (4.16), the determinant $D_{H/H_0}(z)$ is continuous in $z \in \rho(H_0)$ (it is, of course, even analytic in z, but we do not use this fact). By (4.17) with $p = 1$, one has $D_{H/H_0}(z) \to 0$ as $\operatorname{Im} z \to +\infty$. Let us fix the branch of $\arg D_{H/H_0}(z)$ by

$$\arg D_{H/H_0}(z) \to 0 \text{ as } \operatorname{Im} z \to +\infty. \tag{6.2}$$

By Propositions 4.2, 4.4, for $p = 1$ and a.e. $\lambda \in \mathbb{R}$, the Assumptions 4.2 and 4.3 hold true. Therefore, for a.e. $\lambda \in \mathbb{R}$ the function $\mu(\cdot; \lambda, H, H_0)$ is well defined and belongs to $L_1(0, 2\pi)$.

Theorem 6.1. *Assume* (2.5) *and* (4.10) *with* $p = 1$, *define the function* D_{H/H_0} *by* (6.1) *and fix the branch of* $\arg D_{H/H_0}$ *by* (6.2). *Then for a.e.* $\lambda \in \mathbb{R}$ *the limit* $\lim_{y \to +0} \arg D_{H/H_0}(\lambda + iy)$ *exists and*

$$\begin{aligned} \lim_{y \to +0} \arg D_{H/H_0}(\lambda + iy) &= -\frac{1}{2} \int_0^{2\pi} \mu(\theta; \lambda, H, H_0) d\theta \\ &= \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} \operatorname{index} \left(\Xi(J^{-1} + A(\lambda + i0) + tB(\lambda + i0)), \Xi(J^{-1}) \right). \end{aligned} \tag{6.3}$$

remark. A similar reasoning shows that under the hypothesis of Theorem 6.1, one has for all $z \in \mathbb{C}_+$

$$\arg D_{H/H_0}(z) = \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} \operatorname{index} \left(\Xi(J^{-1} + A(z) + tB(z)), \Xi(J^{-1}) \right).$$

This formula might be of an independent interest, although we do not need it in this paper.

Recalling the Krein's formula (1.3), (1.4) for the SSF, we see that for $G \in \mathfrak{S}_2(\mathcal{H}, \mathcal{K})$, the first equation in (6.3) implies (1.12). The second equation (in the case $J^2 = I$) gives the representation (1.7), which was originally obtained in [11].

6.2. Proof of Theorem 6.1

1. First let us prove that

$$\det M(z; H, H_0) = \overline{D_{H/H_0}(z)}/D_{H/H_0}(z), \quad z \in \mathbb{C}_+. \tag{6.4}$$

One has:

$$\begin{aligned}
\overline{D_{H/H_0}(z)}/D_{H/H_0}(z) &= \det\big((I + JT(\overline{z}))(I + JT(z))^{-1}\big) \\
&= \det\big((I + JT(z) - 2iJB(z))(I + JT(z))^{-1}\big) \\
&= \det\big(I - 2iJB(z)(I + JT(z))^{-1}\big) \\
&= \det\Big(I - 2iB^{1/2}(z)(I + JT(z))^{-1}JB^{1/2}(z)\Big) \\
&= \det S(z; H_0, G, J).
\end{aligned}$$

Finally, note that, by Theorem 4.1,

$$\det S(z; H_0, G, J) = \det M(z; H, H_0).$$

2. It follows from (6.4) that

$$\arg D_{H/H_0}(z) = -\frac{1}{2} \arg\det M(z; H, H_0),$$

where the branches are fixed by (6.2) and by the condition

$$\arg\det M(z; H, H_0) \to 0 \text{ as } \operatorname{Im} z \to +\infty. \tag{6.5}$$

Now let U be the mapping (4.8) (for $p = 1$) and $\gamma = \operatorname{ext}(\eta_1 \circ U)$ (recall that η_1 has been introduced in §3.4, and ext — in §3.5). Note that for any $W \in Y_1$,

$$\det W = \exp\left(i \int_0^{2\pi} f(e^{i\theta}) d\theta\right), \quad f \in \pi_1^{-1}(\eta_1(W)).$$

Thus, it is clear that with the choice (6.5) of the branch, one has

$$\arg\det M(z; H, H_0) = \int_0^{2\pi} \widetilde{\gamma}(e^{i\theta}; t) d\theta, \quad z = \lambda + i(1 - t)t^{-1},$$

where $\widetilde{\gamma}$ is the lift of γ with the initial condition $\widetilde{\gamma}(0) = 0$. This proves the first of the equalities (6.3). The second one follows from Theorem 5.1 after the change of variables $t = \cot(\theta/2)$. ∎

7. THE INVARIANCE PRINCIPLE FOR μ

7.1. Statement of results

Let H_0 and H be self-adjoint operators in a Hilbert space $\mathcal{H}$. Fix $\lambda \in \mathbb{R}$. In this section we prove the invariance principle (1.13) for the function μ. For the sake of convenience of notation, we shall prove it in the following form:

$$\mu(\theta; f_1(\lambda), f_1(H), f_1(H_0)) = \mu(\theta; f_2(\lambda), f_2(H), f_2(H_0)), \quad \theta \in (0, 2\pi). \tag{7.1}$$

The functions f_1, f_2 in (7.1) are supposed to satisfy Assumption 1.1 (with λ from (7.1) and with the same $\Omega \supset \sigma(H_0) \cup \sigma(H)$ for f_1 and f_2).

Theorem 7.1. *Let $\Omega \subset \mathbb{R}$ be a Borel set, $\sigma(H_0) \cup \sigma(H) \subset \Omega$, and let the functions f_1, f_2 satisfy Assumption 1.1 with $\lambda \in \Omega$. Let the two pairs of operators $f_j(H_0), f_j(H)$, $j = 1, 2$, satisfy Assumption 4.2(i) (for $p = \infty$). Then:*

(i) *Assumption 4.3 (for $p = \infty$) holds true for the pair $f_1(H_0), f_1(H)$ at the point $f_1(\lambda)$ if and only if it holds true for the pair $f_2(H_0), f_2(H)$ at the point $f_2(\lambda)$.*

(ii) *If for $j = 1, 2$ Assumption 4.3 (for $p = \infty$) holds true for the pair $f_j(H_0), f_j(H)$ at the point $f_j(\lambda)$, then*

$$\begin{aligned} &\mathrm{X}_\infty\text{-}\lim_{y \to +0} \eta_\infty\big(M(f_1(\lambda) + iy; f_1(H), f_1(H_0))\big) \\ &\qquad = \mathrm{X}_\infty\text{-}\lim_{y \to +0} \eta_\infty\big(M(f_2(\lambda) + iy; f_2(H), f_2(H_0))\big). \end{aligned} \tag{7.2}$$

Suppose that under the hypothesis of Theorem 7.1, the two pairs of operators $f_j(H_0), f_j(H)$, $j = 1, 2$, satisfy the Assumption 4.2(ii) (for $p = \infty$). Then $\mu(\cdot; f_j(\lambda), f_j(H), f_j(H_0))$ is well defined for $j = 1, 2$. The relation (7.2) leads to the invariance principle (7.1) modulo $\mathbb{Z}$. In order to obtain the invariance principle in the full scale, we have to replace Assumption 4.2 by a pair of slightly more restrictive conditions.

For $z \in \mathbb{C}$, $z \notin \mathbb{R}_- := \{z \mid \operatorname{Im} z = 0, \operatorname{Re} z < 0\}$, let us fix the branch of $\arg z$, say, by

$$\arg z \in (-\pi, \pi), \quad z \in \mathbb{C} \setminus \mathbb{R}_-. \tag{7.3}$$

Assumption 7.4. *For a pair of self-adjoint operators H_0, H, one has:*
(i) *for any $z \in \mathbb{C}_+$,*

$$\arg(H - zI) - \arg(H_0 - zI) \in \mathfrak{S}_\infty(\mathcal{H}); \tag{7.4}$$

(ii) *for any $\lambda \in \mathbb{R}$,*

$$\lim_{y\to+\infty} \|\arg(H - (\lambda + iy)I) - \arg(H_0 - (\lambda + iy)I)\| = 0. \tag{7.5}$$

Proposition 7.1. *If for the pair H_0, H Assumption 7.4(i) holds, then Assumption 4.2(i) holds. If Assumption 7.4(ii) holds, then Assumption 4.2(ii) holds.*

Theorem 7.2. *Let $\Omega \subset \mathbb{R}$ be a Borel set, $\sigma(H_0) \cup \sigma(H) \subset \Omega$, and let the functions f_1, f_2 satisfy Assumption 1.1 with $\lambda \in \Omega$. Let, for $j = 1, 2$, the pair of operators $f_j(H_0), f_j(H)$ satisfy Assumption 7.4 and Assumption 4.3 (for $p = \infty$) at the point $f_j(\lambda)$. Then the invariance principle* (7.1) *holds.*

Let us give a sufficient condition for Assumption 7.4.

Theorem 7.3. *Let the operators H_0, G, J be as described in §2.2; assume* (2.5) *and let $H = H(H_0, G, J)$. Then Assumption 7.4 holds for the pair H_0, H.*

7.2. Auxiliary statements

Lemma 7.1. *Let $M_j = M_j^* \in \mathcal{B}(\mathcal{H})$, $j = 0, 1$. Then, for any $t \in \mathbb{R}$,*
(i) *one has*

$$\left\| e^{itM} - e^{itM_0} \right\| \leq |t| \, \|M - M_0\| \, ;$$

(ii) *if $M - M_0 \in \mathfrak{S}_\infty$, then $e^{itM} - e^{itM_0} \in \mathfrak{S}_\infty$.*

Proof Immediately follows from the representation

$$e^{itM} - e^{itM_0} = ie^{itM} \int_0^t e^{-isM}(M - M_0)e^{isM_0} ds. \quad \blacksquare$$

Recall that we have fixed the branch of the argument by (7.3).

LEMMA 7.2. *Let the functions* f_1, f_2 *satisfy Assumption 1.1 at a point* $\lambda \in \Omega$. *Then, for any self-adjoint operator* H *such that* $\sigma(H) \subset \Omega$, *one has*

$$\lim_{y \to +0} \| \arg(f_2(H) - f_2(\lambda)I - iyf_2'(\lambda)I) \\ - \arg(f_1(H) - f_1(\lambda)I - iyf_1'(\lambda)I)\| = 0. \quad (7.6)$$

Proof. 1. First let us denote $g_j(x) = (f_j(x) - f_j(\lambda))/f_j'(\lambda)$, $j = 1,2$ and without loss of generality assume that $\lambda = 0$. Clearly, we get $g_j(0) = 0$, $g_j'(0) = 1$, $j = 1, 2$, and we have to prove that

$$\lim_{y \to +0} \| \arg(g_2(H) - iyI) - \arg(g_1(H) - iyI)\| = 0,$$

which reduces to

$$\lim_{y \to +0} \sup_{x \in \mathbb{R}} |\arg(g_2(x) - iy) - \arg(g_1(x) - iy)| = 0. \quad (7.7)$$

It is sufficient to prove the following two relations:

$$\lim_{y \to +0} \sup_{|x| > \delta} |\arg(g_2(x) - iy) - \arg(g_1(x) - iy)| = 0 \quad \text{for any } \delta > 0, \quad (7.8)$$

$$\lim_{x \to 0} \sup_{y > 0} |\arg(g_2(x) - iy) - \arg(g_1(x) - iy)| = 0. \quad (7.9)$$

2. Let us prove (7.8). Clearly, by Assumption 1.1(ii), one has

$$\sup_{|x| > \delta} (1/|g_j(x)|) < \infty, \quad j = 1, 2.$$

Thus, as $y \to +0$,

$$\arg(g_2(x) - iy) - \arg(g_1(x) - iy) \\ = \arg(1 - (iy/g_2(x))) - \arg(1 - (iy/g_1(x))) = O(y)$$

uniformly in $|x| > \delta$.

3. Let us prove (7.9). By Assumption 1.1(i), one has for $x \to 0$:

$$\arg(g_2(x) - iy) - \arg(g_1(x) - iy) = \arg(x + o(x) - iy) - \arg(x + o(x) - iy) \\ = \arg(1 - i(y/x) + o(1)) - \arg(1 - i(y/x) + o(1)) = o(1)$$

uniformly in $y > 0$. ∎

7.3. Proof of Proposition 7.1 and Theorems 7.1, 7.2

1. *Proof of Proposition 7.1* First note that

$$\begin{aligned} & M(z;H,H_0)-I \\ &= \exp(-2i\arg(H-zI))\,(\exp(2i\arg(H_0-zI))-\exp(2i\arg(H-zI)))\,. \end{aligned}$$

Thus, by Lemma 7.1(ii), (7.4) implies (4.5) (with $p=\infty$). The inclusion (4.5) is equivalent to (4.2). Similarly, by Lemma 7.1(i), (7.5) implies (4.6) (with $p=\infty$), and (4.6) is equivalent to (4.3). ▮

2. *Proof of Theorem 7.1* As in the proof of Lemma 7.2, we can reduce the problem to the case $\lambda=0$, $f_j(0)=0$, $f_j'(0)=1$, $j=1,2$. Further, for $x\in\mathbb{R}$ and $y>0$ denote

$$\begin{aligned} A(x;y) = & \frac{(f_2(x)+iy)(f_1(x)-iy)}{(f_2(x)-iy)(f_1(x)+iy)} \\ & \qquad = \exp\left(2i\arg(f_1(x)-iy)-2i\arg(f_2(x)-iy)\right). \end{aligned}$$

One has

$$M(iy;f_2(H),f_2(H_0)) = A(H;y)M(iy;f_1(H),f_1(H_0))(A(H_0;y))^*.$$

By Lemma 7.2 and Lemma 7.1(i),

$$\lim_{y\to+0}\|A(H;y)-I\| = \lim_{y\to+0}\|A(H_0;y)-I\| = 0.$$

Therefore,

$$\lim_{y\to+0}\|M(iy;f_2(H),f_2(H_0))-M(iy;f_1(H),f_1(H_0))\|=0.$$

By Lemma 3.2, this proves the theorem. ▮

3. *Proof of Theorem 7.2:* 1. For $j=1,2$, let U_j be the mapping (4.8) (for $p=\infty$), corresponding to the pair of operators $f_j(H_0), f_j(H)$ and the spectral parameter $f_j(\lambda)$. Let $\gamma_j=\mathrm{ext}(\eta_\infty\circ U_j)$ (recall that η_∞ has been introduced in §3.4, and ext — in §3.5). Clearly, $\gamma_1(0)=\gamma_2(0)$. By Theorem 7.1, $\gamma_1(1)=\gamma_2(1)$. Below we explicitly construct a homotopy between γ_1 and γ_2. By Proposition 3.7, the existence of a homotopy between γ_1 and γ_2 implies that

$$\mathrm{sf}(z;U_1)=\mathrm{sf}(z;U_2),\quad z\in\mathbb{T}\setminus\{1\},$$

and (7.1) follows.

2. As in the proof of Lemma 7.2, we reduce the problem to the case when $\lambda=0$, $f_j(0)=0$, $f_j'(0)=1$, $j=1,2$. Further, for $x\in\mathbb{R}$ and $t\in(0,1)$

denote

$$h_j(x;t) := \arg(f_j(x) - i(1-t)t^{-1}), \quad j = 1, 2.$$

For $x \in \mathbb{R}$, $s \in [0,1]$ and $t \in (0,1)$ denote

$$A(x;t,s) := \exp\bigl(2is(h_1(x;t) - h_2(x;t))\bigr),$$
$$M(t,s) := A(H;t,s)M(i(1-t)t^{-1}; f_1(H), f_1(H_0))(A(H_0;t,s))^*.$$

It is straightforward to see that

$$\begin{aligned} M(t,0) &= M(i(1-t)t^{-1}; f_1(H), f_1(H_0)), \\ M(t,1) &= M(i(1-t)t^{-1}; f_2(H), f_2(H_0)). \end{aligned} \tag{7.10}$$

3. Let us check that $M(t,s) - I \in \mathfrak{S}_\infty(\mathcal{H})$ for all $(t,s) \in (0,1) \times [0,1]$. By Assumption 7.4(i), one has $h_j(H;t) - h_j(H_0;t) \in \mathfrak{S}_\infty(\mathcal{H})$ for all $t \in (0,1)$ and $j = 1, 2$. By Lemma 7.1(ii), this implies that for $j = 1, 2$,

$$\exp(2ish_j(H;t)) - \exp(2ish_j(H_0;t)) \in \mathfrak{S}_\infty(\mathcal{H}), \quad (t,s) \in (0,1) \times [0,1],$$

and therefore

$$A(H;t,s) - A(H_0;t,s) \in \mathfrak{S}_\infty(\mathcal{H}), \quad (t,s) \in (0,1) \times [0,1].$$

From here it is easy to deduce that $M(t,s) - I \in \mathfrak{S}_\infty(\mathcal{H})$.

4. Define the mapping $\Gamma : [0,1] \times [0,1] \to X_\infty$ by

$$\begin{aligned} &\Gamma(t,s) = \eta_\infty(M(t,s)), \quad t \neq 0, 1; \\ &\Gamma(0,s) = 0; \\ &\Gamma(1,s) = \gamma_1(1)(= \gamma_2(1)). \end{aligned}$$

Let us prove that Γ is a homotopy between γ_1 and γ_2. By (7.10), $\Gamma(t,0) = \gamma_1(t)$ and $\Gamma(t,1) = \gamma_2(t)$ for all $t \in [0,1]$. It remains to check that the mapping Γ is continuous.

5. First let us check that the mapping

$$(0,1) \times [0,1] \ni (t,s) \mapsto M(t,s) - I \in \mathfrak{S}_\infty(\mathcal{H})$$

is continuous. By Proposition 4.1(iii), $M(i(1-t)t^{-1}; f_1(H), f_1(H_0))$ depends continuously on $t \in (0,1)$ in the operator norm. It can also be checked explicitly that the mapping

$$(0,1) \times [0,1] \ni (t,s) \mapsto A(\cdot;t,s) \in C(\mathbb{R})$$

is continuous and therefore $A(H;t,s)$ and $A(H_0;t,s)$ depend continuously on (t,s) in the operator norm.

6. Let us check the continuity of Γ at $t=0$. Let us prove that

$$\lim_{t\to+0}\sup_{s\in[0,1]}\|M(t,s)-I\|=0.$$

Assumption 7.4(ii) implies that

$$\lim_{y\to+\infty}\|M(iy;f_1(H),f_1(H_0))-I\|=0,$$

and therefore it suffices to prove that

$$\lim_{t\to+0}\sup_{s\in[0,1]}\|A(H;t,s)-A(H_0;t,s)\|=0,\quad j=1,2.$$

By Lemma 7.1(i), the last relation follows again from Assumption 7.4(ii).

7. Let us check the continuity of Γ at $t=1$. Let us prove that

$$\lim_{t\to1-}\sup_{s\in[0,1]}\|M(t,s)-M(t,0)\|=0. \tag{7.11}$$

It follows from Lemma 7.2 and Lemma 7.1(i) that

$$\lim_{t\to1-}\sup_{s\in[0,1]}\|A(H_0;t,s)-I\|=\lim_{t\to1-}\sup_{s\in[0,1]}\|A(H;t,s)-I\|=0.$$

This implies (7.11). By Lemma 3.2, it follows that Γ is continuous at $t=1$. ▮

7.4. Proof of Theorem 7.3

Lemma 7.3. *Let H_0 be a self-adjoint operator in $\mathcal{H}$ and K be a compact operator. Then, for any $r>0$ and $\psi\in\mathcal{H}$ one has*

$$\int_r^\infty\left\|K(|H_0|+I)^{1/2}(H_0-itI)^{-1}\psi\right\|^2dt\le C_{7.12}(r;K)\left\|\psi\right\|^2, \tag{7.12}$$

where

$$\lim_{r\to\infty}C_{7.12}(r;K)=0. \tag{7.13}$$

Proof. 1. Below we prove the following two facts:

(i) the relations (7.12), (7.13) hold for any finite rank operator K;
(ii) for any bounded operator K and any $\psi \in \mathcal{H}$ one has

$$\int_1^\infty \left\| K(|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi \right\|^2 dt \le C_{7.14} \|K\|^2 \|\psi\|^2 , \qquad (7.14)$$

where $C_{7.14}$ is a universal constant.

Approximating a compact operator K by finite rank operators, one obtains the assertion of the lemma from (i), (ii).

2. Let us prove (i). Clearly, it is sufficient to consider a rank one operator $K = (\cdot, \varphi)\chi$, $\|\varphi\| = \|\chi\| = 1$. Let $d\mu_\varphi(\lambda) := d(E_{H_0}((-\infty, \lambda))\varphi, \varphi)$ be the spectral measure of H_0, associated with the vector φ. One has:

$$\begin{aligned} &\int_r^\infty \left\| K(|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi \right\|^2 dt \\ &\quad = \int_r^\infty \left| ((|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi, \varphi) \right|^2 dt \\ &\quad \le \|\psi\|^2 \int_r^\infty \left\| (|H_0| + I)^{1/2}(H_0 + itI)^{-1}\varphi \right\|^2 dt \\ &\quad = \|\psi\|^2 \int_r^\infty dt \int_{\mathbb{R}} \frac{|\lambda| + 1}{\lambda^2 + t^2} d\mu_\varphi(\lambda) = \|\psi\|^2 \int_{\mathbb{R}} F(\lambda, r) d\mu_\varphi(\lambda), \end{aligned}$$

where

$$F(\lambda, r) = (|\lambda| + 1) \int_r^\infty \frac{dt}{\lambda^2 + t^2} = \frac{|\lambda| + 1}{|\lambda|} \tan^{-1}(|\lambda|/r).$$

Clearly,

$$C_{7.15} := \sup_{r>1} \sup_{\lambda \in \mathbb{R}} F(\lambda, r) < \infty, \qquad (7.15)$$

and $\lim_{r\to\infty} F(\lambda, r) = 0$ for any $\lambda \in \mathbb{R}$. Therefore,

$$\lim_{r\to\infty} \int_{\mathbb{R}} F(\lambda, r) d\mu_\varphi(\lambda) = 0$$

and we arrive at (7.12), (7.13) with $C_{7.12} = \int_{\mathbb{R}} F(\lambda, r) d\mu_\varphi(\lambda)$.

3. Let us prove (ii). As above, one has:

$$\begin{aligned} &\int_1^\infty \left\| K(|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi \right\|^2 dt \\ &\quad \le \|K\|^2 \int_1^\infty \left\| (|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi \right\|^2 dt \\ &\quad = \|K\|^2 \int_{\mathbb{R}} F(\lambda, 1) d\mu_\psi(\lambda) \le C_{7.15} \|K\|^2 \|\psi\|^2 , \end{aligned}$$

and we get (7.14) with $C_{7.14} = C_{7.15}$. ▌

Proof of Theorem 7.3: 1. First of all, note that the conditions (2.5) are invariant under the linear transformations $H_0 \mapsto aH_0 + bI$, $a, b \in \mathbb{R}$. Thus, it is sufficient to prove (7.4) with $z = i$ and (7.5) — with $\lambda = 0$.

Next, we will use the integral representation

$$\arg(x - iy) = -(\pi/2) + \operatorname{Im} \int_y^\infty \frac{x}{x - it} \frac{dt}{t}, \quad x \in \mathbb{R}, \quad y > 0.$$

In view of this representation, it is sufficient to prove that under the assumptions (2.5), one has

$$\int_1^R [(H - itI)^{-1} - (H_0 - itI)^{-1}]dt \in \mathfrak{S}_\infty(\mathcal{H}) \quad \text{for any } R > 0, \tag{7.16}$$

$$\lim_{r\to\infty} \sup_{R\geq r} \left\| \int_r^R [(H - itI)^{-1} - (H_0 - itI)^{-1}]dt \right\| = 0. \tag{7.17}$$

By (2.10), the inclusion (4.2) (with $p = \infty$) holds for all $z \in \rho(H_0) \cap \rho(H)$. From here we get (7.16). Thus, it remains to prove (7.17).

2. Let us prove (7.17). First, for brevity we denote $K := G(|H_0|+I)^{-1/2}$. Using (2.10) and Lemma 7.3, we obtain the following estimate for any $\psi, \varphi \in \mathcal{H}$:

$$\begin{aligned} &\left| \int_r^R [((H - itI)^{-1}\varphi, \psi) - ((H_0 - itI)^{-1}\varphi, \psi)]dt \right| \\ &\quad \leq \int_r^R \left\|(J^{-1} + T(it))^{-1}\right\| \left\|G(H_0 - itI)^{-1}\varphi\right\| \left\|G(H_0 - itI)^{-1}\psi\right\| dt \\ &\quad \leq \sup_{t\geq r} \left\|(J^{-1} + T(it))^{-1}\right\| \left(\int_r^\infty \left\|K(|H_0| + I)^{1/2}(H_0 - itI)^{-1}\varphi\right\|^2 dt \right)^{1/2} \\ &\qquad \times \left(\int_r^\infty \left\|K(|H_0| + I)^{1/2}(H_0 - itI)^{-1}\psi\right\|^2 dt \right)^{1/2} \\ &\quad \leq \sup_{t\geq r} \left\|(J^{-1} + T(it))^{-1}\right\| \|\varphi\| \|\psi\| C_{7.12}(r, K), \end{aligned}$$

which, by (7.13) and (4.17) (with $p = \infty$), proves (7.17). ▌

8. MAIN RESULT

Theorems 5.1, 6.1 and 7.2 imply the following statement, which is the central result of this paper.

THEOREM 8.1. *Let the operators H_0, G, J be as described in §2.2; assume* (2.5) *and let $H = H(H_0, G, J)$. Suppose that for an open interval $\delta \subset \mathbb{R}$ the inclusion* (4.11) *holds. Further, let $\Omega \subset \mathbb{R}$ be a Borel set, $\sigma(H_0) \cup \sigma(H) \subset \Omega$, and let a function f satisfy Assumption 1.1 for all $\lambda \in \delta$. Suppose that*

$$f(H) - f(H_0) \in \mathfrak{S}_1(\mathcal{H}).$$

Then for a.e. $\lambda \in \delta$, the representation (1.10) *holds true.*

Proof. First note that the limit $T(\lambda+i0)$ exists in $\mathfrak{S}_\infty(\mathcal{K})$ by Proposition 4.4 and the pair $\Xi(J^{-1} + A(\lambda+i0) + tB(\lambda+i0)), \Xi(J^{-1})$ is Fredholm by Theorem 5.1. Further, by Theorem 7.3, both the pair H_0, H, and the pair $f(H_0)$, $f(H)$ satisfy Assumption 7.4. By Proposition 4.4, the pair H_0, H satisfies Assumption 4.3 (for $p = \infty$) for a.e. $\lambda \in \delta$ and the pair $f(H_0)$, $f(H)$ satisfies Assumption 4.3 (for $p = \infty$) for a.e. $\lambda \in \mathbb{R}$. Thus, we can apply Theorem 7.2, which yields

$$\mu(\theta; f(\lambda), f(H), f(H_0)) = \mu(\theta; \lambda, H, H_0), \qquad \text{a.e. } \lambda \in \delta.$$

By Theorem 5.1, one has

$$\mu(\theta; \lambda, H, H_0) = \operatorname{index}\big(\Xi(J^{-1}), \Xi(J^{-1} + A(\lambda+i0) + \cot(\theta/2)B(\lambda+i0))\big).$$

Applying Theorem 6.1 to the pair $f(H_0)$, $f(H)$, we get

$$\lim_{y\to+0} \arg \Delta_{f(H)/f(H_0)}(\lambda'+iy) = -\frac{1}{2}\int_0^{2\pi} \mu(\theta; \lambda', f(H), f(H_0))d\theta, \text{ a.e. } \lambda' \in \mathbb{R}.$$

Combining the last three equalities and the Krein's formula (1.3) and making the change of variables $t = \cot(\theta/2)$ in the resulting integral, we get (1.10). ∎

As in §5.1, for the perturbations of a definite sign the representation (1.10) takes the form

$$\xi(f(\lambda); f(H), f(H_0)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} \operatorname{rank} E_{A(\lambda+i0)+tB(\lambda+i0)}((-\infty,-1)), \quad J = I, \tag{8.1}$$

$$\xi(f(\lambda); f(H), f(H_0)) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} \operatorname{rank} E_{A(\lambda+i0)+tB(\lambda+i0)}([1,\infty)), \quad J = -I. \tag{8.2}$$

The representations (8.1), (8.2) have been proven in [20] in the following particular case. It was assumed that the operator H_0 is semibounded from below and $f(\lambda) = (\lambda - a)^{-l}$, $l > 0$, $a < \inf(\sigma(H_0) \cup \sigma(H))$. Instead of (4.11), it was supposed that $G(H_0 - aI)^{-m} \in \mathfrak{S}_2$ for some $m > 0$. The proof was heavily based upon the particular form of the function f and used the results of [14].

Note that the SSF is non-negative in (8.1) and non-positive in (8.2). This fact itself is already non-trivial. In the case $f(\lambda) = \lambda$, it has been proven by M. G. Krein in the original paper [15], but very few generalizations for $f(\lambda) \neq \lambda$ have been known so far (see [26, §8.10] for the discussion).

9. APPENDIX: ADDITIONAL PROPERTIES OF THE FUNCTION μ

Here we prove formula (1.11) and explain the relation of the function $\mu(\cdot; \lambda, H, H_0)$ to the eigenvalue counting functions of the operators H_0 H. These results have not been used above and are given only in order to clarify the links between the function μ and the standard objects of the spectral theory of perturbations.

9.1. The function μ and the spectrum of the scattering matrix

Let the operators H_0, G, J be as described in §2.2; assume (2.5) and let $H = H(H_0, G, J)$. Fix an interval Δ in the absolutely continuous spectrum of H_0. Below we recall a criterion for existence of the scattering matrix $\mathcal{S}(\lambda; H, H_0)$ for a.e. $\lambda \in \Delta$, which can be found, e.g., in [26, §5.8]. For technical reasons, we suppose that $\operatorname{Ker} G = \{0\}$; this will simplify the statement below.

PROPOSITION 9.1. *Suppose that for a.e. $\lambda \in \Delta$, the limit*

$$\operatorname*{n-lim}_{y \to +0} T(\lambda + iy)$$

exists and $0 \in \rho(J^{-1} + T(\lambda + i0))$. Then the local wave operators $W_\pm(H, H_0; \Delta)$ exist and are complete. For a.e. $\lambda \in \Delta$, the scattering matrix $\mathcal{S}(\lambda; H, H_0)$ is given by

$$\mathcal{S}(\lambda; H, H_0) = I - 2\pi i Z(\lambda; G)(J^{-1} + T(\lambda + i0))^{-1} Z^*(\lambda; G), \qquad (9.1)$$

where the operator $Z(\lambda; G)$ satisfies the relation

$$\pi Z^*(\lambda; G) Z(\lambda; G) = B(\lambda + i0).$$

In this situation, clearly, $\mathcal{S}(\lambda; H, H_0) - I \in \mathfrak{S}_\infty$. Note that under the hypothesis of Proposition 9.1, the Assumptions 4.2 and 4.3 hold for $p = \infty$ and a.e. $\lambda \in \Delta$.

Further, by (4.24) and Theorem 4.1, one has

$$\eta_\infty(\mathcal{S}(\lambda; H, H_0)) = \eta_\infty(S(\lambda + i0; H_0, G, J)) = \lim_{y\to+0} \eta_\infty(M(\lambda + iy; H, H_0)).$$

Thus, we see that under the hypothesis of Proposition 9.1, for a.e. $\lambda \in \Delta$ the relation (1.11) holds true.

9.2. The function μ on the discrete spectrum

1. Let H_0, H be self-adjoint operators in $\mathcal{H}$, satisfying Assumption 4.2 (with $p = \infty$). If $\lambda \in \mathbb{R} \setminus (\sigma(H_0) \cup \sigma(H))$, then, obviously, Assumption 4.3 is fulfilled and $M(\lambda; H, H_0) = I$. Therefore, $\mu(\theta; \lambda, H, H_0)$ equals to an integer constant. Below we discuss the relation of this constant to the eigenvalue counting functions of H_0 and H. First we need notation, similar to (3.5), but for self-adjoint operators. For $\lambda_1, \lambda_2 \in \mathbb{R}$ and $H = H^*$ we put

$$N(\lambda_1, \lambda_2; H) = \begin{cases} \operatorname{rank} E_H([\lambda_1, \lambda_2)), & \lambda_2 > \lambda_1, \\ 0, & \lambda_2 = \lambda_1, \\ -\operatorname{rank} E_H([\lambda_2, \lambda_1)), & \lambda_1 > \lambda_2. \end{cases}$$

Recall that Assumption 4.2 implies that $\sigma_{ess}(H) = \sigma_{ess}(H_0)$.

Theorem 9.1. *Let $[\lambda_1, \lambda_2] \cap \sigma_{ess}(H_0) = \emptyset$ and $\{\lambda_1, \lambda_2\} \subset \rho(H) \cap \rho(H_0)$. Then, for all $\theta \in (0, 2\pi)$,*

$$\mu(\theta; \lambda_2, H, H_0) - \mu(\theta; \lambda_1, H, H_0) = N(\lambda_1, \lambda_2; H) - N(\lambda_1, \lambda_2; H_0). \quad (9.2)$$

2. Let

$$H : [0, 1] \ni \alpha \mapsto H(\alpha)$$

be a family of self-adjoint operators in $\mathcal{H}$, which satisfies the following assumptions:

$$(H(\alpha) - zI)^{-1} - (H(0) - zI)^{-1} \in \mathfrak{S}_\infty(\mathcal{H}), \quad \forall z \in \mathbb{C}_+, \quad \alpha \in [0, 1], \quad (9.3)$$

the map $[0, 1] \ni \alpha \mapsto (H(\alpha) - zI)^{-1} \in \mathcal{B}(\mathcal{H})$ is continuous for all $z \in \mathbb{C}_+$, (9.4)

$$\lim_{y\to+\infty} \sup_{\alpha\in[0,1]} y \left\| (H(\alpha) - (\lambda + iy)I)^{-1} - (H(0) - (\lambda + iy)I)^{-1} \right\| = 0, \quad \forall \lambda \in \mathbb{R}. \quad (9.5)$$

By (9.3), the essential spectra of all the operators $H(\alpha)$ coincide. Suppose that $\Delta \subset \mathbb{R} \setminus \sigma_{ess}(H(\alpha))$. Below we explain that for $\lambda \in \Delta$ the function

$\mu(\theta;\lambda,H(1),H(0))$ can be considered as the spectral flow of the family H through the point λ.

In order to define the spectral flow of the family H, let us repeat (without proofs) the basic steps of the construction of §3. First let us fix a function space $\widetilde{X}$ where the function $\mathrm{sf}(\lambda;H)$, $\lambda\in\Delta$, will belong to. Let $\widetilde{X}$ be the set of left continuous bounded non-decreasing functions $f:\Delta\to\mathbb{Z}$. There is a lot of freedom in choosing the topology in $\widetilde{X}$; let us consider $\widetilde{X}$ with the topology, say, induced by the embedding $\widetilde{X}\subset L_1(\Delta)$ (we could instead take $L_p(\Delta)$ with any $p<\infty$). Consider the equivalence relation

$$f\sim g \iff \exists n\in\mathbb{Z}:\forall x\in\Delta,\quad f(x)=g(x)+n.$$

Let X be the quotient space $\widetilde{X}/\sim$, and let $\pi:\widetilde{X}\to X$ be the corresponding projection. In the natural way one defines a topology in X and checks that $\pi:\widetilde{X}\to X$ is a covering.

Further, note that for every $\alpha\in[0,1]$ and $\lambda_0\in\Delta\cap\rho(H(\alpha))$, the function $N(\lambda_0,\cdot;H(\alpha))$ belongs to $\widetilde{X}$. Define the mapping $\gamma:[0,1]\to X$ by

$$\gamma(\alpha)=\pi\big(N(\lambda_0,\cdot;H(\alpha))\big),\quad \lambda_0\in\Delta\cap\rho(H(\alpha)).$$

This definition does not depend on the choice of λ_0. Since all the eigenvalues of $H(\alpha)$ depend continuously on α, it follows that γ is continuous. Let $\widetilde{\gamma}$ be a lift of γ to $\widetilde{X}$. Then we put

$$\mathrm{sf}(\lambda;H):=\widetilde{\gamma}(\lambda;1)-\widetilde{\gamma}(\lambda;0),\quad \lambda\in\Delta. \tag{9.6}$$

As in §3.5(3), it is easy to see that

$$\begin{aligned}\mathrm{sf}(\lambda;H)=&\ \langle\text{the number of eigenvalues of } H(\alpha) \text{ that cross } \lambda \text{ leftwards}\rangle\\ &-\langle\text{the number of eigenvalues of } H(\alpha) \text{ that cross } \lambda \text{ rightwards}\rangle\end{aligned} \tag{9.7}$$

as α grows from 0 to 1, whenever the r.h.s. is well defined.

It follows from Theorem 9.1 that $\mathrm{sf}(\lambda;H)$ and $\mu(\theta;\lambda,H(1),H(0))$ differ by a function (of λ), which is identically equal to an integer number. The following theorem shows that this number equals zero.

Theorem 9.2. *The mapping*

$$[0,1]\ni\alpha\mapsto\mu(\theta;\cdot,H(\alpha),H(0))\in L_1(\Delta) \tag{9.8}$$

is continuous.

Thus, the mapping (9.8) is a lift of γ and therefore,

$$\mu(\theta;\lambda,H(1),H(0))=\mathrm{sf}(\lambda;H),\quad \lambda\in\Delta. \tag{9.9}$$

As a typical example, consider the family $H(\alpha) = H(H_0, \sqrt{\alpha}G, J)$, where the operators H_0, G, J satisfy (2.5). It is easy to see that in this case the assumptions (9.3)–(9.5) hold. Moreover, the eigenvalues of $H(\alpha)$ in the gaps depend analytically on α, and therefore the r.h.s. of (9.7) is well defined (see, e.g., [23]).

9.3. Proofs of Theorems 9.1, 9.2

Proof of Theorem 9.1: 1. Let us first prove that if $[\lambda_1, \lambda_2] \subset \rho(H_0) \cap \rho(H)$, then

$$\mu(\theta; \lambda_1, H, H_0) = \mu(\theta; \lambda_2, H, H_0). \tag{9.10}$$

For $j = 1, 2$, let $\gamma_j : [0, 1] \to X_\infty$ be the mapping

$$\gamma_j(0) = 0,$$
$$\gamma_j(t) = \eta_\infty(M(\lambda_j + i(1-t)t^{-1}; H, H_0)), \quad t \in (0, 1].$$

We need to check that γ_1 and γ_2 are homotopic. Define the mapping $\Gamma : [0, 1] \times [\lambda_1, \lambda_2] \to X_\infty$ by

$$\Gamma(0, \lambda) = 0, \quad \lambda \in [\lambda_1, \lambda_2];$$
$$\Gamma(t, \lambda) = \eta_\infty(M(\lambda + i(1-t)t^{-1}; H, H_0)), \quad (t, \lambda) \in (0, 1] \times [\lambda_1, \lambda_2].$$

Similarly to the proof of Theorem 7.2, one easily checks that Γ is a homotopy between γ_1 and γ_2.

2. It remains to check that for all $\lambda \in \mathbb{R} \setminus \sigma_{ess}(H_0)$, one has

$$\mu(\theta; \lambda + 0, H, H_0) - \mu(\theta; \lambda - 0, H, H_0) = \operatorname{rank} E_H(\{\lambda\}) - \operatorname{rank} E_{H_0}(\{\lambda\}). \tag{9.11}$$

Without the loss of generality assume that $\lambda = 0$. Choose $\varepsilon > 0$ small enough so that there is no spectrum of H and H_0 in $[-\varepsilon, 0) \cup (0, \varepsilon]$. We are going to prove that

$$\mu(\theta; \varepsilon, H, H_0) - \mu(\theta; -\varepsilon, H, H_0) = \operatorname{rank} E_H(\{0\}) - \operatorname{rank} E_{H_0}(\{0\}).$$

In order to do this, consider the path $\beta_1 : [0, \pi] \to X_\infty$,

$$\beta_1(t) = \eta_\infty(M(\varepsilon e^{it}; H, H_0)).$$

Clearly, $\beta_1(0) = \beta_1(\pi) = 0$. Further, consider the paths $\gamma_\pm : [0, 1] \to X_\infty$,

$$\gamma_\pm(0) = 0,$$
$$\gamma_\pm(t) = \eta_\infty(M(\pm\varepsilon + i(1-t)t^{-1}; H, H_0)), \quad t \in (0, 1].$$

It is easy to see that the catenation $\gamma_+ \cdot \beta_1$ is homotopic to γ_-. Therefore, it is sufficient to prove that

$$\widetilde{\beta}_1(\theta;\pi) - \widetilde{\beta}_1(\theta;0) = \operatorname{rank} E_{H_0}(\{0\}) - \operatorname{rank} E_H(\{0\}), \tag{9.12}$$

where $\widetilde{\beta}_1$ is a lift of β_1.

3. In order to prove (9.12), we are going to check that β_1 is homotopic to the following path $\beta_2 : [0,\pi] \to X_\infty$:

$$\beta_2(t) := \eta_\infty\big((E_H(\mathbb{R}\setminus\{0\}) + e^{-2it}E_H(\{0\}))(E_{H_0}(\mathbb{R}\setminus\{0\}) + e^{2it}E_{H_0}(\{0\}))\big).$$

It is clear that for a lift $\widetilde{\beta}_2$ of β_2, one has

$$\widetilde{\beta}_2(\theta;\pi) - \widetilde{\beta}_2(\theta;\pi) = \operatorname{rank} E_{H_0}(\{0\}) - \operatorname{rank} E_H(\{0\}),$$

which implies (9.12).

4. The homotopy $\Gamma : [0,\pi]\times[0,1] \to X_\infty$ between β_1 and β_2 is given by

$$\begin{aligned}
\Gamma(t,s) &= \eta_\infty(U(t,s)),\\
U(t,s) &= \left(\frac{H - s\varepsilon e^{-it}I}{H - s\varepsilon e^{it}I}E_H(\mathbb{R}\setminus\{0\}) + e^{-2it}E_H(\{0\})\right)\\
&\quad\times\left(\frac{H_0 - s\varepsilon e^{it}I}{H_0 - s\varepsilon e^{-it}I}E_{H_0}(\mathbb{R}\setminus\{0\}) + e^{2it}E_{H_0}(\{0\})\right) \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 9.2: 1. First let us prove the following statement. Fix $\lambda \in \rho(H_0)$ and consider $\mu(\theta;\lambda, H(\alpha), H(0))$ as the function of α. Let $\delta = [\alpha_1,\alpha_2]$ be an interval such that $\lambda \in \rho(H(\alpha))$ for all $\alpha \in \delta$. Then

$$\mu(\theta;\lambda, H(\alpha_1), H(0)) = \mu(\theta;\lambda, H(\alpha_2), H(0)).$$

For $j = 1,2$ let U_j be the mapping (4.8) (with $p = \infty$) for the pair $H(0)$, $H(\alpha_j)$, and let $\gamma = \operatorname{ext}(\eta_\infty \circ U)$. We need to prove that γ_1 and γ_2 are homotopic. Using (9.3)–(9.5), one easily checks that the mapping $\Gamma : [0,1]\times\delta \to X_\infty$, given by

$$\begin{aligned}
&\Gamma(0,\alpha) = 0,\\
&\Gamma(t,\alpha) = \eta_\infty(M(\lambda + i(1-t)t^{-1}; H(\alpha), H(0))),
\end{aligned}$$

is a homotopy between γ_1 and γ_2.

2. Fix $\alpha_0 \in [0,1]$; let the neighbourhood $\omega \subset [0,1]$ of α_0 be small enough so that there exists $\lambda_0 \in \Delta$, $\lambda_0 \in \rho(H(\alpha))$ for all $\alpha \in \omega$. As we have seen above, one has

$$\mu(\theta;\lambda_0, H(\alpha), H(0)) = \mu(\theta;\lambda_0, H(\alpha_0), H(0)), \quad \alpha \in \omega.$$

Therefore, by Theorem 9.1,

$$\begin{aligned}
&\mu(\theta;\lambda,H(\alpha),H(0))-\mu(\theta;\lambda,H(\alpha_0),H(0))\\
&\quad=(\mu(\theta;\lambda,H(\alpha),H(0))-\mu(\theta;\lambda_0,H(\alpha),H(0)))\\
&\quad-(\mu(\theta;\lambda,H(\alpha_0),H(0))-\mu(\theta;\lambda_0,H(\alpha_0),H(0)))\\
&\quad=N(\lambda_0,\lambda;H(\alpha))-N(\lambda_0,\lambda;H(\alpha_0)).
\end{aligned}$$

Since there are only finitely many eigenvalues of $H(\alpha)$ in Δ and they depend continuously on t, we conclude that

$$\lim_{\alpha\to\alpha_0}\|N(\lambda_0,\cdot;H(\alpha))-N(\lambda_0,\cdot;H(\alpha_0))\|_{L_1(\Delta)}=0.$$

This implies (9.8). ▮

ACKNOWLEDGMENT

The author is grateful to K. A. Makarov and O. L. Safronov for useful discussions. The research was partially supported by the EPSRC grants no. GR/M00549 and no. GR/N19328.

REFERENCES

1. M. F. Atiyah, V. K. Patodi, I. M. Singer, Spectral asymmetry and Riemannian geometry. III, *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 71–99.
2. J. Avron, R. Seiler, B. Simon, The index of a pair of projections, *Funct. anal.* **120** (1994), 220–237.
3. M. Sh. Birman, Discrete spectrum in the gaps of a continuous one for perturbations with large coupling constant, *Adv. in Sov. Math.* **7** (1991), 57–73.
4. M. Sh. Birman, On existence conditions for wave operators (in Russian), *Dokl. Akad. Nauk SSSR* **143**, no.3 (1962), 506–509. English transl. in *Soviet Math. Dokl.* **3** (1962), 408–411.
5. Birman, M. Sh., Entina, S. B.: The stationary approach in abstract scattering theory (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **31**, no. 2 (1967), 401–430; English translation in *Math. USSR Izv.* **1**, no. 1 (1967), 391–420.
6. M. Sh. Birman and M. G. Krein, On the theory of wave operators and scattering operators (in Russian), *Dokl. Akad. Nauk SSSR* **144** (1962), 475-478; English transl. in *Soviet Math. Dokl.* **3** (1962).
7. M. Sh. Birman and D. R. Yafaev, The spectral shift function. The work of M. G. Krein and its further development (in Russian), *Algebra i Analiz* **4** (1992), no.5, 1–44; English translation in *St. Petersburg Math. J.* **4** (1993), no.5.
8. M. Sh. Birman and D. R. Yafaev, Spectral properties of the scattering matrix (in Russian), *Algebra i Analiz* **4** (1992), no.6, 1–27; English translation in *St. Petersburg Math. J.* **4** (1993), no.6.
9. P. M. Fitzpatrick, J. Peisachowicz, L. Recht, Spectral flow and bifurcation of critical points of strongly-indefinite functionals. Part I. General Theory, *Journal of Functional Analysis* **162** (1999), 52–95.

10. F. Gesztesy, K. A. Makarov, S. N. Naboko, The spectral shift operator, *in* Mathematical Results in Quantum Mechanics, J. Dittrich, P. Exner, M. Tater (eds.), *Operator Theory: Advances and Applications,* **108** Birkhäuser, Basel, 1999, pp. 59–90.
11. F. Gesztesy, K. A. Makarov, The Ξ operator and its relation to Krein's spectral shift function, J. Anal. Math. **81** (2000), 139–183.
12. T. Kato, "Perturbation theory for linear operators," Springer, 1966.
13. T. Kato, Wave operators and similarity for some self-adjoint operators, *Math. Ann.* **162**, 258–279 (1966).
14. L. S. Koplienko, On the theory of the spectral shift function (in Russian), *Problemy Mat. Fiz.* **9** (1971), 62–79.
15. M. G. Krein, On the trace formula in perturbation theory (in Russian), *Mat. Sb.* **33 (75)**, no. 3 (1953), 597–626.
16. V. B. Lidski, The proper values of the sum and product of symmetric matrices (in Russian), *Dokl. Akad. Nauk SSSR* **75** (1950), 769–772.
17. W. S. Massey, "Algebraic topology: an introduction," Graduate texts in mathematics, **56**. Springer–Verlag, New York–Heidelberg, 1967.
18. S. N. Naboko, On boundary problems for analytic operator valued functions with positive imaginary part (in Russian), *Zap. Nauchn. Seminarov LOMI* **157** (1987), 55–69.
19. S. N. Naboko, Nontangential boundary values of operator R-functions in a half-plane (in Russian), *Algebra i Analiz* **1**, no. 5 (1989), 197–222. English translation in: *Leningrad Math. J.* **1**, no. 5 (1990), 1255–1278.
20. A. B. Pushnitski, Representation for the spectral shift function for perturbations of a definite sign (in Russian), *Algebra i Analiz* **9**, no. 6 (1997), 197–213. English translation in: *St. Petersburg Math. J.* **9**, no. 6 (1998), 1181–1194.
21. A. B. Pushnitski, Spectral shift function of the Schrödinger operator in the large coupling constant limit, *Commun. in PDE* **25**, no. 3-4 (2000), 703–736.
22. J. Robbin, D. Salamon, The spectral flow and the Maslov index, *Bull. London Math. Soc.* **27** no.1 (1995), 1–33.
23. O. L. Safronov, Discrete spectrum in a gap of the continuous spectrum for variable sign perturbation with large coupling constant (in Russian), *Algebra i Analiz* **8**, no. 2 (1996), 162–194; English translation in *St. Petersburg Math. J.* **8**, no. 2 (1997), 307–331.
24. O. L. Safronov , Spectral shift function in the large coupling constant limit, to appear in J. Functional Analysis.
25. A. V. Sobolev, Efficient bounds for the spectral shift function, *Ann. Inst. H. Poincare Phys. Theor.* **58** (1993) no. 1, 55–83.
26. Yafaev, D. R. "Mathematical scattering theory. General theory," Amer. Math. Soc., Providence, RI, 1992.