The cubic Szegő equation and Hankel operators

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Dispersion is a very important property when studying dynamics of Hamiltonian partial differential equations. For nonlinear equations where dispersion fails or is not strong enough, our current toolbox is actually extremely reduced, and even existence of global dynamics is an open problem in many cases. The cubic Szegő equation is a simple model of a nontrivial Hamiltonian equation with lack of dispersion, which can be derived from some nonlinear wave equation in one space dimension. The important feature of the cubic Szegő equation is that it admits a Lax pair, namely a structure which allows to reduce its dynamics to spectral theory of some operators. In the special case of the cubic Szegő equation, these operators are Hankel operators and are well known in harmonic analysis and control theory. The purpose of this mini–course is to describe the links between the cubic Szegő equation and Hankel operators which are inherited from this Lax pair structure. Applications will be given to energy transfer to high frequencies, and to inverse spectral problems for Hankel operators. The required background is basic real and complex analysis, basic functional analysis, a little of theory of finite dimensional Hamiltonian systems, but no special knowledge on PDEs or Hankel operators is necessary.

Tentative contents.

Lecture 1. Generalities on dispersive and non dispersive partial differential equations. The cubic half-wave and Szegő equations.

Lecture 2. Hankel operators, the Lax pair structure and an explicit formula for the solution of the initial value problem.

Lecture 3. Finite dimensional invariant manifolds, action-angle variables and inverse spectral problems for finite rank generic Hankel operators.

Lecture 4. Generalizations to infinite rank and spectral multiplicity.