Characterizing the Spectrum of Non-Self-Adjoint Operators

Simon Chandler-Wilde, University of Reading

We focus in this talk on bounded linear operators on $X = \ell^p(\mathbb{Z}, U)$, where $U = \mathbb{C}$ (in which case this is the standard ℓ^p sequence space) or, more generally, U is a complex Banach space. We study the class of operators whose matrix representation is a band matrix, and the norm closure of this class (the band-dominated operators). We start by displaying two examples of this class, the first example that of non-self-adjoint discrete-Schrödinger-type operators (an example with $U = \mathbb{C}$), the second example a strongly singular boundary integral operator on the graph of a Lipschitz function (an example with $1 < p < \infty$ and $U = L^p[0,1]$, so that X is isometrically isomorphic to $L^p(\mathbb{R})$). We then review part of the recent work of the author, Marko Lindner (Chemnitz), and Ratchanikorn Chonchaiya (Reading), in which a key role is played by the so-called *operator spectrum* of a band-dominated operator A on X, defined as the set of *limit operators* of the operator A, representing the behaviour of the operator towards infinity. We characterize the essential spectrum of a band-dominated operator A on X as the set of its eigenvalues as an operator on $\ell^\infty(\mathbb{Z}, U)$; this result holds in the case when U is finite-dimensional and in the case when U is infinite dimensional and $A = I + B$, where I is the identity and B is locally compact.

As a concrete example of application of this result we consider its application to a specific non-self-adjoint random tridiagonal operator considered by Feinberg and Zee (*Phys Rev E*, 1999); though our results, like those of Davies (*Comm Math Phys*, 2001) rely on the weaker and deterministic notion of *pseudo-ergodicity* (which holds almost surely for the random operator). This particular random operator has only two non-zero diagonals, the first sub- and super-diagonals, whose entries are iid ± 1's. Holz, Orland and Zee (*J Phys A*, 2003) have conjectured that the spectrum of this operator has a fractal dimension between 1 and 2. We apply our characterization to disprove this conjecture, showing that the spectrum contains the unit disk. But, intriguingly, discrete fractal structures play a part in the proof, and our numerics suggest a fractal boundary of the spectrum and fractal structure in the density functions of eigenvalue distributions of associated finite random matrices.