AUTOMORPHISM 2-GROUP OF A WEIGHTED PROJECTIVE STACK

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Abstract. For a given sequence of positive integers $(n_0, \cdots, n_r)$ we define the *weighted projective general linear 2-group* $\mathrm{PGL}(n_0, \cdots, n_r)$ as a crossed-module in the category of schemes and show that it is a model for (i.e., is naturally homotopy equivalent to) the gr-stack of self-equivalences of the weighted projective stack of weight $(n_0, \cdots, n_r)$. We also give an explicit description of the structure of $\mathrm{PGL}(n_0, \cdots, n_r)$.

1. Introduction

To study group actions on stacks is a difficult matter. Even to write down what a group action is can often be hopelessly complicated. The difficulty lies in the fact that symmetries of a stack $\mathcal{X}$ are encoded by its *self-equivalences* (the 1-morphisms) as well as the *natural transformations* between them (the 2-morphisms). In more technical terms, in contrast to the case of a usual scheme X where the symmetries form a sheaf of groups $\mathrm{Aut}\, X$, the symmetries of a stack $\mathcal{X}$ form what is called a *gr-stack*. If we denote this gr-stack of self-equivalences by $\mathcal{A}ut\mathcal{X}$, an action of a group (or more correctly, a 2-group) G on $\mathcal{X}$ is given by a *weak* functor $G \to \mathcal{A}ut\mathcal{X}$ and consists of cumbersome coherence diagrams involving 1-morphisms and 2-morphisms. There is yet another layer of complication: two such actions should be regarded the "same" if there is a transformation between them.

Based on results of [No2, No3], in [No5] we develop a strategy to tackle the problem of classifying actions of a given 2-group scheme $\mathfrak{G}$ on a stack $\mathcal{X}$. This is briefly outlined in §5. The main input for carrying out this approach is to find a crossed-module model for the gr-stack $\mathcal{A}ut\mathcal{X}$. Once such a crossed-module is provided, to write down an action of $\mathfrak{G}$ on $\mathcal{X}$ reduces to some standard group theory which involves solving a certain group extension problem.

The difficulty, however, is that finding a crossed-module model for $\mathcal{A}ut\mathcal{X}$ is not always straightforward (it is standard that such a model always exists).

In this paper, we present an approach for finding a crossed-module model for $\mathcal{A}ut\mathcal{X}$ for certain quotient stacks $\mathcal{X}$. We illustrate our method by explicitly calculating the auto-equivalence gr-stack in an important special case, that of the weighted projective stacks.[1] The outcome is what is called a *weighted projective general linear 2-group* $\mathrm{PGL}(n_0, \cdots, n_r)$; see §6. This is a crossed-module (in schemes).

Weighted projective general linear 2-groups $\mathrm{PGL}(n_0, \cdots, n_r)$ were originally introduced in [BeNo] in the case where all schemes were over $\mathbb{C}$. In this paper we study $\mathrm{PGL}(n_0, \cdots, n_r)$ over an arbitrary base scheme S. In Theorem 6.1, we show that the gr-stack associated to $\mathrm{PGL}_S(n_0, \cdots, n_r)$ is equivalent to the gr-stack of

[1] The same method can be used to calculate the auto-equivalence gr-stacks of all 1-dimensional smooth Deligne-Mumford stacks over $\mathbb{C}$, but we will not get into that.

self-equivalences of $\mathcal{P}_S(n_0,\cdots,n_r)$, generalizing the result of [BeNo]. We then go on to make explicit the structure of $\mathrm{PGL}_S(n_0,\cdots,n_r)$; see Theorem 7.7. In light of the strategy developed in [No5] (see §5), these two results enable one to study actions of group schemes (or 2-group schemes, for that matter) on weighted projective stacks in an explicit manner. They are also expected to be useful in the representation theory of 2-groups. Proposition 4.7 helps in calculating the self-equivalence gr-stack of the quotient stack of an abelian group scheme acting on an arbitrary scheme. This result may be of independent interest.

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2. Notation and terminology

Our notation for 2-groups and crossed-modules is that of [No2] and [No3], to which the reader is referred to for more on 2-group theory relevant to this work. In particular, we use mathfrak letters $\mathfrak{G}$, $\mathfrak{H}$ for 2-groups or crossed-modules.

For us a stack is a presheaf of groupoids (and not a category fibered in groupoids) over a Grothendieck site which satisfies the decent condition. We use mathcal letters $\mathcal{X}$, $\mathcal{Y}$,... for stacks.

Given a presheaf of groupoids $\mathcal{X}$ over a site, its stackification is denoted by $\mathcal{X}^a$. We use the same notation for the sheafification of a presheaf of sets (or groups).

The m^{th} general linear group scheme over $\operatorname{Spec} R$ is denoted by $\mathrm{GL}(m,R)$. When $R=\mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. The corresponding projectivized general linear group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0,n_1,\cdots,n_r)$ for a weighted projective general linear 2-group (§6) because in the latter case we always assume $r\geq 1$.

3. Review of 2-groups and crossed-modules

A (strict) *2-group* is a group object in the category of groupoids. Equivalently, a 2-group is a strict monoidal groupoid $\mathfrak{G}$ in which every object has a strict inverse; that is, multiplication by an object induces an isomorphism of $\mathfrak{G}$ onto itself. A morphism of 2-groups is, by definition, is a strict monoidal functor.

Definition 3.1. A *pseudo 2-group* is a strict monoidal category in which multiplication by any object induces an equivalence of categories. A morphism of pseudo 2-groups is a strict monoidal functor.

It is easy to show that the underlying category of a pseudo 2-group is a groupoid. So a pseudo 2-group is, in particular, a monoid object in the category of groupoids. Note that a 2-group is a monoid object in the category of groupoids in which multiplication by any object is an *isomorphism* of groupoids, whereas a pseudo 2-group is a monoid object in the category of groupoids in which multiplication by any object is an *equivalence* of groupoids.

The set of isomorphism classes of objects in a (pseudo) 2-group $\mathfrak{G}$ is denoted by $\pi_1\mathfrak{G}$; this is a group. The automorphism group of the identity object $1 \in \operatorname{Ob}\mathfrak{G}$ is denoted by $\pi_2\mathfrak{G}$; this is an abelian group.

Pseudo 2-groups and strict monoidal functors between them form a category **Ps2Gp** which contains the category **2Gp** of 2-groups as a full subcategory.[2] Morphisms in **Ps2Gp** induce group homomorphisms on π_1 and π_2. In other words, we have functors $\pi_1, \pi_2\colon \mathbf{Ps2Gp} \to \mathbf{Gp}$; the functor π_2 indeed lands in the full subcategory of abelian groups. A morphism between pseudo 2-groups is called an *equivalence* if the induced homomorphisms on π_1 and π_2 are isomorphisms. Note that an equivalence may not have an inverse.

The following lemma is straightforward.

Lemma 3.2. *Let $f\colon \mathfrak{H} \to \mathfrak{G}$ be a morphism of pseudo 2-groups. Then f, viewed as a morphism of underlying groupoids, is fully faithful if and only if $\pi_1 f\colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$ is injective and $\pi_2 f\colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$ is an isomorphism. It is an equivalence of groupoids if an only if both $\pi_1 f$ and $\pi_2 f$ are isomorphisms.*

A *crossed-module* $\mathfrak{G} = [\partial\colon G_2 \to G_1]$ consists of a pair of groups G_1, G_2, a group homomorphism $\partial\colon G_2 \to G_1$, and a (right) action of G_1 on G_2, denoted $-^a$, which lifts the conjugation action of G_1 on the image of ∂ and descends the conjugation action of G_2 on itself. In other words, the following axioms are satisfied:

- $\forall \alpha, \beta \in G_2,\ \beta^{\partial(\alpha)} = \alpha^{-1}\beta\alpha$;
- $\forall \beta \in G_2, \forall a \in G_1,\ \partial(\beta^a) = a^{-1}\partial(\beta)a$.

It is easy to see that the kernel of ∂ is a central (in particular abelian) subgroup of G_2; we denote this abelian group by $\pi_2\mathfrak{G}$. The image of ∂ is always a normal subgroup of G_1; we denote the cokernel of ∂ by $\pi_1\mathfrak{G}$. A morphism of crossed-modules is a pair of group homomorphisms which commute with the ∂ maps and respect the actions. Such a morphism induces group homomorphisms on π_1 and π_2.

Crossed-modules and morphisms between them form a category, which we denote by **CrossedMod**. We have functors $\pi_1, \pi_2\colon \mathbf{CrossedMod} \to \mathbf{Gp}$; the functor π_2 indeed lands in the full subcategory of abelian groups. A morphism in **CrossedMod** is said to be an equivalence if it induces isomorphisms on π_1 and π_2. Note that an equivalence may not have an inverse.

There is a natural equivalence of categories $\mathbf{2Gp} \simeq \mathbf{CrossedMod}$; see [No2], §3.3. This equivalence is compatible with the functors π_1 and π_2. This way, we can think of a crossed-module as a 2-group, and vice versa. For this reason, we will

[2]Both **Ps2Gp** and **2Gp** are *2-categories* but we will ignore the 2-morphisms for the time being and only look at the underlying 1-category.

sometimes use the term 2-group for an object that is actually a crossed-module. We hope that this will not cause any confusion.

4. 2-groups over a site and gr-stacks

First a few words on terminology. For us a stack is presheaf of groupoids (and not a category fibered in groupoids) over a Grothendieck site which satisfies the descent condition. This may be a bit unusual for algebraic geometers who are used to categories fibered in groupoids, but it makes the exposition simpler. Of course, it is standard that this point of view is equivalent to the one via categories fibered in groupoids. Just to recall how this equivalence works, to any category fibered in groupoids $\mathcal{X}$ one can associate a presheaf $\underline{\mathcal{X}}$ of groupoids over C which is defined as follows. By definition, $\underline{\mathcal{X}}$ is the presheaf that assigns to an object $U \in \mathsf{C}$ the groupoids $\underline{\mathcal{X}}(U) := \mathrm{Hom}(\underline{U}, \mathcal{X})$, where $\underline{U}$ stands for the presheaf of sets represented by U and Hom is computed in the category of stacks over C. Conversely, to any presheaf of groupoids one associates a category fibered in groupoids defined via Grothendieck construction. For more on this we refer the reader to [Ho], especially §5.2.

4.1. Presheaves of pseudo 2-groups over C. Let C be a Grothendieck site. Let $\mathbf{Ps2Gp}_{\mathsf{C}}$ be the category of presheaves of pseudo 2-groups over C; that is, the category of contravariant functors from C to $\mathbf{Ps2Gp}$. We define $\mathbf{2Gp}_{\mathsf{C}}$ and $\mathbf{CrossedMod}_{\mathsf{C}}$ analogously. The category $\mathbf{2Gp}_{\mathsf{C}}$ is a full subcategory of $\mathbf{Ps2Gp}_{\mathsf{C}}$. There is a natural equivalence of categories $\mathbf{2Gp}_{\mathsf{C}} \simeq \mathbf{CrossedMod}_{\mathsf{C}}$. In particular, we can think of a presheaf of crossed-modules as a presheaf of 2-groups.

Let $\mathcal{X}$ be a presheaf of groupoids over C. To $\mathcal{X}$ we associate a presheaf of pseudo 2-groups $\mathcal{A}ut\mathcal{X} \in \mathbf{Ps2Gp}_{\mathsf{C}}$ which parameterizes auto-equivalences of $\mathcal{X}$. By definition, $\mathcal{A}ut\mathcal{X}$ is the functor which associates to an object U in C the pseudo 2-group of self-equivalences of $\mathcal{X}_U$, where $\mathcal{X}_U$ is the restriction of $\mathcal{X}$ to the comma category C_U. (The 'comma category', or the 'over category', C_U is the category of objects in C over U.) Notice that in the case where $\mathcal{X}$ is a stack, $\mathcal{A}ut\mathcal{X}$, viewed as a presheaf of groupoids, is also a stack. Indeed, $\mathcal{A}ut\mathcal{X}$ is almost a group object in the category of stacks over C. To be more precise, $\mathcal{A}ut\mathcal{X}$ is a gr-stack in the sense of Definition 4.1 below.

Let $\underline{\mathfrak{G}} \in \mathbf{Ps2Gp}_{\mathsf{C}}$ be a presheaf of pseudo 2-groups on C. We define $\pi_1^{pre}\underline{\mathfrak{G}}$ to be the presheaf $U \mapsto \pi_1\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_1\underline{\mathfrak{G}}$ the sheaf associated to $\pi_1^{pre}\underline{\mathfrak{G}}$. Similarly, $\pi_2^{pre}\underline{\mathfrak{G}}$ is defined to be the presheaf $U \mapsto \pi_2\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_2\underline{\mathfrak{G}}$ the sheaf associated to $\pi_2^{pre}\underline{\mathfrak{G}}$.

We define $\pi_1\underline{\mathfrak{G}}$ and $\pi_2\underline{\mathfrak{G}}$ for a presheaf of crossed-modules $\underline{\mathfrak{G}} \in \mathbf{CrossedMod}_{\mathsf{C}}$ in a similar manner. The equivalence of categories between $\mathbf{2Gp}_{\mathsf{C}}$ and $\mathbf{CrossedMod}_{\mathsf{C}}$ respects π_1^{pre}, π_1, π_2^{pre} and π_2. Lemma 3.2 remains valid in this setting if instead of π_1 and π_2 we use π_1^{pre} and π_2^{pre}.

4.2. gr-stacks over C. We recall the definition of a gr-stack from [Br]. We modify Breen's definition by assuming that our gr-stacks are *strictly* associative. This is all we will need because the gr-stack $\mathcal{A}ut\mathcal{X}$ of self-equivalences of a stack $\mathcal{X}$ (indeed, any presheaf of groupoids) has this property, and that is all we are concerned with in this paper.

Definition 4.1 ([Br], page 19)**.** Let C be a Grothendieck site. By a (strict) *gr-stack* over C we mean a stack $\mathcal{G}$ that is a (strict) monoid object in the category of stacks over C and for which weak inverses exist. All our gr-stacks will be strict, so from now on we will drop the adjective strict.

The condition on existence of weak inverses means that for every $U \in \mathrm{Ob}\,\mathsf{C}$ and every object a in the groupoid $\mathcal{G}(U)$, multiplication by a induces an equivalence of categories from $\mathcal{G}(U)$ to itself (or equivalently, an equivalence of stacks from $\mathcal{X}_U$ to itself). This condition is equivalent to saying that, for every $U \in \mathrm{Ob}\,\mathsf{C}$, $\mathcal{X}(U)$ is a pseudo 2-group. More compactly, it is equivalent to

$$\mathcal{G} \times \mathcal{G} \stackrel{(pr,mult)}{\longrightarrow} \mathcal{G} \times \mathcal{G}$$

being an equivalence of stacks.

A morphism of gr-stacks is, by definition, a morphism of stacks that (strictly) respects the monoidal structure. Let $\mathbf{grSt}_{\mathsf{C}}$ be the category of gr-stacks and (strict) homomorphisms between them. There are natural functors

$$\mathbf{Ps2Gp}_{\mathsf{C}} \to \mathbf{grSt}_{\mathsf{C}} \quad \text{and} \quad \mathbf{CrossedMod}_{\mathsf{C}} \to \mathbf{grSt}_{\mathsf{C}}.$$

The former is simply the stackification functor that sends a presheaf of groupoids to the associated stack; note that since the stackification functor preserves products, we can carry over the monoidal structure from the presheaf of groupoids to its stackification. The latter functor is obtained from the former by using the natural functor $\mathbf{CrossedMod}_{\mathsf{C}} \to \mathbf{Ps2Gp}_{\mathsf{C}}$. Given a presheaf of crossed-modules $[\partial\colon \underline{G_2} \to \underline{G_1}]$, the associated gr-stack has as underlying stack the quotient stack $[\underline{G_1}/\underline{G_2}]$, where $\underline{G_2}$ acts on $\underline{G_1}$ by multiplication on the right (via ∂).

Definition 4.2. Let $\mathcal{X}$ be a presheaf of groupoids over C. We define $\pi^{pre}\mathcal{X}$ to be the presheaf that sends an object U in C to the set of isomorphism classes in $\mathcal{X}(U)$. We denote the sheaf associated to $\pi^{pre}\mathcal{X}$ by $\pi\mathcal{X}$. For a global section e of $\mathcal{X}$, we define $\underline{\mathrm{Aut}}_{\mathcal{X}}(e)$ to be sheaf associated to the presheaf that sends an object U in C to the group of automorphisms, in the groupoid $\mathcal{X}(U)$, of the object e_U; note that when $\mathcal{X}$ is a stack this presheaf is already a sheaf and no sheafification is needed.

It is clear that π^{pre}, π and $\underline{\mathrm{Aut}}$ are functorial in $\mathcal{X}$.

Definition 4.3. Let $\mathcal{G}$ be a gr-stack. We define $\pi_1^{pre}\mathcal{G} := \pi^{pre}\mathcal{G}$, and $\pi_2^{pre}\mathcal{G} := \underline{\mathrm{Aut}}(e)$, where e is the identity section of $\mathcal{G}$. We define $\pi_1\mathcal{G}$ and $\pi_2\mathcal{G}$ to be sheafifications of $\pi_1^{pre}\mathcal{G}$ and $\pi_2^{pre}\mathcal{G}$, respectively.

4.3. **Equivalences of gr-staks.** There are two ways of defining the notion of equivalence between gr-stacks. One way is to regard them as stacks and use the usual notion of equivalence of stacks. The other way is to regard them as presheaves of pseudo 2-groups and use π_1 and π_2. The next lemma shows that these two definitions agree.

Lemma 4.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be gr-stacks, and let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of gr-stacks. Then, the following are equivalent:*

- (i) *f is an equivalence of stacks.*
- (ii) *The induced maps $\pi_1 f\colon \pi_1^{pre}\mathcal{H} \to \pi_1^{pre}\mathcal{G}$ and $\pi_2 f\colon \pi_2^{pre}\mathcal{H} \to \pi_2^{pre}\mathcal{G}$ are isomorphisms of presheaves of groups.*
- (iii) *The induced maps $\pi_1 f\colon \pi_1\mathcal{H} \to \pi_1\mathcal{G}$ and $\pi_2 f\colon \pi_2\mathcal{H} \to \pi_2\mathcal{G}$ are isomorphisms of sheaves of groups.*

Proof. The only non-trivial implication is $(iii) \Rightarrow (ii)$. In the proof we will use the following standard fact from closed model category theory.

Theorem ([Hi], Theorem 3.2.13)**.** Let $\mathcal{M}$ be a closed model category, L a localizaing class of morphisms in $\mathcal{M}$, and $\mathcal{M}_L$ the localized model category. Let $\mathcal{X}$ and $\mathcal{Y}$ be fibrant objects (i.e. L-local objects) in $\mathcal{M}_L$, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathcal{M}$ that is a weak equivalence in the localized model structure $\mathcal{M}_L$ (that is, f is an L-local weak equivalence). Then, f is a weak equivalence in $\mathcal{M}$.

We will apply the above theorem with $\mathcal{M}$ being the model structure on the category $\mathbf{Gpd}_{\mathsf{C}}$ of presheaves of groupoids on C in which weak equivalences are morphisms that induce isomorphisms (of presheaves of groups) on π_1^{pre} and π_2^{pre}, and fibrations are objectwise. We take L to be the class of hypercovers. The weak equivalences in the localized model structure will then be the ones inducing isomorphism (of sheaves of groups) on π_1 and π_2. The main reference for this is [Ho].

Let us now prove $(iii) \Rightarrow (ii)$. It is shown in [Ho] that $\mathcal{G}$ and $\mathcal{H}$ are L-local objects (see §5.2 and §7.3 of [ibid.]). By hypothesis, f induces isomorphisms (of sheaves) on π_1 and π_2, so it is a weak equivalence in the localized model structure. Therefore, since $\mathcal{G}$ and $\mathcal{H}$ are L-local, f is already a weak equivalence in the the non-localized model structure. This exactly means that $\pi_1(f)\colon \pi_1^{pre}(\mathcal{H}) \to \pi_1^{pre}(\mathcal{G})$ and $\pi_2(f)\colon \pi_2^{pre}(\mathcal{H}) \to \pi_2^{pre}(\mathcal{G})$ are isomorphisms of presheaves. □

Lemma 4.5. *Let $\mathcal{X}$ be a presheaf of groupoids over C and $\varphi\colon \mathcal{X} \to \mathcal{X}^a$ its stackification. Then we have the following:*

- (i) *The induced morphism $\pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is an isomorphism of sheaves of sets.*
- (ii) *For every global section e of $\mathcal{X}$, the natural map $\underline{\mathrm{Aut}}_{\mathcal{X}}(e) \to \underline{\mathrm{Aut}}_{\mathcal{X}^a}(e)$ is an isomorphism of sheaves of groups.*

Proof. This is a simple sheaf theory exercise. We include the prove of (i). Proof of (ii) is similar.

First we prove that $\pi\varphi\colon \pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is injective. Let $U \in \mathrm{Ob}\,\mathsf{C}$, and let x, y be element in $\pi\mathcal{X}(U)$ such that $\pi\varphi(x) = \pi\varphi(y)$. We have to show that $x = y$. By passing to a cover of U, we may assume x and y lift to objects $\bar{x}$ and $\bar{y}$ in $\mathcal{X}(U)$. We will show that there is an open cover of U over which $\bar{x}$ and $\bar{y}$ become isomorphic. Since $\varphi(\bar{x})$ and $\varphi(\bar{y})$ become equal in $\pi(\mathcal{X}^a)$, there is a cover $\{U_i\}$ of U such that there is an isomorphism $\alpha_i\colon \varphi(\bar{x}|_{U_i}) \xrightarrow{\sim} \varphi(\bar{y}|_{U_i})$ in the groupoid $\mathcal{X}^a(U_i)$, for every i. By replacing $\{U_i\}$ with a finer cover, we may assume that α_i come from $\mathcal{X}(U_i)$. (More precisely, $\alpha_i = \varphi(\beta_i)$, where β_i is a morphism in the groupoid $\mathcal{X}(U_i)$.) This implies that, for every i, $\bar{x}|_{U_i}$ and $\bar{y}|_{U_i}$ are isomorphic as objects of the groupoid $\mathcal{X}(U_i)$. This is exactly what we wanted to prove.

Having proved the injectivity, to prove the surjectivity it is enough to show that every object x in $\pi(\mathcal{X}^a)(U)$ is in the image of φ, possibly after replacing U by an open cover. By choosing an appropriate cover, we may assume x lifts to $\mathcal{X}^a(U)$. Since $\mathcal{X}^a$ is the stackification of $\mathcal{X}$, we may assume, after refining our cover, that x is in the image of $\mathcal{X}(U) \to \mathcal{X}^a(U)$. The claim is now immediate. □

Lemma 4.6. *Let $\underline{\mathfrak{G}} = [\underline{G_2} \to \underline{G_1}]$ be a presheaf of crossed-modules, and let $\mathcal{G} = [\underline{G_1}/\underline{G_2}]$ be the corresponding gr-stack. Then, we have natural isomorphisms of sheaves of groups $\pi_i\underline{\mathfrak{G}} \xrightarrow{\sim} \pi_i\mathcal{G}$, $i = 1, 2$.*

Proof. Apply Lemma 4.5. □

4.4. **Crossed-modules in schemes and self-equivalences of quotient stacks.** We now specialize to the case where C is $\mathbf{Sch}_S$, the big site of schemes over a base scheme S, endowed with a subcanonical topology (say, étale, Zariski, fppf, fpqc,...). We define a *crossed-module in* S*-schemes* $[\partial\colon G_2 \to G_1]$ to be a pair of S-group schemes G_1, G_2, an S-group scheme homomorphism $\partial\colon G_2 \to G_1$, and a (right) action of G_1 on G_2 satisfying the axioms of a crossed-module. These are precisely the representable objects in $\mathbf{CrossedMod}_{\mathbf{Sch}_S}$; in other words, a crossed-module in schemes $[\partial\colon G_2 \to G_1]$ gives rise to a presheaf of crossed-modules

$$U \mapsto [\partial(U)\colon G_2(U) \to G_1(U)].$$

We will abuse the terminology and call a crossed-module in schemes over S simply a *2-group scheme over* S.

Proposition 4.7. *Let S be a base scheme. Let H be an abelian group scheme over S, acting on a S-scheme X, and let $\mathcal{X} = [X/H]$ be the quotient stack. Let G be those automorphisms of X that commute with the H action; this is a sheaf of groups on $\mathbf{Sch}_S$. We have the following:*

- (i) *With the trivial action of G on H, the natural map $H \to G$ becomes a crossed-modules in $\mathbf{Sch}_S$-schemes.*
- (ii) *Let $\mathcal{G}$ be the gr-stack associated to $[H \to G]$. Then, there is a natural morphism of gr-stacks $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$. Furthermore, this morphism induces an isomorphism of sheaves of groups $\pi_2\mathcal{G} \xrightarrow{\sim} \pi_2(\mathcal{A}ut\mathcal{X})$.*
- (iii) *Assume that $X \to S$ has geometrically connected fibers and $\mathcal{X} \to S$ is proper. Then $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ is fully faithful (as a morphism of presheaves of groupoids). In particular, the induced map $\pi_1\mathcal{G} \xrightarrow{\sim} \pi_1(\mathcal{A}ut\mathcal{X})$ of sheaves of groups is injective.*

Proof. Let $\underline{\mathfrak{G}}$ denote the presheaf of 2-groups associated to $[H \to G]$. It is enough to construct a morphism of presheaves of 2-groups $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ and show that it has the required properties. Stackification of this map gives us the desired map (Lemma 4.6).

Construction of the morphism $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ is precisely as in ([BeNo], Lemma 8.2), and so are the proofs of (i) and (ii). Also, in the presence of the hypothesis on having geometrically connected fibers, the proof in [ibid.] of (iii) can be repeated word by word with $\mathbb{C}$ replaced by a connected scheme U. This implies that $\underline{\mathfrak{G}}(U) \to \mathcal{A}ut\mathcal{X}_U$ is fully faithful (Lemma 3.2) for an arbitrary U. From this it follows that $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ is fully faithful; it remains fully faithful after stackification (because $\mathcal{A}ut\mathcal{X}$ is a stack). □

5. Interlude: 2-group actions on stacks

The main objective of this paper is to find a crossed-module model for the gr-stack $\mathcal{A}ut\mathcal{X}$ of self-equivalences of a stack $\mathcal{X}$ (in our case, a weighted projective stack $\mathcal{P}(n_0, n_1, \cdots, n_r)$). It is perhaps not so clear to the reader why at all it is important to find such a crossed-module model. To provide a motivation for the reader, we explain how such a model can be used to study 2-group actions on stacks. This is based on the method developed in [No5]. Since [ibid.] is not yet available, in this section we give a synopsis of some of the ideas involved in [ibid.]. The point is that, the input needed to run the method developed in [ibid.] is a crossed-module model for $\mathcal{A}ut\mathcal{X}$.

The material in this section is independent of the rest of the paper and is only meant to be motivational.

5.1. **What is a 2-group action on a stack?** Let $\mathcal{X}$ be a stack over a base scheme S, and let G be a group scheme over S. For simplicity, let us first assume that G is a constant discrete group scheme. What is an action of G on $\mathcal{X}$? If $\mathcal{X}$ was a scheme, this would mean assignment of an automorphism $\phi_g \colon \mathcal{X} \to \mathcal{X}$, for every $g \in G$, such that $\phi_{gh} = \phi_g \circ \phi_h$. In the case where $\mathcal{X}$ is a stack, this is, however, not the correct definition. Let us explain why.

1. The requirement $\phi_{gh} = \phi_g \circ \phi_h$ is not natural. Instead of asking for equality, one should require existence of 2-morphisms $\alpha_{g,h} \colon \phi_{gh} \Rightarrow \phi_g \circ \phi_h$ satisfying certain coherence conditions.

2. Even the requirement of the assignment $g \mapsto \phi_g$ is too restrictive! For instance, assume G is isomorphic to a quotient of groups H/K. Then, any action of H on $\mathcal{X}$, as in (**1**), for which ϕ_k is 2-isomorphic to the identity for all $k \in K$ should also be regarded as an action of G on $\mathcal{X}$.

3. There is a notion of a natural transformation between two actions ϕ and ϕ and ϕ' which consists of a collection of 2-morphisms $\theta_g \colon \phi_g \Rightarrow \phi'_g$ satisfying certain coherence conditions. Any two actions that are related by a natural transformation should be regraded as being the "same".

4. Assume that we are given a covering $\{U_i\}$ of S (in the given topology of S). Assume that, for every U_i, we have an action of G on $\mathcal{X}|_{U_i}$. Assume that the actions are compatible, in the sense of (**3**), on the double intersections $U_i \cap U_j$. This should then give rise to a global action of G on $\mathcal{X}$.

Let us make two further remarks: 1) More complications arise when G is a group scheme (or a sheaf of groups) over S; 2) What is the correct thing to study is the action of a *gr-stack* $\mathfrak{G}$ on a stack $\mathcal{X}$ – this adds extra complications.

In [No5] we propose a definition for the action of a gr-stack $\mathfrak{G}$ on a stack $\mathcal{X}$ in a way that it addresses the above difficulties. Briefly put, an action of $\mathfrak{G}$ on $\mathcal{X}$ is a *weak morphism of gr-stacks* $\mathfrak{G} \to \mathcal{A}ut\mathcal{X}$, as defined in [ibid.]. Two such actions are regarded as the "same" if there is a *natural transformation* between them [ibid.].

We will not give the definition of a weak morphism in this paper. Instead, in the next section, we quote a general result which gives a simple classification of weak morphisms between 2-groups, and also of natural transformations between them. It is this result that has been our main motivation for looking for crossed-module models for $\mathcal{A}ut\mathcal{X}$.

5.2. **Classification of weak morphisms between 2-groups.** We recall a definition from [No3].

Definition 5.1 ([No3], Definition 8.1)**.** Let $\mathfrak{G} = [\varphi \colon G_2 \to G_1]$ and $\mathfrak{H} = [\psi \colon H_2 \to H_1]$ be crossed-modules. By a *butterfly* from $\mathfrak{G}$ to $\mathfrak{H}$ we mean a commutative diagram of groups

$$\begin{array}{ccccc} G_2 & & & & H_2 \\ & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \\ \psi \downarrow & & E & & \downarrow \varphi \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ G_1 & & & & H_1 \end{array}$$

in which both diagonal sequences are complexes, and the NE-SW sequence, that is, $H_2 \to E \to G_1$, is short exact. We require that ρ and σ satisfy the following compatibility with actions. For every $x \in E$, $\alpha \in H_2$, and $\beta \in G_2$,

$$\iota(\alpha^{\rho(x)}) = x^{-1}\iota(\alpha)x, \quad \kappa(\beta^{\sigma(x)}) = x^{-1}\kappa(\beta)x.$$

A *morphism* between two butterflies $(E, \rho, \sigma, \iota, \kappa)$ and $(E', \rho', \sigma', \iota', \kappa')$ is a morphism $f : E \to E'$ commuting with all four maps (it is easy to see that such an f is necessarily an isomorphism). We define $\mathcal{M}(\mathfrak{G}, \mathfrak{H})$ to be the groupoid of butterflies from $\mathfrak{G}$ to $\mathfrak{H}$.

The definition quoted above is only given for the case of crossed-modules in sets. The exact same definition can be made for crossed-modules over a site C. More precisely, $\mathfrak{G}$ and $\mathfrak{H}$ are now crossed-modules in sheaves of groups, and the sequence $H_2 \to E \to G_1$ is a short exact sequence of sheaves.

The significance of this definition is due to the following result from [No5].

Theorem 5.2. *Let $\mathfrak{G} = [G_2 \to G_1]$ and $\mathfrak{H} = [H_2 \to H_1]$ be crossed-modules of sheaves of groups over the cite C with $\mathcal{G} = [G_1/G_2]$, and let $\mathcal{H} = [H_1/H_2]$ be the corresponding quotient gr-stacks. Then, there is a natural equivalence*

$$\mathcal{M}(\mathfrak{G}, \mathfrak{H}) \cong \mathrm{Hom}_{\mathrm{weak}}(\mathcal{G}, \mathcal{H})$$

where the groupoid on the right hand side has weak morphisms of gr-stacks as its objects and natural transformations as morphisms.

In particular, if $[H_2 \to H_1]$ is a crossed-module model for the gr-stack $\mathcal{A}ut\mathcal{X}$ of auto-equivalences of a stack $\mathcal{X}$, then an action of $\mathcal{G} = [G_1/G_2]$ on $\mathcal{X}$ is the same thing as an isomorphism class of a butterfly as in Definition 5.1. In other words, to give an action of $\mathcal{G}$ on $\mathcal{X}$, we need to find an extension E of G_1 by H_2, together with group homomorphisms $\kappa : G_2 \to E$ and $\rho : E \to G_1$ satisfying the conditions required in Definition 5.1.

6. Weighted projective general linear 2-groups

In this section we introduce our main objects of interest, the weighted projective general linear 2-group schemes, and prove that they model the self-equivalences of weighted projective stacks (Theorem 6.1). In fact, since the construction of the weighted projective stacks, and also of the weighted projective general linear 2-group schemes, commutes with base change, it will be enough to work over $\mathbb{Z}$.

We begin with some notation. We denote the multiplicative group scheme over $\operatorname{Spec}\mathbb{Z}$ by $\mathbb{G}_{m,\mathbb{Z}}$, or simply $\mathbb{G}_m$. The affine $(r+1)$-space over a base scheme S is denoted by $\mathbb{A}^{r+1}_S$; when the base scheme is $\operatorname{Spec} R$ it is denoted by $\mathbb{A}^{r+1}_R$, and when the base scheme is $\operatorname{Spec}\mathbb{Z}$ simply by $\mathbb{A}^{r+1}$. Since r will be fixed throughout this section, we will usually denote $\mathbb{A}^{r+1}_S - \{0\}$ by $\mathbb{U}_S$. We will abbreviate $\mathbb{U}_{\operatorname{Spec} R}$ and $\mathbb{U}_{\operatorname{Spec}\mathbb{Z}}$ to $\mathbb{U}_R$ and $\mathbb{U}$, respectively. We fix a Grothendieck topology on $\mathbf{Sch}_S$ that is not coarser than Zariski.

Let $n_0, n_1, \cdots, n_r$ be a sequence of positive integers, and consider the weight $(n_0, n_1, \cdots, n_r)$ action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$. (That is, for every scheme T, an element $t \in \mathbb{G}_m(T)$ acts on $\mathbb{U}_T$ by multiplication by $(t^{n_0}, t^{n_1}, \cdots, t^{n_r})$.) The quotient stack of this action is called the *weighted projective stack* of weight $(n_0, n_1, \cdots, n_r)$ and is denoted by $\mathcal{P}_{\mathbb{Z}}(n_0, n_1, \cdots, n_r)$, or simply by $\mathcal{P}(n_0, n_1, \cdots, n_r)$. The **weighted**

projective general linear 2-group scheme $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ is defined to be the 2-group scheme associated to the crossed-module

$$[\partial\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}],$$

where $G_{n_0,n_1,\cdots,n_r}$ is the group scheme, over $\mathbb{Z}$, of all $\mathbb{G}_m$-equivariant (for the above weighted action) automorphisms of $\mathbb{U}$. More precisely, the T-points of $G_{n_0,n_1,\cdots,n_r}$ are automorphisms

$$f\colon \mathbb{U}_T \to \mathbb{U}_T$$

that commute with the $\mathbb{G}_m$-action. The homomorphism $\partial\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}$ is the one induced from the $\mathbb{G}_m$-action itself. We take the action of $G_{n_0,n_1,\cdots,n_r}$ on $\mathbb{G}_m$ to be trivial. The associated gr-stack is denoted by $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$, and is called the *projective general linear* gr-*stack* of weight $(n_0, n_1, \cdots, n_r)$.

The following theorem says that a weighted projective general linear 2-group scheme is a model for the gr-stack of self-equivalences of the corresponding weighted projective stack. A special case of this theorem (namely, the case where the base scheme is $\mathbb{C}$) was proved in ([BeNo], Theorem 8.1). We briefly sketch how the proof in [ibid.] can be modified to cover the general case.

Theorem 6.1. *Let $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ be the gr-stack of automorphisms of the weighted projective stack $\mathcal{P}(n_0, n_1, \cdots, n_r)$. Then, the natural map*

$$\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$$

is an equivalence of gr-stacks. In particular, we have isomorphisms of sheaves of groups

$$\pi_1\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_1\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_1 \mathrm{PGL}(n_0, n_1, \cdots, n_r),$$

$$\pi_2\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_2\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_2 \mathrm{PGL}(n_0, n_1, \cdots, n_r) \cong \mu_d,$$

where $d = \gcd(n_0, n_1, \cdots, n_r)$ and μ_d stands for the multiplicative group scheme of d^{th} roots of unity.

To prove the above theorem we use the following result.

Proposition 6.2. *Let $\mathcal{P} = \mathcal{P}_S(n_0, n_1, \cdots, n_r)$, where $S = \operatorname{Spec} R$ is the spectrum of a local ring. Then every line bundle on $\mathcal{P}$ is of the form $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

Proof. We only sketch the proof (due to A. Vistoli). Details can be found in [No4]. In the proof we use stacky versions of Grothendieck's base change and semicontinuity results ([Ha], III. Theorem 12.11). We will assume R is Noetherian.

In the case where R is a field, the assertion is easy to prove using the fact that the Picard group of $\mathcal{P}$ is isomorphic to the Weil divisor class group. To prove the general case, let x be the closed point of $S = \operatorname{Spec} R$. Let $\mathcal{L}$ be a line bundle on $\mathcal{P}$. After twisting with on appropriate $\mathcal{O}(d)$, we may assume $\mathcal{L}_x \cong \mathcal{O}$. We will show that $\mathcal{L}$ is trivial. We have $H^1(\mathcal{P}_x, \mathcal{L}_x) = H^1(\mathcal{P}_x, \mathcal{O}_x) = 0$. Hence, by semicontinuity, $H^1(\mathcal{P}_y, \mathcal{L}_y) = 0$ for every point y of S. Base change implies that $R^1 f_*(\mathcal{L}) = 0$, and that $R^0 f_*(\mathcal{L}) = f_*(\mathcal{L})$ is locally free (necessarily of rank 1). Therefore, $f_*(\mathcal{L})$ is free of rank 1 and, by base change, $H^0(\mathcal{P}_y, \mathcal{L}_y)$ is 1-dimensional as a $k(y)$-vector space, for every y in S. In fact, this is true for every tensor power $\mathcal{L}^{\otimes n}$, $n \in \mathbb{Z}$. So, $\mathcal{L}_y$ is trivial for every y in S. (Note that, when k is a field, $\dim_k H^0(\mathcal{P}_k(n_0, n_1, \cdots, n_r), \mathcal{O}(d))$ is equal to the number of solutions of the equation $a_1 n_0 + a_2 n_1 + \cdots + a_r n_r = d$ in non-negative integers a_i.)

Now let s be a generating section of $f_*(\mathcal{L}) \cong R$. It follows that $f^*(s)$ is a generating section of $\mathcal{L}$. So $\mathcal{L}$ is trivial. □

Proof of Theorem 6.1. We apply Proposition 4.7 with $S = \operatorname{Spec}\mathbb{Z}$, $X = \mathbb{A}^{r+1} - \{0\}$, and $H = \mathbb{G}_m$. This implies that $\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ is a fully faithful morphism of stacks. That is, for every scheme U, the morphism of groupoids $\mathcal{PGL}(n_0, n_1, \cdots, n_r)(U) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)(U)$ is fully faithful. All that is left to show is that it is essentially surjective. Since $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$ and $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ are both stacks, it is enough to prove this for $U = \operatorname{Spec} R$, where R is a local ring. In this case, we know by Proposition 6.2 that $\operatorname{Pic}\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \mathbb{Z}$. We can now proceed exactly as in ([BeNo], Theorem 8.1).

The isomorphisms stated at the end of the theorem follow from Lemma 4.4 and Lemma 4.6. □

7. Structure of $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$

In this section we give detailed information about the structure of the group $G_{n_0,n_1,\cdots,n_r}$. We show that, as a group scheme over an arbitrary base, it splits as a semi-direct product of a reductive group scheme and a unipotent group scheme. The reductive part is a product of a copies of the general linear groups. The unipotent part is a successive semi-direct product of vector groups; see Theorem 7.7.

Throughout this section, the action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$ means the weight $(n_0, n_1, \cdots, n_r)$ action. To shorten the notation, we denote the group $G_{n_0,n_1,\cdots,n_r}$ by G. The rank m general linear group scheme over $\operatorname{Spec} R$ is denoted by $\mathrm{GL}(m, R)$. When $R = \mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. We always assume $r \geq 1$. The corresponding projectivized group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ for a weighted projective general linear 2-group as in the latter case we have at least two variables.

We begin with a simple lemma.

Lemma 7.1. *Let R be an arbitrary ring, and let f be a global section of the structure sheaf of $\mathbb{U}_R = \mathbb{A}_R^{r+1} - \{0\}$, $r \geq 1$. Then f extends uniquely to a global section of $\mathbb{A}_R^{r+1}$.*

Proof. Let $U_i = \operatorname{Spec} R[x_0, \cdots, x_r, x_i^{-1}]$ and consider the covering $\mathbb{U}_R = \cup_{i=1}^n U_i$. We show that the restriction $f_i := f|_{U_i}$ is a polynomial for every i. To see this, observe that, except possibly for x_i, all variables occur with positive powers in f_i. To show that x_i also occurs with a positive power, pick some $j \neq i$ and use the fact that x_i occurs with a positive power in $f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$.

Therefore, for every i, f_i actually lies in $R[x_0, \cdots, x_r, x_i^{-1}]$. Since $f_j|_{U_i} = f_i|_{U_j}$, it is obvious that all f_i are actually the same and provide the desired extension of f to $\mathbb{U}_R$. □

From now on, we will use a slightly different notation with indices. Namely, we assume that the weights are $m_1 < m_2 < \cdots < m_t$, with each m_i appearing exactly $r_i \geq 1$ times in the weight sequence (so in the previous notation we would have $r + 1 = r_1 + \cdots + r_t$). We denote the corresponding projective general linear 2-group by $\mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t)$. We use the coordinates x_j^i, $1 \leq i \leq t$, $1 \leq j \leq r_i$, for $\mathbb{A}^{r+1}$. We think of x_j^i as a variable of degree m_i. We will usually abbreviate the sequence $x_1^i, \cdots, x_{r_i}^i$ to $\mathbf{x}^i$. Similarly, a sequence $F_1^i, \cdots, F_{r_i}^i$ of polynomials is abbreviated to $\mathbf{F}^i$.

Let R be a ring. The following proposition tells us how a $\mathbb{G}_{m,R}$-equivariant automorphisms of $\mathbb{U}_R$ looks like.

Proposition 7.2. *Let $F\colon \mathbb{U}_R \to \mathbb{U}_R$ be a $\mathbb{G}_m$-equivariant map. Then F is of the form $(\mathbf{F}^i)_{1\leq i\leq t}$, where for every i, each component $F^i_j \in R[x^i_j; 1\leq i\leq t, 1\leq j\leq r_i]$ of $\mathbf{F}^i$ is a weighted homogeneous polynomial of weight m_i.*

Proof. The fact that components of F are polynomial follows from Lemma 7.1. The statement about homogeneity of F^i_j is a simple exercise in polynomial algebra and is left to the reader. □

In the above proposition, each F^i_j can be written in the form $F^i_j = L^i_j + P^i_j$, where L^i_j is linear in the variables $x^i_1, \cdots, x^i_{r_i}$, and P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. Let $L_F := (\mathbf{L}^i)_{1\leq i\leq t}$ be the linear part of F. It is again a $\mathbb{G}_m$-equivariant endomorphism of $\mathbb{U}$.

Proposition 7.3. *Let F be as in the Proposition 7.2. The assignment $F \mapsto L_F$ respects composition of endomorphisms. In particular, if F is an automorphism, then so is L_F.*

Proof. This follows from direct calculation, or, alternatively, by using the fact that L_F is simply the derivative of F at the origin. □

Corollary 7.4. *There is a natural split homomorphism*

$$\phi\colon G \to \mathrm{GL}(r_1)\times \mathrm{GL}(r_2)\times\cdots\times \mathrm{GL}(r_t).$$

Next we give some information about the structure of the kernel U of ϕ. It consists of endomorphisms $F = (F^i_j)_{i,j}$, where F^i_j has the form

$$F^i_j = x^i_j + P^i_j.$$

Here, P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. Indeed, it is easily seen that, for an arbitrary choice of the polynomials P^i_j, the resulting endomorphism F is automatically invertible. So, to give such an $F \in U$ is equivalent to giving an arbitrary collection of polynomials $\{P^i_j\}_{1\leq i\leq t, 1\leq j\leq r_i}$ such that each P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. So, from now on we switch the notation and denote such an element of U by $(P^i_j)_{i,j}$.

Proposition 7.5. *For each $1 \leq a \leq t$, let $U_a \subseteq U$ be the set of those endomorphisms $F = (P^i_j)_{i,j}$ for which $P^i_j = 0$ whenever $i \neq a$. Let K_a denote the set of monomials of degree m_a in variables x^i_j, $i < a$, and let k_a be the cardinality of K_a. (In other words, k_a is the number of solutions of the equation*

$$\sum_{i=1}^{a-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_a$$

in non-negative integers $z_{i,j}$.) Then we have the following:

- *(i) U_a is a subgroup of U and is canonically isomorphic to the vector group scheme $\mathbb{A}^{r_a}\otimes \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a k_a}$. (Note: U_1 is trivial.)*
- *(ii) If $a < b$, then U_a normalizes U_b.*
- *(iii) The groups U_a, $1 \leq i \leq t$, generate U and we have $U_a \cap U_b = \{1\}$ if $a \neq b$.*

Proof of (i). The action of $(P^i_j)_{i,j} \in U_a$ on $\mathbb{A}^{r+1}$ is given by

$$(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) \longmapsto (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t).$$

So, if $\mathbb{A}^{K_a}$ stands for the vector group scheme on the basis K_a, there is a canonical isomorphism

$$U_a \cong \bigoplus_{i=1}^{r_a} \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}.$$

Proof of (ii). Let $G = (Q^i_j)_{i,j}$ be an element in U_a and $F = (P^i_j)_{i,j}$ an element in U_b. By (i), the inverse of G is $G^{-1} = (-Q^i_j)_{i,j}$. Let us analyze the effect of the composite $G \circ F \circ G^{-1}$ on $\mathbb{A}^{r+1}$:

$$\begin{aligned}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) \quad &\overset{G^{-1}}{\longmapsto} \quad (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) \\ &\overset{F}{\longmapsto} \quad (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t) \\ &\overset{G}{\longmapsto} \quad (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t).\end{aligned}$$

Here the polynomial R^b_k, $1 \leq k \leq r_b$, is obtained from P^b_k by substituting the variables x^a_j with the polynomial $x^a_j - Q^a_j$.

Proof of (iii). Easy. □

Part (ii) implies that each U_a acts by conjugation on each of U_{a+1}, $U_{a+2}, \cdots$, U_t.[3] To fix the notation, in what follows we let the conjugate of an automorphism f by an automorphism g to be $g \circ f \circ g^{-1}$.

Notation. Let $\{U_a\}_{a=1}^t$ be a family of subgroups of a group U which satisfies the following properties: 1) Each U_a normalizes every U_b with $a < b$; 2) No two distinct U_a intersect; 3) The U_a generate U. In this case, we say that U is a successive semi-direct product of the U_a, and use the notation $U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1$.

The following is an immediate corollary of Proposition 7.5.

Corollary 7.6. *There is a natural decomposition of* U *as a semi-direct product*

$$U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1,$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ *is the group introduced in Proposition 7.5. (Note that* U_1 *is trivial.)*

In the next theorem we use the notation $\mathbb{A}^m$ for two things. One that has already appeared is the affine group scheme of dimension m. When there is a group scheme G involved, we also use the notation $\mathbb{A}^m$ for the trivial representation of G on $\mathbb{A}^m$.

Theorem 7.7. *There is a natural decomposition of* G *as a semi-direct product*

$$G \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes \big(\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t)\big),$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ *and* k_a *is as in Proposition 7.5. (Note that* U_1 *is trivial.) Furthermore, for every* $1 \leq a \leq t$*, the action of* $\mathrm{GL}(r_a)$ *leaves each* U_b *invariant. We also have the following:*

(i) *When* $a > b$ *the induced action of* $\mathrm{GL}(r_a)$ *on* U_b *is trivial.*

[3] All group actions in this section are assumed to be on the left.

(*ii*) *When* $a = b$ *the induced action of* $\mathrm{GL}(r_a)$ *on* U_a *is naturally isomorphic to the representation* $\rho \otimes \mathbb{A}^{K_a}$*, where* ρ *is the standard representation of* $\mathrm{GL}(r_a)$ *and* K_a *is as in Proposition 7.5. (Recall that* U_a *is canonically isomorphic to* $\mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}$*.)*

(*iii*) *When* $a < b$ *the action of* $\mathrm{GL}(r_a)$ *on* U_b *is naturally isomorphic to the representation*

$$\bigoplus_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{r_b d_l} \otimes \hat{\rho}^{\otimes l}.$$

Here $\hat{\rho}$ *stands for the inverse transpose of* ρ*, and* d_l *is the number of monomials of degree* m_b *in variables* x^i_j*,* $i < b$*,* $i \neq a$*; so* d_l *also depends on* a *and* b*. (In other words,* d_l *is the number of solutions of the equation*

$$\sum_{\substack{i=1 \\ i \neq a}}^{b-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_b - l m_a$$

in non-negative integers $z_{i,j}$*.)*

Proof. Let $g \in \mathrm{GL}(r_a)$ and $F \in U_b$. As in the proof of Proposition 7.5.i, we analyze the effect of the composite $g \circ F \circ g^{-1}$ on $\mathbb{A}^{r+1}$. The element $g \in \mathrm{GL}(r_a)$ acts on $\mathbb{A}^{r+1}$ as follows: it leaves every component x^j_i invariant if $i \neq a$ and on the coordinates $x^a_1, \cdots, x^a_{r_a}$ it acts linearly (like the action of an $r_a \times r_a$ matrix on a column vector).

Proof of (i). The effect of $g \in \mathrm{GL}(r_a)$ only involves the variables $x^a_1, \cdots, x^a_{r_a}$ and does not see any other variable, whereas the effect of $F \in U_b$ only involves the variables x^i_j, $i \leq b$. Since $b < a$, these two are independent of each other. That is, F and g commute.

Proof of (ii). Assume $F = (P^i_j)_{i,j}$; so $P^i_j = 0$ if $i \neq a$. The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{aligned} (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) &\xmapsto{g^{-1}} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^t) \\ &\xmapsto{F} (\mathbf{x}^1, \cdots, \mathbf{y}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t) \\ &\xmapsto{g} (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{Q}^a, \cdots, \mathbf{x}^t). \end{aligned}$$

Here, y^a_j is the linear combination of $x^a_1, \cdots, x^a_{r_a}$, the coefficients being the entries of the j^{th} row of the matrix g^{-1}. Similarly, Q^a_j is the linear combination of $P^a_1, \cdots, P^a_{r_a}$, coefficients being the entries of the j^{th} row of the matrix g.

Proof of (iii). Assume $F = (P^i_j)_{i,j}$; so $P^i_j = 0$ if $i \neq b$. Let $\mathbf{y}^a$ be as in (ii). The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{aligned} (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) &\xmapsto{g^{-1}} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) \\ &\xmapsto{F} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t) \\ &\xmapsto{g} (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t). \end{aligned}$$

Here the polynomials R^b_k, $1 \leq k \leq r_b$, are obtained from P^b_k by substituting the variable x^a_j with y^a_j.

Let λ be the representation of $\mathrm{GL}(r_a)$ on the space V of homogenous polynomials of degree m_b which acts as follows: it takes a polynomial $P \in V$ and substitutes the variables x^a_j, $1 \leq j \leq r_a$, with y^a_j. From the description above, we see that the representation of $\mathrm{GL}(r_a)$ on U_b is a direct sum of r_b copies of λ. We will show that

$$\lambda \cong \bigoplus_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{d_l} \otimes \hat{\rho}^{\otimes l}.$$

To obtain the above decomposition, simply note that a polynomial in V can be uniquely written in the form

$$\sum_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} S_l T_l,$$

where T_l is a homogenous polynomial of degree lm_a in variables $x^a_1, \cdots, x^a_{r_a}$, and S_l is a homogenous polynomial of degree $m_b - lm_a$ in the rest of the variables. The action of $\mathrm{GL}(r_a)$ leaves S_l intact and acts on T_l by the l^{th} power of the inverse transpose of the standard representation. □

The actions of various pieces in the above semi-direct product decomposition, though explicit, are tedious to write down, except for small values of t. We give some examples.

Remark 7.8. It is perhaps useful to put the above result in the general context of algebraic group theory. Recall that every algebraic group G over a *field* fits in a short exact sequence

$$1 \to U \to G \to G_{red} \to 1,$$

where U is the unipotent radical of G and G_{red} is reductive. The sequence is not split in general. In our case, however, the group scheme $G_{n_0,n_1,\cdots,n_r}$ admits such a short sequence over an arbitrary base. Furthermore, the sequence is split.

The general theory of unipotent groups tells us that any unipotent group over a *perfect field* admits a filtration whose graded pieces are vector groups. This filtration splits, but only in the category of schemes (i.e., the splitting maps may not be group homomorphisms). In our case, however, the group scheme U admits such a filtration over an arbitrary base. Furthermore, the filtration is split group theoretically.

Let us see some explicit examples.

Example 7.9. *Weight sequence* $m < n, m \nmid n$. In this case we have $t = 2$, and $r_1 = r_2 = 1$ and $k_1 = 0$. So $G \cong \mathbb{G}_m \times \mathbb{G}_m$.

Example 7.10. *Weight sequence* $m < n, m \mid n$. In this case we have $t = 2$, $r_1 = r_2 = 1$, and $k_1 = 1$. So we have

$$G \cong \mathbb{A} \rtimes (\mathbb{G}_m \times \mathbb{G}_m).$$

The action of an element $(\lambda_1, \lambda_2) \in \mathbb{G}_m \times \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by

$$(\lambda_1, \lambda_2) \cdot a = \lambda_2 \lambda_1^{-\frac{n}{m}} a.$$

More explicitly, an element in G is map of the form

$$(x, y) \mapsto (\lambda_1 x, \lambda_2 y + a x^{\frac{n}{m}}).$$

Note the similarity with the group of 2×2 lower-triangular matrices.

Example 7.11. *Weight sequence* $n = m$. We obviously have $G \cong \mathrm{GL}(2)$.

Example 7.12. *Weight sequence* $1,2,3$. First we determine U. A typical element in U is of the form

$$(x,y,z) \mapsto (x, y+ax^2, z+bx^3+cxy).$$

We have $U_2=\mathbb{A}$ and $U_3=\mathbb{A}^2$. The action of an element $a \in U_2$ on an element $(b,c) \in U_3$ is given by $(b-ac,c)$. That is, a acts on $U_3=\mathbb{A}^2$ by the matrix

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 2} \rtimes \mathbb{A}$. Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 2} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1,\lambda_2,\lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a,b,c) \in U$ is given by $(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-3}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c)$.

Example 7.13. *Weight sequence* $1,2,4$. An element in U has the general form

$$(x,y,z) \mapsto (x, y+ax^2, z+bx^4+cx^2y+dy^2).$$

We have $U_2=\mathbb{A}$ and $U_3=\mathbb{A}^3$. The action of an element $a \in U_2$ on an element $(b,c,d) \in U_3$ is given by the matrix

$$\begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 3} \rtimes \mathbb{A}$.

Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 3} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1,\lambda_2,\lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a,b,c,d) \in U$ is given by

$$(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-4}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c, \lambda_2^{-2}\lambda_3 d).$$

Next we look at $\mathrm{PGL}(m_1:r_1, m_2:r_2,\cdots,m_t:r_t)$. Recall that, as a crossed-module, this is given by $[\partial\colon \mathbb{G}_m \to G]$, where ∂ is the obvious map coming from the action of $\mathbb{G}_m$ on $\mathbb{A}^{r+1}$, and the action of G on $\mathbb{G}_m$ is the trivial one.

Observe that the map ∂ factors though the component $\mathrm{GL}(r_1)\times\cdots\times\mathrm{GL}(r_t)$ of G. So, let us define L to be the cokernel of the following map:

$$\mathbb{G}_m \xrightarrow{(\overbrace{\lambda^{m_1},\ldots,\lambda^{m_1}}^{r_1},\ldots\ldots,\overbrace{\lambda^{m_t},\ldots,\lambda^{m_t}}^{r_t})} \mathrm{GL}(r_1)\times\cdots\times\mathrm{GL}(r_t).$$

From Theorem 7.7 we immediately obtain the following.

Proposition 7.14. *Let L be the group defined in the previous paragraph, and let $k=\gcd(m_1,\cdots,m_t)$. We have natural isomorphisms of group schemes*

$$\begin{aligned} \pi_1\,\mathrm{PGL}(m_1:r_1,m_2:r_2,\cdots,m_t:r_t) &\cong U_t \rtimes\cdots\rtimes U_2 \rtimes U_1 \rtimes L, \\ \pi_2\,\mathrm{PGL}(m_1:r_1,m_2:r_2,\cdots,m_t:r_t) &\cong \mu_k. \end{aligned}$$

Our final result is that, if all weights are distinct (that is, $r_i=1$), then the corresponding projective general linear 2-group is split.

Proposition 7.15. *Let* $\{m_1, \cdots, m_t\}$ *be distinct positive integers, and consider the projective general linear 2-group* $\mathrm{PGL}(m_1, m_2, \cdots, m_t)$. *Then, the projection map* $G \to \pi_1 \mathrm{PGL}(m_1, m_2, \cdots, m_t)$ *splits. In particular,* $\mathrm{PGL}(m_1, m_2, \cdots, m_t)$ *is split. That is, it is completely classified by its homotopy group schemes:*

$$\begin{aligned} \pi_1 \mathrm{PGL}(m_1, \cdots, m_t) &\cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes (\mathbb{G}_m)^{t-1}, \\ \pi_2 \mathrm{PGL}(m_1, \cdots, m_t) &\cong \mu_k. \end{aligned}$$

Proof. By Theorem 7.7 and Proposition 7.14 we know that $G \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes (\mathbb{G}_m)^t$ and $\pi_1 \mathrm{PGL}(m_1, m_2, \cdots, m_t) \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes L$, where L is the cokernel of the map

$$\alpha\colon \mathbb{G}_m \xrightarrow{(\lambda^{m_1}, \cdots, \lambda^{m_t})} (\mathbb{G}_m)^t.$$

So it is enough to show that the image of μ is a direct factor. Note that if we divide all the m_i by their greatest common divisor, the image of α does not change. So, we may assume $\gcd(m_1, \cdots, m_t) = 1$. Let M be a $t \times t$ integer matrix whose determinant is 1 and whose first column is $(m_1, \cdots, m_t)$. The matrix M gives rise to an isomorphism $\mu\colon (\mathbb{G}_m)^t \to (\mathbb{G}_m)^t$ whose restriction to the subgroup $\mathbb{G}_m \times \{1\}^{t-1} \cong \mathbb{G}_m$ is naturally identified with α. The subgroup $\mu(\{1\} \times (\mathbb{G}_m)^{t-1}) \subset (\mathbb{G}_m)^t$ is the desired complement of the image of α. $\square$

Corollary 7.16. *Let* m, n *be distinct positive integers, and let* $k = \gcd(m, n)$. *Then* $\mathrm{PGL}(m, n)$ *is a split 2-group. That is, it is classified by its homotopy groups:*

$$\begin{aligned} \pi_1 \mathrm{PGL}(m, n) &\cong \begin{cases} \mathbb{G}_m, & \text{if } m < n, m \nmid n \\ \mathbb{A} \rtimes \mathbb{G}_m, & \text{if } m < n, m \mid n \end{cases} \\ \pi_2 \mathrm{PGL}(m, n) &\cong \mu_k. \end{aligned}$$

(In the case $m \mid n$*, the action of* $\mathbb{G}_m$ *on* $\mathbb{A}$ *in the cross product* $\mathbb{A} \rtimes \mathbb{G}_m$ *is simply the multiplication action.)*

Proof. Everything is clear, except perhaps a clarification is in order regarding the parenthesized statement. Observe that the $\mathbb{G}_m$ appearing in the cross product $\mathbb{A} \rtimes \mathbb{G}_m$ is indeed the cokernel of the map

$$\alpha\colon \mathbb{G}_m \xrightarrow{(\lambda^m, \lambda^n)} (\mathbb{G}_m)^2,$$

which is naturally identified with the subgroup $\{1\} \times \mathbb{G}_m \subset (\mathbb{G}_m)^2$. Therefore, by the formula of Example 7.10, the action of an element $\lambda \in \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by λa. $\square$

Finally, for the sake of completeness, we include the following.

Proposition 7.17. *The 2-group* $\mathrm{PGL}(k, k, \cdots, k)$, k *appearing* t *times, is given by the following crossed-module:*

$$[\mathbb{G}_m \xrightarrow{(\lambda^k, \cdots, \lambda^k)} \mathrm{GL}(t)].$$

We have $\pi_1 \mathrm{PGL}(k, \cdots, k) \cong \mathrm{PGL}(t)$ *and* $\pi_2 \mathrm{PGL}(k, \cdots, k) \cong \mu_k$. *In particular,* $\mathrm{PGL}(1, 1, \cdots, 1)$, 1 *appearing* t *times, is equivalent to the group scheme* $\mathrm{PGL}(t)$.

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