MAPPING STACKS OF TOPOLOGICAL STACKS

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ABSTRACT. We prove that the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ of topological stacks $\mathcal{X}$ and $\mathcal{Y}$ is again a topological stack if $\mathcal{Y}$ admits a groupoid presentation $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are compact topological spaces. If Y_0 and Y_1 are only locally compact, we show that $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is a paratopological stack. In particular, it has a classifying space (hence, a natural weak homotopy type). We also show that the weak homotopy type of the mapping stack $\mathrm{Map}(Y,\mathcal{X})$ does not change if we replace $\mathcal{X}$ by its classifying space, provided that Y is a paracompact topological space. As an example, we describe the loop stack of the classifying stack $\mathcal{B}G$ of a topological group G in terms of twisted loop groups of G.

CONTENTS

1. INTRODUCTION

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological stacks. The purpose of these notes is to show that under a mild condition on $\mathcal{Y}$ the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is a paratopological stack and, in particular, admits a classifying space.

There are various classes of stacks to which this result applies. For example, for arbitrary $\mathcal{X}$, we can take $\mathcal{Y}$ to be coming from a: Lie groupoid, orbifold, action of a locally compact group on a locally compact space (e.g., classifying stack $\mathcal{B}G$ of locally compact group), complex-of-groups, Artin stack of finite type over complex numbers, foliation on a manifold, and so on.

In the case where $\mathcal{X}$ and $\mathcal{Y}$ are orbifolds, the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ has been studied by Chen. One of the main results of [Ch] is that in this case $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is again an orbifold. To our knowledge, this is the only general result previously known about the mapping stacks being topological, and its proof is quite nontrivial. Another known case is when $\mathcal{Y} = S^1$, in which case the free loop stack $\mathrm{Map}(S^1,\mathcal{X}) =: \mathcal{L}\mathcal{X}$ is shown to be a topological stack in [LuUr].

The mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ can be defined for arbitrary stacks $\mathcal{X}$ and $\mathcal{Y}$, and it is functorial in both $\mathcal{X}$ and $\mathcal{Y}$ (§ 3). However, it does not in general admit a groupoid presentation, even if $\mathcal{X}$ and $\mathcal{Y}$ do. Therefore, $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ may not always be a topological stack.

In the case where $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ coincides with the usual mapping space with the compact-open topology. If we are given groupoid presentations $\mathbb{X} = [X_1 \rightrightarrows X_0]$ and $\mathbb{Y} = [X_1 \rightrightarrows X_0]$ for $\mathcal{X}$ and $\mathcal{Y}$, the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ parameterizes the Hilsum-Skandalis morphisms from $\mathbb{Y}$ to $\mathbb{X}$. In the case $\mathcal{X} = \mathcal{B}G$, where G is an arbitrary topological group, the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{B}G)$ classifies principal G-bundles over $\mathcal{Y}$.

Our first main result shows that, under a locally compactness condition on $\mathcal{Y}$, the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ admits a classifying space (Definition 2.2), hence also a natural weak homotopy type; see Theorems 4.2 and 4.4.

Theorem 1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological stacks, and let* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *be their mapping stacks. If $\mathcal{Y}$ admits a presentation by a groupoid $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are compact topological spaces, then* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is again a topological stack. If Y_0 and Y_1 are only locally compact, then* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is paratopological (Definition 2.3).*

The theorem implies that the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ has a *classifying space* (Definition 2.2). In other words, there exists a topological space V and a map $\varphi\colon V \to \mathrm{Map}(\mathcal{Y},\mathcal{X})$ that is a *universal weak equivalence*, i.e., φ has the property that for every map $T \to \mathrm{Map}(\mathcal{Y},\mathcal{X})$ from a topological space T, the base extension $\varphi_T\colon V_T \to T$ of φ is a weak equivalence of topological spaces. (In fact, whenever T is paracompact, we can arrange so that φ_T admits a section and V_T has a fiberwise strong deformation retraction over the image of this section.) In the case where we have the compactness condition on $\mathcal{Y}$, we can arrange so that φ is also an epimorphism (thereby, giving rise to a presentation for $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ by the topological groupoid $[U \rightrightarrows V]$, where $U = V \times_{\mathrm{Map}(\mathcal{Y},\mathcal{X})} V$).

As is discussed in [No2] in detail, existence of such a classifying space $\varphi\colon X \to \mathcal{X}$ is crucial for doing algebraic topology on $\mathcal{X}$, as it allows one to translate problems about the stack $\mathcal{X}$ to ones about the space X (e.g., by pull-back along φ). For example, classifying spaces of stacks have been used extensively in [BGNX] to develop intersection theory on loop stacks of differentiable stacks. Another application appears in [EbGi] in which the same method is used to produce new classes in the singular homology (with coefficients) of moduli stacks $\bar{\mathfrak{M}}_{g,n}$ of curves.

As the terminology suggests, in the case where $\mathcal{X}$ is the quotient stack of a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$, the Haefliger classifying space of $\mathbb{X}$ is indeed a classifying space for $\mathcal{X}$ in the sense discussed above. Therefore, the weak homotopy type of $\mathcal{X}$ is the same as the weak homotopy type of the Haefliger classifying space. This raises the question of *homotopy invariance* of mapping stacks. For example, one could ask whether the loops stack $L\mathcal{X}$ has the same weak homotopy type as the loop space LX of the classifying space of $\mathcal{X}$. In §6 we give an affirmative answer to this question. More generally, we prove the following (see Corollary 6.5).

Theorem 1.2. *Let $\mathcal{X}$ be a topological stack, and let X be a classifying space for it. Let Y be a paracompact topological space. Then, there is a natural (universal) weak equivalence* $\mathrm{Map}(Y,X) \to \mathrm{Map}(Y,\mathcal{X})$. *That is,* $\mathrm{Map}(Y,X)$ *is a classifying space for* $\mathrm{Map}(Y,\mathcal{X})$.

As another application of Theorem 1.1, we see in §5.3 that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks factorizes as a composition $f = p_f \circ i_f$ of a closed embedding $i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$ which admits a strong deformation retraction followed by a Hurewicz fibration $p_f\colon \tilde{\mathcal{X}} \to \mathcal{Y}$; see Proposition 5.2. In particular, one can define the homotopy fiber of f as a topological stack.

Finally, we study the loop stack of the classifying stack $\mathcal{B}G$ of a topological group G. We prove the following.

Theorem 1.3. *Let G be a topological group (not necessarily connected). Then, there are natural weak homotopy equivalences*

$$LBG \cong L\mathcal{B}G \cong \coprod_{i\in C_G} BL_{(\alpha_i)}G.$$

Here, BG is the Milnor classifying space of G, C_G is the set of conjugacy classes of $\pi_0 G$, and $L_{(\alpha_i)}G$ are twisted loop groups. In particular, when G is discrete, there are natural weak homotopy equivalences

$$LBG \cong \mathcal{I}\mathcal{B}G \cong G \times_G EG,$$

where $\mathcal{I}\mathcal{B}G$ is the inertia stack and $G \times_G EG$ is the Borel construction on the conjugation action of G on itself. (The left equivalence is indeed an equivalence of stacks.)

There is also a similar description for the weak homotopy type of the pointed loop stack $\Omega\mathcal{B}G$; see Theorem 7.7.

Conventions. Throughout the notes, CGTop stands for the category of compactly generated topological spaces. All topological spaces will be assumed to be compactly generated. All stacks considered are over CGTop.

2. Topological and paratopological stacks

In this section, we recall some basic facts and definitions from [No1] and [No2].

By a **topological stack** we mean a stack $\mathcal{X}$ over CGTop which is equivalent to the quotient stack of a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ with X_1 and X_0 (compactly generated) topological spaces.

There are two classes of stacks that are of special interest to us in this paper: paratopological stacks (that are more general than topological stacks) and Hurewicz topological stacks (that are special types of topological stacks). We will discuss them shortly.

2.1. **Classifying space of a stack.** The following notion plays an important role in the homotopy theory of topological stacks.

Definition 2.1 ([No2], Definition 5.1)**.** A representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called a **universal weak equivalence** if the base extension $f_T\colon \mathcal{X}_T \to T$ of f to an arbitrary topological space T is a weak equivalence of topological spaces.

Definition 2.2. Let $\mathcal{X}$ be a stack whose diagonal $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is representable. By a **classifying space** for $\mathcal{X}$ we mean a topological space X equipped with a universal weak equivalence $\varphi\colon X \to \mathcal{X}$.

In [No2], a stack $\mathcal{X}$ which admits a classifying space is called a *homotopical stack*. The classifying space of a homotopical stack $\mathcal{X}$ is unique up to a unique isomorphism in the weak homotopy category of topological spaces. It is functorial (in the weak homotopy category) and is a model for the weak homotopy type of $\mathcal{X}$.

2.2. **Paratopological stacks.**

Definition 2.3 ([No2], Definition 9.1)**.** We say that a stack $\mathcal{X}$ is **paratopological** if it satisfies the following conditions:

- **A1**. Every map $T \to \mathcal{X}$ from a topological space T is representable (equivalently, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable);
- **A2**. There exists a morphism $X \to \mathcal{X}$ from a topological space X such that for every morphism $T \to \mathcal{X}$, with T a paracompact topological space, the base extension $T \times_{\mathcal{X}} X \to T$ is an epimorphism (i.e., admits local sections).

We denote the 2-categories of stacks, topological stacks, and paratopological stacks by $\mathfrak{St}$, $\mathfrak{TopSt}$, and $\mathfrak{ParSt}$, respectively. We have full inclusions of 2-categories

$$\mathfrak{TopSt} \subset \mathfrak{ParSt} \subset \mathfrak{St}.$$

The following lemma says that a paratopological stacks looks like a topological stack in the eye of a paracompact topological space.

Lemma 2.4 ([No2], Lemma 9.2)**.** *Let $\mathcal{X}$ be a stack such that the diagonal $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is representable. Then, $\mathcal{X}$ is paratopological if and only if there exists a topological stack $\bar{\mathcal{X}}$ and a representable morphism $p\colon \bar{\mathcal{X}} \to \mathcal{X}$ such that for every paracompact topological space T, p induces an equivalence of groupoids $\bar{\mathcal{X}}(T) \to \mathcal{X}(T)$.*

Proof. We just indicate how $\bar{\mathcal{X}}$ is constructed. Let $X \to \mathcal{X}$ be as in Definition 2.3 (**A2**). Set $X_0 := X$ and $X_1 := X \times_{\mathcal{X}} X$. The quotient stack $\bar{\mathcal{X}}$ of the groupoid $[X_1 \rightrightarrows X_0]$ has the desired property. □

Note that paracompactness of T does not play a role in the above proof; if in Definition 2.3 (**A2**) the space T is required to belong to a certain class of spaces (e.g., paracompact, CW complex, etc.), then Lemma 2.4 will be true with T in the same class of spaces.

Definition 2.5 ([No2], Definition 5.1)**.** A representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called **parashrinkable** if for every morphism $T \to \mathcal{Y}$ from a paracompact topological space T, the base extension $f_T\colon \mathcal{X}_T \to T$ of f over T admits a section s and a fiberwise deformation retraction of $\mathcal{X}_T$ onto $s(T)$.

Proposition 2.6 ([No2], Proposition 9.4)**.** *For every paratopological stack $\mathcal{X}$, there exists a parashrinkable morphism $\varphi\colon X \to \mathcal{X}$ from a topological space X. If $\mathcal{X}$ is a topological stack, then such a φ can be chosen to be an epimorphism.*

Observe that a parashrinkable morphism is a universal weak equivalence (Definition 2.1). Therefore, the space X in the above proposition is a classifying space for $\mathcal{X}$ in the sense of Definition 2.2. Also, note that every topological stack $\mathcal{X}$ is a paratopological stack, hence admits a classifying space by the above proposition. In fact, it is shown in ([No2], Theorem 6.1) that the Haefliger classifying space of a groupoid presentation for $\mathcal{X}$ is a classifying space for $\mathcal{X}$ in the sense of Definition 2.2. Let us record this fact for future reference.

Proposition 2.7 ([No2], Theorem 6.1)**.** *Let $\mathcal{X}$ be a topological stack and $\mathbb{X} = [X_1 \rightrightarrows X_0]$ a groupoid presentation for it. Let $B\mathbb{X}$ denote the Haefliger classifying space of $\mathbb{X}$. Then there is a natural parashrinkable morphism*

$$\varphi\colon B\mathbb{X} \to \mathcal{X}.$$

In particular, for every topological group G, there is a parashrinkable morphism

$$\varphi\colon BG \to \mathcal{B}G,$$

where BG is the Milnor classifying space of G.

2.3. **Hurewicz topological stacks.** In the 2-category of topological stacks colimits are quite ill-behaved. (For example, homotopies between maps with target $\mathcal{X}$ a topological stack do not always glue.) The situation is slightly better in the 2-category of *Hurewicz topological stacks* (e.g., see Proposition 5.1). This makes Hurewicz topological stacks more appropriate for certain homotopy theoretic manipulations.

To give the definition of a Hurewicz topological stack, we recall some standard definitions. A *Hurewicz fibration* is a continuous map of topological spaces which has the homotopy lifting property for all topological spaces. A map $f\colon X \to Y$ of topological spaces is a *local Hurewicz fibration* if for every $x \in X$ there are opens $x \in U$ and $f(x) \in V$ such that $f(U) \subseteq V$ and $f|_U\colon U \to V$ is a Hurewicz fibration. An important example is that of a topological submersion: a map $f : X \to Y$, such that locally U is homeomorphic to $V \times \mathbb{R}^n$, for some n.

Definition 2.8. A topological stack $\mathcal{X}$ is called **Hurewicz** if it is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $[X_1 \rightrightarrows X_0]$ whose source and target maps are local Hurewicz fibrations.

Hurewicz topological stacks form a full sub 2-category of $\mathfrak{TopSt}$.

2.4. **Limits of topological stacks.** We will need the following fact in the proof of Theorem 4.4.

Proposition 2.9. *The 2-categories $\mathfrak{TopSt}$, $\mathfrak{ParSt}$, and the 2-category of Hurewicz topological stacks are closed under finite limits. The 2-category of stacks with representable diagonal is closed under arbitrary limits. The 2-category $\mathfrak{ParSt}$ is closed under arbitrary fiber products. In particular, the product of an arbitrary family of paratopological stacks is paratopological.*

Proof. The statements about $\mathfrak{TopSt}$ and $\mathfrak{ParSt}$ are proved in [No2], Lemmas 9.12 and 9.13. The same argument used in [ibid.] proves that Hurewicz topological stacks are closed under finite limits. The statement about stacks with representable diagonal is ([No2], Lemmas 9.11). □

3. Generalities on mapping stacks

We begin by recalling the definition of the mapping stack. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks over CGTop. We define the stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$, called the **mapping stack** from $\mathcal{Y}$ to $\mathcal{X}$, by the rule

$$T \in \mathsf{CGTop} \quad \mapsto \quad \mathrm{Hom}(T \times \mathcal{Y}, \mathcal{X})\,,$$

where Hom denotes the groupoid of stack morphisms. This is easily seen to be a stack. Note that we have a natural equivalence of groupoids

$$\mathrm{Map}(\mathcal{Y},\mathcal{X})(*) \cong \mathrm{Hom}(\mathcal{Y},\mathcal{X}),$$

where $*$ is a point. In particular, the underlying set of the coarse moduli space of $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is the set of 2-isomorphism classes of morphisms from $\mathcal{Y}$ to $\mathcal{X}$.

It follows from the exponential law for mapping spaces that when X and Y are spaces, then $\mathrm{Map}(Y,X)$ is representable by the usual mapping space from Y to X (endowed with the compact-open topology).

The mapping stacks are functorial in both variables.

Lemma 3.1. *The mapping stacks* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *are functorial in* $\mathcal{X}$ *and* $\mathcal{Y}$. *That is, we have natural functors* $\mathrm{Map}(\mathcal{Y},-)\colon \mathfrak{St} \to \mathfrak{St}$ *and* $\mathrm{Map}(-,\mathcal{X})\colon \mathfrak{St}^{op} \to \mathfrak{St}$. *(Note: in* $\mathfrak{St}^{op}$ *we only invert the direction of 1-morphisms.)*

The exponential law holds for mapping stacks.

Lemma 3.2. *For stacks* $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z}$ *we have a natural equivalence of stacks*

$$\mathrm{Map}(\mathcal{Z}\times\mathcal{Y},\mathcal{X}) \cong \mathrm{Map}(\mathcal{Z},\mathrm{Map}(\mathcal{Y},\mathcal{X})).$$

4. Mapping stacks of topological stacks

In this section we show that, for a fairly large class of topological stacks $\mathcal{X}$ and $\mathcal{Y}$, the machinery developed in [No2] can be used to associate a classifying space (hence, a homotopy type) to the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$.

The first main result of this section (Theorem 4.2) shows that if $\mathcal{X}$ and $\mathcal{Y}$ are topological stacks, then $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is again a topological stack, provided that $\mathcal{Y}$ has a groupoid presentation $[Y_1 \rightrightarrows Y_0]$ in which both Y_1 and Y_0 are compact topological spaces. The compactness assumption is somewhat restrictive, although it is enough for many applications (e.g., loop stacks). In Theorem 4.4 we show that $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is a *paratopological* stack under the weaker assumption that Y_0 and Y_1 are only locally compact. By the results of [No2], this allows one to associate a classifying space to $\mathrm{Map}(\mathcal{Y},\mathcal{X})$.

Lemma 4.1. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be stacks. Assume that* $\mathcal{Y}$ *is topological and that the diagonal* $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ *is representable. Then, for every topological space* S, *every morphism* $S \to \mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is representable. (Equivalently,* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *has a representable diagonal.)*

Proof. First note that we can reduce to the case where $\mathcal{Y} = Y$ is a topological space. This is possible because by ([No1], Proposition 3.19) the mapping stack $\mathrm{Map}([Y_0/Y_1],\mathcal{X})$ can be written as the limit of a (finite) diagram produced out of $\mathrm{Map}(Y_0,\mathcal{X})$ and $\mathrm{Map}(Y_1,\mathcal{X})$; see Proposition 2.9.

Let $p\colon S \to \mathrm{Map}(Y,\mathcal{X})$ and $q\colon T \to \mathrm{Map}(Y,\mathcal{X})$ be arbitrary morphisms from topological spaces S and T. We need to show that $T \times_{\mathrm{Map}(Y,\mathcal{X})} S$ is a topological space. Let $\tilde{p}\colon S\times Y \to \mathcal{X}$ be the defining map for p, and $\tilde{q}\colon T\times Y \to \mathcal{X}$ the one for q. Set $Z := (T \times Y) \times_{\mathcal{X}} (S \times Y)$. This is a topological space which sits in the following 2-cartesian diagram:

$$\begin{array}{ccc} \mathrm{Map}(Y, Z) & \longrightarrow & \mathrm{Map}(Y, S \times Y) \\ \downarrow & & \downarrow \tilde{p}_* \\ \mathrm{Map}(Y, T \times Y) & \xrightarrow{\tilde{q}_*} & \mathrm{Map}(Y, \mathcal{X}) \end{array}$$

The claim now follows from the fact that p and q factor through $\tilde{p}_*$ and $\tilde{q}_*$, respectively. □

Theorem 4.2. *Let $\mathcal{X}$ and $\mathcal{K}$ be topological stacks. Assume that $\mathcal{K} \cong [K_0/K_1]$, where $[K_1 \rightrightarrows K_0]$ is a topological groupoid with K_0 and K_1 compact. Then,* $\mathrm{Map}(\mathcal{K}, \mathcal{X})$ *is a topological stack.*

Proof. First note that we can reduce to the case where $\mathcal{K} = K$ is a compact topological space. This is possible because by ([No1], Proposition 3.19) the mapping stack $\mathrm{Map}([K_0/K_1], \mathcal{X})$ can be written as the limit of a finite diagram produced out of $\mathrm{Map}(K_0, \mathcal{X})$ and $\mathrm{Map}(K_1, \mathcal{X})$; use Proposition 2.9.

In view of Lemma 4.1, all we need to do is to find an epimorphism $R \to \mathrm{Map}(K, \mathcal{X})$ from a topological space R. First some notation. Let $\mathbb{Y} = [Y_1 \rightrightarrows Y_0]$ and $\mathbb{X} = [X_1 \rightrightarrows X_0]$ be topological groupoids. We define $\mathrm{Hom}(\mathbb{Y}, \mathbb{X})$ to be the space of continuous groupoid morphisms from $\mathbb{Y}$ to $\mathbb{X}$. This is topologized as a subspace of $\mathrm{Map}(Y_1, X_1) \times \mathrm{Map}(Y_0, X_0)$, and it represents the (set-valued) functor

$$T \in \mathsf{Top} \quad \mapsto \quad \text{groupoid morphisms } \; T \times \mathbb{Y} \to \mathbb{X},$$

where $T \times \mathbb{Y}$ stands for the groupoid $[T \times Y_1 \rightrightarrows T \times Y_0]$. In particular, we have a universal family of groupoid morphisms $\mathrm{Hom}(\mathbb{Y}, \mathbb{X}) \times \mathbb{Y} \to \mathbb{X}$.

Let J be the set of all finite open covers of K. For $\alpha \in J$, let U_α denote the disjoint union of the open sets appearing in the open cover α. There is a natural map $U_\alpha \to K$. Let $\mathbb{K}_\alpha := [U_\alpha \times_K U_\alpha \rightrightarrows U_\alpha]$ be the corresponding topological groupoid. Note that the quotient stack of $\mathbb{K}_\alpha$ is K. Fix a groupoid presentation $\mathbb{X} = [X_1 \rightrightarrows X_0]$ for $\mathcal{X}$, and let $\pi \colon X_0 \to \mathcal{X}$ be the corresponding atlas for $\mathcal{X}$. Set $R_\alpha = \mathrm{Hom}(\mathbb{K}_\alpha, \mathbb{X})$, with Hom being as above. Let $R = \coprod_\alpha R_\alpha$.

The universal groupoid morphisms $R_\alpha \times \mathbb{K}_\alpha \to \mathbb{X}$ give rise to morphisms $R_\alpha \to \mathrm{Map}(K, \mathcal{X})$. Putting these all together we obtain a morphism $R \to \mathrm{Map}(K, \mathcal{X})$. We claim that this an epimorphism. (Here is where compactness of K gets used.) Let $p \colon T \to \mathrm{Map}(K, \mathcal{X})$ be an arbitrary morphism. We have to show that, for every $t \in T$, there exists an open neighborhood W of t such that $p|_W$ lifts to R. Let $\tilde{p} \colon T \times K \to \mathcal{X}$ be the defining morphism for p. Since $\pi : X_0 \to \mathcal{X}$ is an epimorphism, we can find finitely many open sets V_i of $T \times K$ which cover $\{t\} \times K$ and such that $\tilde{p}|_{V_i}$ lifts to X_0 for every i. We may assume $V_i = K_i \times W$, where K_i are open subsets of K, and W is an open neighborhood of t independent of i. Let $\alpha := \{K_i\}$ be the corresponding open cover of K. Then $p|_W$ lifts to $R_\alpha \subset R$. □

Remark 4.3. In the above proof we implicitly made use of the fact that the cartesian product of a compactly generated topological space with a compact topological space is again compactly generated ([Wh], I.4.14.). (Recall that product in CGTop is, in general, slightly different from the usual cartesian product in Top. This is due to the fact that the product of two compactly generated spaces may no longer be compactly generated. See ([Wh], I.4) for more details.)

We now treat the case of mapping stacks $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ where $\mathcal{Y}$ is no longer compact. We have the following result.

Theorem 4.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be paratopological stacks. Assume that $\mathcal{Y} \cong [Y_0/Y_1]$, where $[Y_1 \rightrightarrows Y_0]$ is a topological groupoid with Y_0 and Y_1 locally compact. Then,* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is a paratopological stack (Definition 2.3). In particular,* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *has a classifying space in the sense of Definition 2.2.*

Proof. We prove the theorem in several steps.

Step 1, $\mathcal{X}$ a paratopological stack, and $\mathcal{Y} = Y$ a compact topological space. If $\mathcal{X}$ is topological, then $\mathrm{Map}(Y,\mathcal{X})$ is topological by Theorem 4.2. For an arbitrary paratopological stack $\mathcal{X}$, pick a topological stack $\bar{\mathcal{X}}$ as in Lemma 2.4. For every paracompact space T, the product $T \times Y$ is paracompact. This implies that, $\mathrm{Map}(Y,\bar{\mathcal{X}})(T) \to \mathrm{Map}(Y,\mathcal{X})(T)$ is an equivalence of groupoids for every paracompact T. Furthermore, by Lemma 4.1, $\mathrm{Map}(Y,\mathcal{X})$ has representable diagonal. It follows from Lemma 2.4 that $\mathrm{Map}(Y,\mathcal{X})$ is a paratopological stack.

Step 2, $\mathcal{X}$ a paratopological stack, and Y a disjoint union of compact topological spaces. Write $Y = \coprod Y_i$, with Y_i compact. It is easy to see that $\mathrm{Map}(\coprod Y_i,\mathcal{X}) \cong \prod \mathrm{Map}(Y_i,\mathcal{X})$. Hence, by Proposition 2.9, $\mathrm{Map}(Y,\mathcal{X})$ is a paratopological stack.

Step 3, $\mathcal{X}$ a paratopological stack, and $\mathcal{Y} = [Y_0/Y_1]$ quotient stack of a topological groupoid $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are disjoint unions of compact topological spaces. As in the proof of Theorem 4.2, the mapping stack $\mathrm{Map}([Y_0/Y_1],\mathcal{X})$ can be written as the limit of a finite diagram produced out of $\mathrm{Map}(Y_0,\mathcal{X})$ and $\mathrm{Map}(Y_1,\mathcal{X})$. Now use Proposition 2.9 and Step (2).

Step 4, $\mathcal{X}$ a paratopological stack, and $\mathcal{Y} = Y$ a locally compact topological space. Choose a collection $\{Y_i\}$ of closed compact subsets of Y whose interiors covers $\mathcal{Y}$, and let Y_0 denote their disjoint union. Let $Y_1 := Y_0 \times_Y Y_0$ be the disjoint union of pairwise intersections of the compact sets Y_i. Then Y is the quotient stack of the groupoid $[Y_1 \rightrightarrows Y_0]$ and the claim follows from Step (3).

Step 5, $\mathcal{X}$ a paratopological stack, and $\mathcal{Y} = [Y_0/Y_1]$ quotient stack of a topological groupoid $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are locally compact topological spaces. Use the same argument as in Step (3). □

Corollary 4.5. *Let $\mathcal{Y}$ be a differentiable stack and $\mathcal{X}$ and arbitrary paratopological stack. Then,* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is a paratopological stack. In particular,* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *admits a classifying space.*

5. Application

In this section we present an application of our results by showing that any morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks has a factorization as a homotopy equivalence followed by a Hurewicz fibration. Along the way, we also point out certain subtleties that one should be aware of when working with mapping stacks. This explains why in certain contexts it is preferable to work with Hurewicz topological stacks as opposed to arbitrary topological stacks.

5.1. **A gluing lemma for mapping stacks.** One main subtlety of working with mapping stacks is that certain intuitive statements about them may not hold in full generality. The following proposition is proved in ([BGNX], Proposition 2.2). The proof is easy and it relies on Proposition 1.3 of [ibid.] (see [No1], Theorem 16.2 for an earlier version of this proposition). We emphasize that the statement may not be true if we do not assume that $\mathcal{X}$ is a *Hurewicz* topological stack.

Proposition 5.1. *Let $j\colon A \to Y$ be a closed embedding of Hausdorff spaces. Assume that j is a local cofibration in the sense that every $x \in A$ has a neighborhood U in Y such that $j|_{A\cap U}\colon A\cap U \to U$ is a Hurewicz cofibration. Let $A \to Z$ be a finite proper map of Hausdorff spaces. Let $\mathcal{X}$ be a Hurewicz topological stack. Then the diagram*

$$\begin{array}{ccc} \mathrm{Map}(Z\vee_A Y,\mathcal{X}) & \longrightarrow & \mathrm{Map}(Y,\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(Z,\mathcal{X}) & \longrightarrow & \mathrm{Map}(A,\mathcal{X}) \end{array}$$

is a 2-cartesian diagram of stacks.

5.2. **Path and loop stacks.** It is immediate from Theorem 4.2 that the **path stack** $P\mathcal{X} = \mathrm{Map}([0,1],\mathcal{X})$ of a topological (resp., paratopological) stack $\mathcal{X}$ is again a topological (resp., paratopological) stack. There is a natural *constant path* morphism $c_{\mathcal{X}}\colon \mathcal{X} \to P\mathcal{X}$. For for every $t \in [0,1]$, there is a natural evaluation map $\mathrm{ev}_t\colon P\mathcal{X} \to \mathcal{X}$, and a natural 2-morphism $\alpha_t\colon \mathrm{ev}_t \circ c_{\mathcal{X}} \Rightarrow \mathrm{id}_{\mathcal{X}}$.

Similarly, we define the **free loop stack** of $\mathcal{X}$ to be $L\mathcal{X} := \mathrm{Map}(S^1,\mathcal{X})$. If $\mathcal{X}$ is topological (resp., paratopological), then so is $L\mathcal{X}$. There is, however, another way of defining the free loop stack. Namely, we can define $L\mathcal{X}$ to be

$$\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},(\mathrm{ev}_0,\mathrm{ev}_1)} P\mathcal{X}.$$

In contrast to the case of topological spaces, these definitions are not expected to be equivalent in general. However, if we assume that $\mathcal{X}$ is a Hurewicz topological stack, then all "reasonable" definitions are equivalent. This is true thanks to Proposition 5.1.

Assume now that $x \in \mathcal{X}$ is a basepoint. The **based loop stack** $\Omega_x\mathcal{X}$ of $\mathcal{X}$ is defined by

$$\Omega_x\mathcal{X} := * \times_{x,\mathcal{X},\mathrm{ev}} L\mathcal{X},$$

where $\mathrm{ev}\colon L\mathcal{X} \to \mathcal{X}$ is evaluation at the basepoint of S^1. Again, as in the previous paragraph, if we do not assume that $\mathcal{X}$ is Hurewicz, there may be more than one way of defining $\Omega_x\mathcal{X}$, and these definitions may not be equivalent. But if $\mathcal{X}$ is Hurewicz, all reasonable definitions will be equivalent.

5.3. **Homotopy fiber of a morphism of stacks.** In classical topology there is a standard procedure for replacing an arbitrary continuous map $f\colon X \to Y$ of topological spaces by a fibration. This construction involves taking path spaces and fiber products, both of which are available to us in the 2-category of (para)topological stacks, thanks to Theorems 4.2, 4.4 and Proposition 2.9.

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Set $\tilde{\mathcal{X}} := \mathcal{X} \times_{f,\mathcal{Y},\mathrm{ev}_1} P\mathcal{Y}$, where $\mathrm{ev}_1\colon P\mathcal{Y} \to \mathcal{Y}$ is the time $t = 1$ evaluation map. Note that if $\mathcal{X}$ and $\mathcal{Y}$ are (para)topological stacks, then so is $\tilde{\mathcal{X}}$. We define $p_f\colon \tilde{\mathcal{X}} \to \mathcal{Y}$ to be the composition $\mathrm{ev}_0 \circ \mathrm{pr}_2$, and

$i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$ to be the map whose first and second components are $\mathrm{id}_{\mathcal{X}}$ and $c_{\mathcal{Y}} \circ f$, respectively.

The proof of the following proposition, as well as a thorough discussion of the notion of fibration of stacks, will appear elsewhere.

Proposition 5.2. *Notation being as above, we have a factorization $f = p_f \circ i_f$ such that:*

- **i.** *the map $p_f\colon \tilde{\mathcal{X}} \to \mathcal{Y}$ is a Hurewicz fibration;*
- **ii.** *the map $i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$ is a closed embedding and it admits a strong deformation retraction $r_f\colon \tilde{\mathcal{X}} \to \mathcal{X}$.*

We remark that the above proposition is quite formal and is valid for all stacks. What is interesting, and that is where we make use of the results of this papers, is that in the case where $\mathcal{X}$ and $\mathcal{Y}$ are (para)topological stacks, $\tilde{\mathcal{X}}$ is again a (para)topological stack. This is important because it means that when working with (para)topological stacks the fibration replacement of morphisms keep us in the 2-category of (para)topological stacks.

We define the **homotopy fiber** over a point $y \in \mathcal{Y}$ of a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of (para)topological stacks to be

$$\mathrm{hFib}_y(f) := * \times_{y,\mathcal{Y},p_f} \tilde{\mathcal{X}},$$

where p_f is as in Proposition 5.2. Since the 2-category of (para)topological stacks is closed under fiber products, the homotopy fiber of a morphisms of (para)topological stacks is again a (para)topological stack.

6. Homotopy invariance of mapping stacks

In this section we study homotopy invariance of the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ under change of $\mathcal{X}$. Although we can not make a statement in full generality, our result (Theorem 6.4) applies to many important classes of examples (e.g., loop stacks). A useful consequence of our invariance theorem is that to calculate the homotopy type of the mapping stack $\mathrm{Map}(Y,\mathcal{X})$, where Y is a paracompact topological space and $\mathcal{X}$ is a paratopological stack, one can replace $\mathcal{X}$ with its classifying space X; see Corollary 6.5 below for the precise statement.

6.1. Weakly parashrinkable morphisms.

There is a class of representable morphisms of stacks which shares most of the nice properties of parashrinkable morphisms but is particularly better-behaved. Although not strictly necessary, we deemed appropriate to formulate our invariance theorem in terms of these morphisms.

Definition 6.1. We say that a representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is **weakly parashrinkable** if for every paracompact topological space T, and every morphism $p\colon T \to \mathcal{Y}$, the space $\mathrm{Map}_p(T,\mathcal{X})$ of lifts of p to $\mathcal{X}$ is weakly contractible and non-empty.

By definition, $\mathrm{Map}_p(T,\mathcal{X})$ is the fiber of the morphism of stacks $f_*\colon \mathrm{Map}(T,\mathcal{X}) \to \mathrm{Map}(T,\mathcal{Y})$ over the point in $\mathrm{Map}(T,\mathcal{Y})$ corresponding to p. Note that $\mathrm{Map}_p(T,\mathcal{X})$ is naturally equivalent to the space of sections of the projection map $T \times_{\mathcal{Y}} \mathcal{X} \to T$. In particular, $\mathrm{Map}_p(T,\mathcal{X})$ is an honest topological space.

Lemma 6.2. *The following statements are true about weakly parashrinkable morphisms:*

1. *Every parashrinkable morphism is weakly parashrinkable.*
2. *Every weakly parashrinkable morphism is a universal weak equivalence.*
3. *Base extension of a weakly parashrinkable morphism is weakly parashrinkable.*

Proof. Proof of (**1**) is easy because the space of sections of the shrinkable morphism $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is clearly contractible. Parts (**2**) and (**3**) are straightforward. $\square$

Remark 6.3. The above discussion can be repeated for *weakly shrinkable* morphisms and *weakly pseudoshrinkable* morphisms ([No2], Definition 5.1) as well. Since we do not need these concepts in this paper, we will avoid further discussion.

6.2. **Invariance theorem.**

Theorem 6.4. *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a representable morphism of stacks. Assume that f is weakly parashrinkable (Definition 6.1). Let Y be a paracompact topological space. Then $f_*\colon \mathrm{Map}(Y,\mathcal{X}') \to \mathrm{Map}(Y,\mathcal{X})$ is a universal weak equivalence. If Y is compact, then f_* is weakly parashrinkable.*

Proof. Let T be a finite CW complex (or any compact topological space). We show that for every morphism $p\colon T \to \mathrm{Map}(Y,\mathcal{X})$, the space $\mathrm{Map}_p(T, \mathrm{Map}(Y,\mathcal{X}'))$ of lifts of p to $\mathrm{Map}(Y,\mathcal{X}')$ is weakly contractible. It is easy to see that this implies that $f_*\colon \mathrm{Map}(Y,\mathcal{X}') \to \mathrm{Map}(Y,\mathcal{X})$ is a universal weak equivalence.

To show that $\mathrm{Map}_p(T, \mathrm{Map}(Y,\mathcal{X}'))$ is weakly contractible, observe that we have a natural homeomorphism

$$\mathrm{Map}_p(T, \mathrm{Map}(Y,\mathcal{X}')) \cong \mathrm{Map}_{\tilde{p}}(T \times Y, \mathcal{X}'),$$

where $\tilde{p}\colon T \times Y \to \mathcal{X}$ is the map corresponding to p under the exponential law (Lemma 3.2). Since $T \times Y$, being the product a compact space and a paracompact space, is paracompact, and since $\mathcal{X}' \to \mathcal{X}$ is weakly parashrinkable, the right hand side of the equation is contractible.

In the case where Y is compact, one can repeat the same argument as above, with T an arbitrary paracompact topological space. One finds that $f_*\colon \mathrm{Map}(Y,\mathcal{X}') \to \mathrm{Map}(Y,\mathcal{X})$ is weakly parashrinkable. $\square$

Corollary 6.5. *Let Y be a paracompact topological space and $\mathcal{X}$ a paratopological stack. Let X be a classifying space for $\mathcal{X}$ with $\varphi\colon X \to \mathcal{X}$ weakly parashrinkable (such an X always exists, see Proposition 2.6). Then,* $\mathrm{Map}(Y,X)$ *is a classifying space for* $\mathrm{Map}(Y,\mathcal{X})$*, hence it represents the weak homotopy type of* $\mathrm{Map}(Y,\mathcal{X})$.

Corollary 6.6. *Let $\mathcal{X}$ be a paratopological stack and X a classifying space for it, with $\varphi\colon X \to \mathcal{X}$ weakly parashrinkable (such an X always exists, see Proposition 2.6). Then, the free loop space LX of X is a classifying space for the free loop space $L\mathcal{X}$ of $\mathcal{X}$ (hence, has the same weak homotopy type). In fact, we can arrange for the map $LX \to L\mathcal{X}$ to be weakly parashrinkable (hence, a universal weak equivalence).*

Of course, there is nothing special about S^1 in the above corollary. One can take any compact topological space instead of S^1.

Corollary 6.7. *Composition of two weakly parashrinkable morphisms is weakly parashrinkable.*

Proof. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be weakly parashrinkable morphisms. Let T be a paracompact topological space and $p\colon T \to \mathcal{Z}$ a morphism. We want to show that $\mathrm{Map}_p(T, \mathcal{X})$ is weakly contractible. We have a cartesian square

$$\begin{array}{ccc} \mathrm{Map}_p(T,\mathcal{X}) & \longrightarrow & \mathrm{Map}(T,\mathcal{X}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Map}_p(T,\mathcal{Y}) & \longrightarrow & \mathrm{Map}(T,\mathcal{Y}) \end{array}$$

By Theorem 6.4, the right vertical arrow is a universal weak equivalence. Hence, so is the left vertical arrow. Since g is weakly parashrinkable, $\mathrm{Map}_p(T, \mathcal{Y})$ is weakly contractible. Therefore, $\mathrm{Map}_p(T, \mathcal{X})$ is weakly contractible. □

The above corollary reveals a main advantage of weakly parashrinkable morphisms over parashrinkable morphisms: they are closed under composition. The same thing is true with weakly shrinkable morphisms and weakly pseudoshrinkable morphisms as well (see Remark 6.3).

Remark 6.8. Homotopy invariance of $\mathrm{Map}(\mathcal{Y}, \mathcal{X})$ with respect to $\mathcal{Y}$ is not an interesting problem. To illustrate this, consider the case where $\mathcal{Y} = \mathcal{B}G$ is the classifying stack of a group G, and $\mathcal{Y}' = |BG|$ is the Milnor classifying space of G. Let $\mathcal{X} = X$ be an arbitrary topological space. There is a natural parashrinkable morphism $BG \to \mathcal{B}G$. The induced map

$$\mathrm{Map}(\mathcal{B}G, X) \to \mathrm{Map}(BG, X)$$

is in general far from being a weak equivalence. The left hand side is nothing but $\mathrm{Map}(*, X) = X$, while the right hand side could be considerably more complicated.

7. An example: loop stack of $\mathcal{B}G$ and twisted loop groups

In this section, we describe the loop stack $L\mathcal{B}G$ of the classifying stack of a topological group G. We show that the homotopy type of $L\mathcal{B}G$ can be calculated in terms of the twisted loop groups of G. In the case where G is discrete, $L\mathcal{B}G$ is equivalent to the inertia stack of $\mathcal{B}G$. See Theorem 7.6.

In what follows, all G-torsors are right G-torsors.

Lemma 7.1. *Let G be a topological group and BG its Milnor classifying space. Let T be a paracompact topological space. Then, every G-torsor on $[0,1] \times T$ is isomorphic to the pull-back of a G-torsor on T.*

Proof. It is well-known that isomorphism classes of G-torsors on a paracompact space T are in natural bijection with homotopy classes of maps $T \to BG$. Apply this fact to T and $[0,1] \times T$, both of which are paracompact, and use the fact that the projection $[0,1] \times T \to T$ is a homotopy equivalence. □

Lemma 7.2. *Let U be a topological space and $f\colon U \to G$ a continuous map. Let T_f be the quotient space of $G \times U \times [0,1]$ obtained by gluing $G \times U \times \{0\}$ to $G \times U \times \{1\}$ along the map $(g, u, 0) \mapsto \big(f(u)g, u, 1\big)$. Then, T_f is naturally a (right) G-torsor over $U \times S^1$. Furthermore, if $f'\colon U \to G$ is another continuous map, then $T_{f'}$ is isomorphic to T_f as a G-torsor if and only if there is a function $\delta\colon U \to G$ such that f' is homotopic to the conjugate $\delta f \delta^{-1}$ of f under δ.*

Proof. A straightforward argument shows that a morphism $T_f \to T_{f'}$ of G-torsors corresponds precisely to a map $\gamma\colon U\times[0,1]\to G$ such that

$$f'(u)\gamma(u,0)=\gamma(u,1)f(u).$$

If such a γ exists, then $h(u,t):=f'(u)\gamma(u,t)\gamma(u,1)^{-1}$ gives a homotopy from $\delta f\delta^{-1}$ to f', where $\delta(u):=\gamma(u,1)$. Conversely, given a homotopy $h(u,t)$ from $\delta f\delta^{-1}$ to f', for some $\delta\colon U\to G$, we define $\gamma(u,t):=f'(u)^{-1}h(u,t)\delta(u)$. □

Let $\pi_0(G)$ denote the group of path components of G, and C_G the set of conjugacy classes of elements of $\pi_0(G)$. Choose a set of representative $\alpha_i\in G$, $i\in C_G$, of conjugacy classes of path components of G. Let T_α be the G-torsor over S^1 obtained from the trivial torsor $G\times[0,1]$ on $[0,1]$ by gluing $G\times\{0\}$ to $G\times\{1\}$ along the map $(g,0)\mapsto(\alpha g,1)$, as in Lemma 7.2. The automorphism group of T_α is isomorphic, as a topological group, to the **twisted loop group** $L_{(\alpha)}G$ ([PrSe], §3.7) associated to the conjugation action $g\mapsto\alpha g\alpha^{-1}$ of α on G. In other words,

$$L_{(\alpha)}G:=\{\gamma\colon\mathbb{R}\to G\mid\gamma(\theta+1)=\alpha\gamma(\theta)\alpha^{-1}\}.$$

(Note that our setting is slightly more general than that of [ibid.] because we are not assuming that G is connected.) In the case where α belongs to the path component of the identity, $L_{(\alpha)}G$ is isomorphic to the loop group $LG=\mathrm{Map}(S^1,G)$. When G is discrete, $L_{(\alpha)}G$ is the centralizer of α.

For every α as above, let $p_\alpha\colon *\to L\mathcal{B}G$ denote the map corresponding to the G-torsor T_α. We have the following.

Lemma 7.3. *Let $p_\alpha\colon *\to L\mathcal{B}G$ be defined as above. Then, we have a natural isomorphism of topological groups*

$$*\times_{p_\alpha,L\mathcal{B}G,p_\alpha}*\cong L_{(\alpha)}G.$$

If $\alpha,\beta\in G$ map to different elements in C_G, then

$$*\times_{p_\alpha,L\mathcal{B}G,p_\beta}*\cong\emptyset.$$

Proof. The first statement follows from fact that the automorphism group of the G-torsor T_α is isomorphic to $L_{(\alpha)}G$ as a topological group. The second statement follows from the fact that, T_α and T_β are isomorphic as G-torsors if and only if the images of α and β in $\pi_0(G)$ belong to the same conjugacy class (see Lemma 7.2). □

Lemma 7.4. *The map $P=\coprod_{i\in C_G}p_{\alpha_i}\colon\coprod_{i\in C_G}*\to L\mathcal{B}G$, where $*$ stands for a point, has the property (**A2**) of Definition 2.3 for every locally contractible paracompact topological space W (in particular, for every finite CW complex W).*

Proof. Pick a map $h\colon W\to L\mathcal{B}G$, and let $T\to W\times S^1$ be the corresponding G-torsor. We want to show that every point $x\in W$ has an open neighborhood U such that $h|_U$ lifts along some $p_{\alpha_i}\colon *\to L\mathcal{B}G$, that is, $T|_{U\times S^1}$ is isomorphic to the G-torsor $U\times T_{\alpha_i}$, for some $i\in C_G$. We claim that any contractible neighborhood U of x has the desired property. By Lemma 7.1, we know that T is of the form T_f for some $f\colon W\to G$ (see Lemma 7.2 for notation). Since U is contractible, $f|_U$ is homotopic to a constant map into some path component of G. Hence, $f|_U$ is homotopic to a conjugate of the constant map $\alpha_i\colon U\to G$, for some α_i. Therefore, by Lemma 7.2, $T|_U=T_{f|_U}$ is isomorphic to $U\times T_{\alpha_i}$ as a G-torsor. In other words, $f|_U$ lifts along $p_{\alpha_i}\colon *\to L\mathcal{B}G$. □

Corollary 7.5. *The map P of Lemma 7.4 induces a natural map*

$$\bar{P}\colon \coprod_{i\in C_G} \mathcal{B}L_{(\alpha_i)}G \to L\mathcal{B}G$$

which has the property of Lemma 2.4 for every locally contractible paracompact topological space W (in particular, for every finite CW complex W). In particular, $\bar{P}$ is a universal weak equivalence.

Proof. For the existence of the map $\bar{P}$ and that it has the property of Lemma 2.4 see the proof of Lemma 2.4 (and the remark after the proof). Observe that we are also using Lemma 7.3. □

Finally, we put together what we have proved about the loop stack of $\mathcal{B}G$ in the following.

Theorem 7.6. *Let G be a topological group. Then, there are natural weak homotopy equivalences*

$$LBG \cong L\mathcal{B}G \cong \coprod_{i\in C_G} BL_{(\alpha_i)}G.$$

In particular, when G is discrete, there are natural weak homotopy equivalences

$$LBG \cong \mathcal{I}\mathcal{B}G \cong G \times_G EG,$$

where $\mathcal{I}\mathcal{B}G$ is the inertia stack and $G \times_G EG$ is the Borel construction applied to the conjugation action of G on itself. (The equivalence $LBG \cong \mathcal{I}\mathcal{B}G$ is indeed an equivalence of stacks. This can be easily verified directly.)

We can describe the weak homotopy type of the based loop stack $\Omega\mathcal{B}G$ in a similar fashion. (Observe that $\mathcal{B}G$ is a Hurewicz topological stack, so by the discussion of § 5.2 the different definitions for the based loop stack agree.) Consider the map $\mathrm{ev}\colon L\mathcal{B}G \to \mathcal{B}G$ which evaluates a loop at its basepoint. The based loop stack $\Omega\mathcal{B}G$ is the fiber of ev over the base point of $\mathcal{B}G$ corresponding to the trivial G-torsor. Note that the composite map

$$\mathrm{ev}\circ\bar{P}\colon \coprod_{i\in C_G} \mathcal{B}L_{(\alpha_i)}G \to \mathcal{B}G$$

is the one induced by the evaluation maps $\mathrm{ev}_0\colon L_{(\alpha_i)}G \to G$, $\gamma \mapsto \gamma(0)$. Therefore, the fiber of $\mathrm{ev}\circ\bar{P}$ over the basepoint of $\mathcal{B}G$ is

$$\coprod_{i\in C_G} \mathcal{B}\Omega_{(\alpha_i)}G,$$

where $\Omega_{(\alpha)}G$ is the subgroup of the twisted loop group $L_{(\alpha)}G$ defined by

$$\Omega_{(\alpha)}G := \{\gamma\colon \mathbb{R} \to G \mid \gamma(\theta+1) = \alpha\gamma(\theta)\alpha^{-1},\ \gamma(0) = 1_G\}.$$

Note that $\Omega_{(\alpha)}G$ only sees the path component of the identity. In other words, $\Omega_{(\alpha)}G = \Omega_{(\alpha)}(G^0)$. In particular, if G is discrete, then $\Omega_{(\alpha)}G$ is trivial.

We have the following 2-cartesian diagram

$$\begin{array}{ccc}
\coprod_{i\in C_G} \mathcal{B}\Omega_{(\alpha_i)}G & \longrightarrow & \coprod_{i\in C_G} \mathcal{B}L_{(\alpha_i)}G \\
{\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle \bar{P}} \\
\Omega\mathcal{B}G & \longrightarrow & L\mathcal{B}G
\end{array}$$

The morphism Q, being the base extension of $\bar{P}$ inherits the same property that $\bar{P}$ does in Corollary 7.5. In particular, Q is a universal weak equivalence. We have the following.

Theorem 7.7. *Let G be a topological group. Then, there are natural weak homotopy equivalences*

$$\Omega BG \cong \Omega \mathcal{B}G \cong \coprod_{i \in C_G} B\Omega_{(\alpha_i)}G.$$

In particular, when G is discrete, $\Omega \mathcal{B}G$ is weakly homotopy equivalent to the discrete set C_G. (The latter is indeed an equivalence of stacks as can be easily verified directly.)

Proof. Everything follows immediately from the discussion above (thanks to Proposition 2.7), except for the weak equivalence $\Omega BG \cong \Omega \mathcal{B}G$. To prove this, consider the 2-commutative diagram

$$\begin{array}{ccc} LBG & \xrightarrow{\varphi_*} & L\mathcal{B}G \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ BG & \xrightarrow[\varphi]{} & \mathcal{B}G \end{array}$$

Let $V \subset BG$ be the fiber of $\varphi \colon BG \to \mathcal{B}G$ over the base point of $\mathcal{B}G$ and $W = \mathrm{ev}^{-1}(V)$ the fiber of $\varphi \circ \mathrm{ev} \colon LBG \to \mathcal{B}G$. Observe that V is contractible (because φ is parashrinkable), and that $\mathrm{ev} \colon LBG \to BG$ is a fibration. Therefore, the inclusion $j \colon \Omega BG \hookrightarrow W$ is a weak homotopy equivalence. On the other hand, by the 2-commutativity of the above diagram, W fits in the 2-cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & \Omega\mathcal{B}G \\ \downarrow & & \downarrow \\ LBG & \xrightarrow[\varphi_*]{} & L\mathcal{B}G \end{array}$$

By Corollary 6.6 and Proposition 2.7, f is a weak equivalence. Therefore, the composition $f \circ j \colon \Omega BG \to \Omega \mathcal{B}G$ is also a weak equivalence. This completes the proof of the theorem. □

One can use Theorem 7.7 and apply the loop space/stack functor repeatedly to find a description of the week homotopy types of the mapping stacks $\mathrm{Map}(S^n, \mathcal{B}G)$ and also the mapping spaces $\mathrm{Map}(S^n, BG)$ in terms of certain twisted mapping groups of G. We will leave it to the interested reader to work out the details.

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