HOMOTOPY TYPES OF TOPOLOGICAL STACKS

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Abstract. We define the notion of *classifying space* of a topological stack and show that every topological stack $\mathcal{X}$ has a classifying space X which is a topological space well-defined up to weak homotopy equivalence. Under a certain paracompactness condition on $\mathcal{X}$, we show that X is actually well-defined up to homotopy equivalence. These results are formulated in terms of functors from the category of topological stacks to the (weak) homotopy category of topological spaces. We prove similar results for (small) diagrams of topological stacks.

1. Introduction

The category of topological stacks accommodates various classes of objects simultaneously: 1- orbifolds, 2- gerbes, 3- spaces with an action of a topological group, 4- Artin stacks, 5- Lie groupoids, 6- complexes-of-groups, 7- foliated manifolds. And, of course, the category of topological spaces is a full subcategory of the category of topological stacks.

In each of the above cases, tools from algebraic topology have been adopted, to various extents, to study the objects in question. For instance, (3) is the subject of equivariant algebraic topology. Behrend has studied singular (co)homology of Lie groupoids in [Be]. Homotopy invariants and (co)homology theories for orbifolds have been developed by Haefliger, Moerdijk, Thurston,.... The case of complexes-of-groups has been studied extensively by Bass, Bridson, Haefliger, Serre, Soulé,...

The notion of *classifying space* introduced in this paper (Definition 6.2) provides a unified way to do algebraic topology on topological stacks. By definition, a classifying space for a topological stack $\mathcal{X}$ is a topological space $\Theta(\mathcal{X})$ togeotehr with a morphism $\varphi\colon \Theta(\mathcal{X}) \to \mathcal{X}$ which is a universal weak equivalence, in the sense that, for every map $T \to \mathcal{X}$ from a topological space T, the base extension $\varphi_T\colon \Theta(\mathcal{X}) \times_{\mathcal{X}} T \to T$ of φ is a weak equivalence of topological spaces.

The crucial feature of the definition of the classifying space is the existence of the map $\varphi\colon \Theta(\mathcal{X}) \to \mathcal{X}$. This map provides a link between the algebraic topology of $\mathcal{X}$ and that of the topological space $\Theta(\mathcal{X})$ (say, by pull-back and push-forward along φ). For an application of this, the reader can consult [BGNX] where this result is used to develop an intersection theory and a theory of Thom classes on stacks; for another application see [EbGi].

One of the main results of this paper is the following.

Theorem 1.1. *Every topological stack $\mathcal{X}$ admits an atlas $\varphi\colon \Theta(\mathcal{X}) \to \mathcal{X}$ which is a classifying space for $\mathcal{X}$.*

The classifying space $\Theta(\mathcal{X})$ turns out to be unique up to weak homotopy equivalenvce. In fact, we have a functor $\Theta\colon \mathfrak{TopSt} \to \mathsf{Top}_{w.e.}$ to the weak homotopy category $\mathsf{Top}_{w.e.}$ of topological spaces which to a topological stack $\mathcal{X}$ associates its *weak homotopy type* $\Theta(\mathcal{X})$; see Theorem 8.2. When $\mathcal{X}$ is the quotient stack of a topological groupoid $\mathbb{X}$, $\Theta(\mathcal{X})$ is the fat geometric realization of the simplical space associated to $\mathbb{X}$. In particular, the above theorem implies that the weak homotopy type of this fat realization is invariant under Morita equivalence of topological groupoids.

In the case where $\mathcal{X}$ has a groupoid presentation with certain paracompactness properties (§8.2), the above functor can be lifted to a functor $\Theta\colon \mathfrak{TopSt} \to \mathsf{Top}_{h.e.}$ which associates to such a topological stack an actual *homotopy type* (Theorem 8.8). This applies to all differentiable stacks and, more generally, to any stack that admits a presentation by a metrizable groupoid; see Proposition 8.5. This result is useful when defining (co)homology theories that are only homotopy invariant. (For example, certain sheaf cohomology theories or certain Čech type theories are only invariant under homotopy equivalences.)

The next main result in the paper is the generalization of Theorem 1.1 to diagrams of topological stacks (§12). The following theorem is a corollary of Theorem 12.1.

Theorem 1.2. *Let $P\colon \mathsf{D} \to \mathfrak{TopSt}$ be a diagram of topological stacks indexed by a small category D. Then, there is a diagram $Q\colon \mathsf{D} \to \mathsf{Top}$ of topological spaces, together with a transformation $\varphi\colon P \Rightarrow Q$, such that for every $d \in \mathsf{D}$ the morphism $Q(d) \to P(d)$ is a universal weak equivalence. Furthermore, Q is unique up to (objectwise) weak equivalence of diagrams.*

This theorem implies that every diagram of topological stacks has a natural weak homotopy type as a diagram of topological spaces. Furthermore, the transformation φ relates the given diagram of stacks with its weak homotopy type, thus allowing one to transport homotopical information back and forth between the diagram and its homotopy type.

The above theorem has various applications. For example, it implies an equivariant version of Theorem 1.1 for the (weak) action of a discrete group. It also allows one to define homotopy types of pairs (triples, and so on) of topological stacks. It is also useful in studying Bredon type homotopy theories for topological stacks.

We also consider the case where arrows in D are labeled by properties of continuous maps, such as: subspace, open (or closed) subspace, proper, finite, and so on. We show (§12.1) that, under certain conditions, if morphisms in a diagram P of topological stacks have the properties assigned by the corresponding labels, then we can arrange so that the morphisms in the homotopy type Q of P also satisfy the same properties. For example, if we take D to be $\{1 \to 2\}$ and label the unique arrow of D by 'closed subspace', this implies that the weak homotopy type of a 'closed pair' $(\mathcal{X}, \mathcal{A})$ of topological stack can be chosen to be a 'closed pair' (X, A) of topological spaces. Furthermore, we have a weak equivalence of pairs $\varphi\colon (X, A) \to (\mathcal{X}, \mathcal{A})$ relating the pair $(\mathcal{X}, \mathcal{A})$ to its weak homotopy type (X, A).

The above results are valid for *arbitrary* topological stacks $\mathcal{X}$. That is, all we require is for $\mathcal{X}$ to have a presentation by a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$. Although this may sound general enough for applications, there appears to be a major class of stacks which does not fall in this category: the mapping stacks $\mathrm{Map}(\mathcal{Y}, \mathcal{X})$ of topological stacks.

Nevertheless, in [No2] we prove that the mapping stacks are not far from being topological stacks. Let us quote a result from [ibid.].

Theorem 1.3. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be topological stacks, and let* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *be their mapping stack. If* $\mathcal{Y}$ *admits a presentation* $[Y_1 \rightrightarrows Y_0]$ *in which* Y_1 *and* Y_0 *are compact, then* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is a topological stack. If* Y_1 *and* Y_0 *are only locally compact, then* $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ *is a paratopological stack.*

Paratopological stacks (Definition 9.1) form an important 2-category of stacks which contains $\mathfrak{TopSt}$ as a full sub 2-category. The advantage of the 2-category of paratopological stacks over the 2-category $\mathfrak{TopSt}$ of topological stacks is that it is closed under arbitrary 2-limits (but we will not prove this here). We show in §9 that our machinery of homotopy theory of topological stacks extends to the category of paratopological stacks. In particular, all mapping stacks $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ have a well-defined (functorial) weak homotopy type, as long as $\mathcal{Y}$ satisfies the locally compactness condition mentioned above.

We believe the category of paratopological stacks is a suitable category for doing homotopy theory in. Some other candidates are also discussed in §9.

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Contents

2. Notation and conventions

Throughout the notes, Top stands for the category of all topological spaces. The localization of Top with respect to the class of weak equivalences is denoted by $\mathsf{Top}_{w.e.}$ (the *weak homotopy category* of spaces). The localization of Top with respect to the class of homotopy equivalences is denoted by $\mathsf{Top}_{h.e.}$ (the *homotopy category* of spaces).

All stacks considered in the paper are over Top.

We will denote groupoids by $\mathbb{X} = [s, t\colon X_1 \rightrightarrows X_0]$. For convenience, we drop s and t from the notation. We usually reserve the letters s and t for the source and target maps of groupoids, unless it is clear from the context that they stand for something else.

Our terminology differs slightly from that of [No1]. A *topological stack* means a stack $\mathcal{X}$ that is equivalent to the quotient stack of a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$; in [ibid.] these are called *pretopological stacks*.

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called an *epimorphism* if it is an epimorphism in the sheaf theoretic sense (i.e., for every topological space W, every object in $\mathcal{Y}(W)$ has a preimage in $\mathcal{X}(W)$, possibly after passing to an open cover of W). In the case where $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, this is equivalent to saying that f admits local sections.

For simplicity, we assume that all 2-categories have invertible 2-morphisms. The category obtained by identifying 2-isomorphic morphisms in a 2-category $\mathfrak{C}$ is denoted by $[\mathfrak{C}]$. We usually use Fraktur symbols for 2-categories and Sans Serif symbols for 1-categories.

3. Torsors for groupoids

We quickly recall the definition of a torsor for a groupoid; see for instance [No1], §12.

Definition 3.1. Let $\mathbb{X} = [R \rightrightarrows X]$ be a topological groupoid, and let W be a topological space. By an $\mathbb{X}$-**torsor** over W we mean a map $p\colon T \to W$ of topological spaces which admits local sections, together with a cartesian morphism of groupoids

$$[T \times_W T \rightrightarrows T] \to [R \rightrightarrows X].$$

By a *trivialization* of an $\mathbb{X}$-torsor $p\colon T \to W$ we mean an open covering $\{U_i\}$ of W, together with a collection of sections $\sigma_i\colon U_i \to T$. To give an $\mathbb{X}$-torsor and a trivialization for it is the same thing as giving a 1-cocycle on W with values in $\mathbb{X}$, as defined below.

Given an open cover $\{U_i\}$ of W of a topological space W, an $\mathbb{X}$-*valued 1-cocycle* on W relative the cover $\{U_i\}$ consists of a collection of continuous maps $a_i\colon U_i \to X$, and a collection of continuous maps $\gamma_{ij}\colon U_i \cap U_j \to R$, such that:

C1. for every i, j, $\quad s \circ \gamma_{ij} = a_i|_{U_i \cap U_j}$ and $t \circ \gamma_{ij} = a_j|_{U_i \cap U_j}$;
C2. for every i, j, k, $\quad \gamma_{ij}\gamma_{jk} = \gamma_{ik}$ as maps from $U_i \cap U_j \cap U_k$ to R.

Equivalently, a 1-cocyle on $\{U_i\}$ is a groupoid morphism $c\colon \mathbb{U} \to \mathbb{X}$, where $\mathbb{U} := [\coprod U_i \cap Uj \rightrightarrows \coprod U_i]$ is the groupoid associated to the covering $\{U_i\}$.

A morphism from a 1-cocycle $\big(\{U_i\}, a_i, \gamma_{ij}\big)$ to a 1-cocycle $\big(\{U'_k\}, a'_k, \gamma'_{kl}\big)$ is a collection of maps $\delta_{ik}\colon U_i \cap U'_k \to R$ such that:

M1. for every i, k, $\quad s \circ \delta_{ik} = a_i|_{U_i \cap U'_k}$ and $t \circ \delta_{ik} = a'_k|_{U_i \cap U'_k}$;

M2. for every i, k, l,

$$\delta_{ik}\gamma'_{kl} = \delta_{il} \colon U_i \cap U'_k \cap U'_l \to R,$$
$$\gamma_{ij}\delta_{ik} = \delta_{jk} \colon U_i \cap U_j \cap U'_k \to R.$$

Equivalently, a morphism from the 1-cocycle $c \colon \mathbb{U} \to \mathbb{X}$ to the 1-cocycle $c' \colon \mathbb{U}' \to \mathbb{X}$ is a morphism of groupoids $\mathbb{U} \coprod \mathbb{U}' \to \mathbb{X}$ whose restrictions to $\mathbb{U}$ and $\mathbb{U}'$ are equal to c and c', respectively. Here, $\mathbb{U} \coprod \mathbb{U}'$ is defined to be the groupoid associated to the covering $\{U_i, U_k\}_{i,k}$ (repetition allowed).

Lemma 3.2. *Let $\mathbb{X}$ be a topological groupoid and W a topological space. Then, 1-cocycles over W and morphisms between them form a groupoid that is naturally equivalent to the groupoid of $\mathbb{X}$-torsors over W. This groupoid is also naturally equivalent to the groupoid $\mathrm{Hom}_{\mathbf{St}}(W, [X/R])$ of stack morphisms from W to the quotient stack $[X/R]$.*

Proof. The last statement can be found in ([No1], §12). We only point out that the torsor $p \colon T \to W$ associated to a morphism $\varphi \colon W \to [X/R]$ is defined via the following 2-cartesian diagram

$$\begin{array}{ccc} T & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ W & \xrightarrow[\varphi]{} & [X/R] \end{array}$$

We explain how to associate an $\mathbb{X}$-torsor to a 1-cocycle $\big(\{U_i\}, a_i, \gamma_{ij}\big)$. Set $T_i := U_i \times_{a_i, X, s} R$. Define T to be $\coprod T_i/_{\sim}$, where $\sim$ is the following equivalence relation: $(w_i, \alpha_i) \sim (w_j, \alpha_j)$, if $w_i = w_j =: w$ and $\alpha_i = \gamma_{ij}(w)\alpha_j$.

The cartesian groupoid morphism $[T \times_W T \rightrightarrows T] \to [R \rightrightarrows X]$ is defined as follows. The map $T \to X$ is defined by $(w_i, \alpha_i) \mapsto t(\alpha_i)$. An element $\big((w_i, \alpha_i), (w_i, \beta_i)\big)$ in $T \times_W T$ is mapped to $\alpha_i^{-1}\beta_i \in R$; this is easily seen to be well-defined (i.e., independent of i).

The rest of the proof is straightforward and is left to the reader. □

Remark 3.3. Our definition of a 1-cocycle is different from that of ([Hae], §2) in that in *loc. cit.* the maps a_i are not part of the data. There is, however, a forgetful map that associates to a 1-cocycle in our sense a cocycle in the sense of Haefliger.

4. Classifying space of a topological groupoid

In this section, we discuss Haefliger's definition of the classifying space of a topological groupoid.

4.1. **Construction of the classifying space of a topological groupoid.** We recall from [Hae] the definition of the (Haefliger-Milnor) **classifying space** $B\mathbb{X}$ and the **universal bundle** $E\mathbb{X}$ of a topological groupoid $\mathbb{X} = [R \rightrightarrows X]$. Our main objective is to show that $\mathbb{E} \to \mathbb{X}$ is an $\mathbb{X}$-torsor, thereby giving rise to a morphism $\varphi \colon B\mathbb{X} \to \mathcal{X}$, where $\mathcal{X}$ is the quotient stack of $\mathbb{X}$. We will see in §6 that $B\mathbb{X}$ is a classifying space for $\mathcal{X}$ in the sense of Definition 6.2.

An element in $E\mathbb{X}$ is a sequence $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$, where $\alpha_i \in R$ are such that $s(\alpha_i)$ are equal to each other, and $t_i \in [0,1]$ are such that all but finitely many of them are zero and $\sum t_i = 1$. As the notation suggests, we set $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) = (t'_0\alpha'_0, t'_1\alpha'_1, \cdots, t'_n\alpha'_n, \cdots)$ if $t_i = t'_i$ for all i and $\alpha_i = \alpha'_i$ if $t_i \neq 0$.

Let $t_i \colon E\mathbb{X} \to [0,1]$ denote the map $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) \mapsto t_i$, and let $\alpha_i \colon t_i^{-1}(0,1] \to R$ denote the map $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) \mapsto \alpha_i$. The topology on $E\mathbb{X}$ is the weakest topology in which $t_i^{-1}(0,1]$ are all open and t_i and α_i are all continuous.

The classifying space $B\mathbb{X}$ is defined to be the quotient of $E\mathbb{X}$ under the following equivalence relation. We say two elements $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$ and $(t'_0\alpha'_0, t'_1\alpha'_1, \cdots, t'_n\alpha'_n, \cdots)$ of $E\mathbb{X}$ are equivalent, if $t_i = t'_i$ for all i, and if there is an element $\gamma \in R$ such that $\alpha_i = \gamma\alpha'_i$. (So, in particular, $t(\alpha_i) = t(\alpha'_i)$ for all i.) Let $p \colon E\mathbb{X} \to B\mathbb{X}$ be the projection map.

The projections $t_i \colon E\mathbb{X} \to [0,1]$ are compatible with this equivalence relation, so they induce continuous maps $u_i \colon B\mathbb{X} \to [0,1]$ such that $u_i \circ p = t_i$. Let $U_i = u_i^{-1}(0,1]$.

Lemma 4.1. *The projection map $p \colon E\mathbb{X} \to B\mathbb{X}$ can be naturally made into a $\mathbb{X}$-torsor.*

Proof. First we show that p admits local sections. Consider the open cover $\{U_i\}$ of $B\mathbb{X}$ defined above. We define a section $U_i \to E\mathbb{X}$ for p by sending the equivalence class of $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$ to $(t_0\alpha_i^{-1}\alpha_0, t_1\alpha_i^{-1}\alpha_1, \cdots, t_i\alpha_i^{-1}\alpha_n, \cdots)$.

Let us now define a cartesian groupoid morphism

$$F \colon [E\mathbb{X} \times_{B\mathbb{X}} E\mathbb{X} \rightrightarrows E\mathbb{X}] \to [R \rightrightarrows X].$$

The effect on the object space is given by the map $f \colon E\mathbb{X} \to X$ which sends $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$ to $s(\alpha_i)$; this is independent of i (by definition). An element in $E\mathbb{X} \times_{B\mathbb{X}} E\mathbb{X}$ is represented by a pair

$$\big((t_0\gamma\alpha_0, t_1\gamma\alpha_1, \cdots, t_n\gamma\alpha_n, \cdots), (t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)\big),$$

for a unique $\gamma \in R$. We send this element to $\gamma \in R$. This is easily verified to be a cartesian morphism of groupoids. □

It follows from Lemma 3.2 that we have a morphism natural $\varphi \colon B\mathbb{X} \to \mathcal{X}$ and the the $\mathbb{X}$-torsor $p \colon E\mathbb{X} \to B\mathbb{X}$ fits in a 2-cartesian diagram

$$\begin{array}{ccc} E\mathbb{X} & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ B\mathbb{X} & \xrightarrow[\varphi]{} & \mathcal{X} \end{array}$$

4.2. **Comparison with the simplicial construction.** There is another way of associating a classifying space to a groupoid $\mathbb{X} = [R \rightrightarrows X]$, namely, by taking the geometric realization of (the simplicial space associated to) it. In this subsection, we compare this construction with Haefliger's and explain why we prefer Haefliger's construction.

First, let us recall the construction of the geometric realization of $\mathbb{X}$. Consider the simplicial space $N\mathbb{X}$ with

$$(N\mathbb{X})_0 = X, \quad \text{and} \quad (N\mathbb{X})_n = \underbrace{R \times_X \times \cdots \times_X \times R}_{n-\text{fold}}, \ n \geq 1.$$

The geometric realization of $\mathbb{X}$ is, by definition, the geometric realization of $N\mathbb{X}$. We will denote it by $|\mathbb{X}|$.

Alternatively, $|\mathbb{X}|$ can be obtained as a quotient space of $E\mathbb{X}$ by declaring that "it is allowed to take common factors in $E\mathbb{X}$." This means that, if an element $\alpha \in R$ appears several times in the sequence $\mathfrak{s} = (t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) \in E\mathbb{X}$, say at indices $i_1, \cdots, i_k$, then we regard $\mathfrak{s}$ as equivalent to any sequence $\mathfrak{s}' \in E\mathbb{X}$ which is obtained from $\mathfrak{s}$ by altering the coefficients $t_{i_1}, \cdots, t_{i_k}$ in a way that $t_{i_1} + \cdots + t_{i_k}$ remains fixed. (Roughly speaking, we are collapsing the subsequence $(t_{i_1}\alpha, \cdots, t_{i_k}\alpha)$ of $\mathfrak{s}$ to a single element $(t_{i_1} + \cdots + t_{i_k})\alpha$.)

It can be shown that there is a universal bundle $|\mathbb{E}|$ over $|\mathbb{X}|$ that is *almost* an $\mathbb{X}$-torsor. We explain how it is defined.

Consider the topological groupoid $\mathbb{E} := [R \times_{s,X,s} R \rightrightarrows R]$. There is a groupoid morphism $p\colon \mathbb{E} \to \mathbb{X}$ induced by the target map $t\colon R \to X$. This induces a map $|p|\colon |\mathbb{E}| \to |\mathbb{X}|$ on the geometric realizations. This will be the structure map of our (almost) torsor. Let us explain how the cartesian morphism

$$[|\mathbb{E}| \times_{|\mathbb{X}|} |\mathbb{E}| \rightrightarrows |\mathbb{E}|] \to [R \rightrightarrows X]$$

is constructed. Viewing X and R as trivial groupoids $[X \rightrightarrows X]$ and $[R \rightrightarrows R]$, we have the following strictly cartesian diagram in the category of topological groupoids:

$$\begin{array}{ccc} \mathbb{E} \times_{\mathbb{X}} \mathbb{E} & \xrightarrow{\lambda} & R \\ {\scriptstyle \mathrm{pr}_1 \text{ or } \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle s \text{ or } t} \\ \mathbb{E} & \xrightarrow[\sigma]{} & X \end{array}$$

In this diagram, the map $\sigma\colon [R \times_{s,X,s} R \rightrightarrows R] \to [X \rightrightarrows X]$ is the one induced by the source map $s\colon R \to X$. The fiber product $\mathbb{E} \times_{\mathbb{X}} \mathbb{E}$ is the strict fiber product of groupoids, and the maps $\mathbb{E} \to \mathbb{X}$ appearing in this fiber product are both p. The morphism λ in the diagram is defined as follows. An object in the groupoid $\mathbb{E} \times_{\mathbb{X}} \mathbb{E}$ is a pair of arrows (γ_1, γ_2) with the same target. Under λ, this will get sent to $\gamma_1\gamma_2^{-1} \in R$. The effect of λ on arrows is now uniquely determined.

The above diagram is indeed a (cartesian) morphism of groupoid objects in the category of topological groupoids. After passing to geometric realizations at all four corners, and noting that taking geometric realizations commutes with fiber products, the above diagram gives rise to the desired cartesian morphism of topological groupoids $\Psi\colon [|\mathbb{E}| \times_{|\mathbb{X}|} |\mathbb{E}| \rightrightarrows |\mathbb{E}|] \to [R \rightrightarrows X]$.

This almost proves that $|p|\colon |\mathbb{E}| \to |\mathbb{X}|$ is an $\mathbb{X}$-torsor. The only thing that is left to check is the existence of local sections. This, however, may not be true in general, unless the source and target maps of the original groupoid $\mathbb{X}$, and also its identity section, are nicely behaved (locally). This prevents $p\colon |\mathbb{E}| \to |\mathbb{X}|$ from being an $\mathbb{X}$-torsor. As a consequence, we do *not* get a morphism $|\mathbb{X}| \to [X/R]$. This explains why we opted for $B\mathbb{X}$ rather than $|\mathbb{X}|$ as a model for the classifying space of $\mathbb{X}$.

Remark 4.2. The above discussion can be summarized by saying that there are quotient maps $q' \colon E\mathbb{X} \to |\mathbb{E}|$ and $q \colon B\mathbb{X} \to |\mathbb{X}|$ inducing a commutative diagram

$$\begin{array}{ccc} [E\mathbb{X} \times_{B\mathbb{X}} E\mathbb{X} \rightrightarrows E\mathbb{X}] & \xrightarrow{Q} & [|\mathbb{E}| \times_{|\mathbb{X}|} |\mathbb{E}| \rightrightarrows |\mathbb{E}|] \\ & \searrow^{\Phi} & \downarrow \Psi \\ & & [R \rightrightarrows X] \end{array}$$

of cartesian groupoid morphisms. The morphism Φ *does* make $E\mathbb{X} \to B\mathbb{X}$ into an $\mathbb{X}$-torsor. In contrast, the morphism Ψ *does not* always make $|\mathbb{E}| \to |\mathbb{X}|$ into an $\mathbb{X}$-torsor. Therefore, the dotted arrow in the following diagram of the corresponding quotient stacks

$$\begin{array}{ccc} B\mathbb{X} & \xrightarrow{q} & |\mathbb{X}| \\ & \searrow^{\varphi} \; \cong & \vdots \\ & & [X/R] \end{array}$$

may not always be filled.

4.3. **The case of a group action.** Let G be a topological group acting continuously on a topological space X (on the right). Recall that $\mathcal{X} = [X/G]$ is the quotient stack of the topological groupoid $\mathbb{X} = [X \times G \rightrightarrows X]$. In this case, $B\mathbb{X}$ is equal to the Borel construction, that is $B\mathbb{X} = X \times_G EG$. The torsor $E\mathbb{X}$ constructed in §4 is equal to $X \times EG$, and $p \colon E\mathbb{X} \to B\mathbb{X}$ is Milnor's universal G-bundle. The cartesian square appearing after the proof of Lemma 4.1 now takes the following form

$$\begin{array}{ccc} X \times EG & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ X \times_G EG & \xrightarrow[\varphi]{} & [X/G] \end{array}$$

5. Shrinkable morphisms

We begin with an important definition.

Definition 5.1. We say that a continuous map $f \colon X \to Y$ of topological spaces is **shrinkable** ([Do], §1.5), if it admits a section $s \colon Y \to X$ such that there is a fiberwise strong deformation retraction of X onto $s(Y)$. We say that f is **locally shrinkable**, if there is an open cover $\{U_i\}$ of Y such that $f|_{U_i} \colon f^{-1}(U_i) \to U_i$ is shrinkable for all i. We say that f is **parashrinkable**, if for every map $T \to Y$ from a paracompact topological space T, the base extension $f_T \colon T \times_Y X \to T$ is shrinkable. If this condition is only satisfied for T a CW complex, we say that f is **pseudoshrinkable**. We say that f is a **universal weak equivalence**, if for every map $T \to Y$ from a topological space Y, the base extension $f_T \colon T \times_Y X \to T$ of f is a weak equivalence.

Definition 5.2. We say that a representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks is locally shrinkable (respectively, parashrinkable, pseudoshrinkable, a universal weak equivalence) if for every map $T \to \mathcal{Y}$ from a topological space Y, the base extension $f_T\colon T \times_Y \mathcal{X} \to T$ of f is so.

Remark 5.3. The above notions do not distinguish 2-isomorphic morphisms of stacks, so they pass to $[\mathfrak{TopSt}]$.

The following lemma clarifies the relation between the above notions.

Lemma 5.4. *The properties introduced in Definition 5.2 are related in the following way:*

$$\begin{array}{ccccccc}
 & & \textit{triv. Hurewicz fib.} & \Rightarrow & \textit{triv. Serre fib.} & & \\
 & \swarrow & & & & \searrow & \\
\textit{shrinkable} & \Rightarrow & \textit{locally shrinkable} & \Rightarrow & \textit{parashrinkable} & \Rightarrow & \textit{pseudoshrinkable} \\
 & & \Downarrow & & & & \Downarrow \\
 & & \textit{epimorphism} & & & & \textit{universal weak eq.}
\end{array}$$

Proof. All the implications are obvious except for the one in the middle and the one on the top-left. The one on the top-left follows from [Do], Corollary 6.2. To prove the middle implication, we have to show that every locally shrinkable $f\colon X \to Y$ with Y paracompact is shrinkable. Dold ([Do], §2.1) proves that if $f\colon X \to Y$ is a continuous map which becomes shrinkable after passing to a numerable ([Do], §2.1) cover $\{U_i\}_{i\in I}$ of Y, then f is shrinkable. Since in our case Y is paracompact, every open cover of Y admits a numerable refinement. So f is shrinkable by Dold's result. □

Lemma 5.5. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a parashrinkable (respectively, pseudoshrinkable) morphism of topological stacks. Let B be a paracompact topological space (respectively, a CW complex). Then, for every morphism $g\colon B \to \mathcal{Y}$, the space of lifts g to $\mathcal{X}$ is non-empty and contractible. In particular, every morphism $g\colon B \to \mathcal{Y}$ has a lift $\tilde{g}\colon B \to \mathcal{X}$ and such a lift is unique up to homotopy.*

Proof. Recall that a lift of g to $\mathcal{X}$ means a map $\tilde{g}\colon B \to \mathcal{X}$ together with a 2-isomorphism $\varepsilon\colon f \circ \tilde{g} \Rightarrow g$.

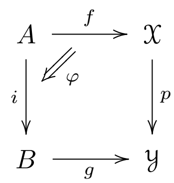
The space of lifts of g is homeomorphic to the space of sections of the map shrinkable map $f_B\colon B \times_{\mathcal{X}} \mathcal{Y} \to B$, hence is contractible. □

The converse of Lemma 5.5 is also true and can be used as an alternative way of defining parashrinkable (respectively, pseudoshrinkable) morphisms.

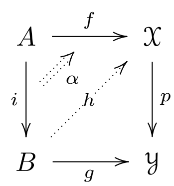
Lemma 5.6. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of topological stacks. Let $f'\colon \mathcal{X}' \to \mathcal{Y}'$ be the base extension of f along g. If f is locally shrinkable (respectively, parashrinkable, pseudoshrinkable, a universal weak equivalence), then so is f'. If g is an epimorphism, and f' is locally shrinkable, then f is also locally shrinkable.*

Proof. Obvious. □

Lemma 5.7. *Consider the 2-commutative square*



in which $p\colon \mathcal{X} \to \mathcal{Y}$ *is a parashrinkable (respectively, a pseaudoshrinkable) morphism of topological stacks and* $i\colon A \hookrightarrow B$ *is a closed Hurewicz cofibration of paracompact topological spaces (respectively, CW complexes). Then, one can find* h *and* α *such that in the diagram*



the upper triangle is 2-commutative and the lower triangle commutes up to a homotopy that leaves A *fixed (i.e., there is a homotopy from* g *to* $p \circ h$ *which after precomposing with* i *becomes 2-isomorphic to the constant homotopy).*

Proof. We can replace p by its base extension $p_B\colon B \times_{\mathcal{X}} \mathcal{Y} \to B$. So, we are reduced to the following situation: we have shrinkable map $p\colon X \to B$ of topological spaces, a subspace $A \subset B$ which is a Hurewicz cofibration, and a section $s\colon A \to X$. We want to extend s to a map $\sigma\colon B \to X$ such that $p \circ \sigma$ is homotopic to the identity map $\mathrm{id}_B\colon B \to B$ via a homotopy that fixes A pointwise.

Since p is shrinkable, it has a section $S\colon B \to X$. Furthermore, $S|_A$ and s are fiberwise homotopic via a homotopy $H\colon A \times [0,1] \to X$. (Fiberwise means that $p \circ H\colon A \times [0,1] \to A$ is equal to the first projection map.) Set

$$C = (A \times [0,1]) \cup_{A \times \{0\}} B.$$

The maps H and S glue to give a map $k\colon C \to X$. The composition $p \circ k\colon C \to B$ is the collapse map that fixes B pointwise and collapses $A \times [0,1]$ onto A via projection.

Since $A \subset B$ is a closed Hurewicz cofibration, the inclusion $C \subset B \times [0,1]$ admits a retraction $r\colon B \times [0,1] \to C$ ([Wh], §I.5.2). Let $r_1\colon B \to C$ be the restriction of r to $B \times \{1\}$. Set $\sigma := k \circ r_1$. It is easy to see that σ has the desired property. □

Remark 5.8. If in the above lemma we switch the roles of the upper and the lower triangles, namely, if we require that the upper triangle is homotopy commutative and the lower one is 2-commutative, then the lemma is true without the cofibration assumptions on i.

Corollary 5.9. *Let* $f\colon \mathcal{X} \to \mathcal{Y}$ *be a pseudoshrinkable morphism of Serre topological stacks (*[No1], *§17; also see Definition 10.1). Then,* f *is a weak equivalence, i.e., it induces isomorphisms on all homotopy groups (as defined in* [No1], *§17).*

Proof. To prove the surjectivity of $\pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{Y}, f(x))$, apply Lemma 5.7 to the case where B is S^n and A is its base point. The injectivity follows by considering $B = \mathbb{D}^{n+1}$ and $A = \partial\mathbb{D}^{n+1}$. □

Lemma 5.10. *Consider a family $f_i\colon \mathcal{X}_i \to \mathcal{Y}$, $i \in I$, of representable morphisms of topological stacks, and let $f\colon \prod_{\mathcal{Y}} \mathcal{X}_i \to \mathcal{Y}$ be their fiber product. (Note that f is well-defined up to a 2-isomorphism.) If all f_i are parashrinkable (respectively, pseudoshrinkable), then so is f. If all f_i are locally shrinkable (respectively, universal weak equivalence), then so is f, provided I is finite.*

Proof. In the parashrinkable case, it is enough to assume that $\mathcal{Y} = Y$ is a paracompact topological space. The result now follows from the fact that an arbitrary fiber product of shrinkable morphisms $X_i \to Y$, $i \in I$, of topological spaces is shrinkable. The case of pseudoshrinkable morphisms is proved analogously.

The case of locally shrinkable maps is also similar, except that for each $i \in I$ we may have a different open cover of Y which makes f_i shrinkable. Since I is finite, choosing a common refinement will make f shrinkable.

The case of universal weak equivalence maps is easily proved by induction (using the fact that the composition of two universal weak equivalences is again a universal weak equivalence). □

6. Classifying space of a topological stack

In this section we prove our first main result (Theorem 6.3). We begin with an important proposition.

Proposition 6.1. *Let $\mathcal{X}$ be a topological stack, and let $\mathbb{X} = [R \rightrightarrows X]$ be a presentation for it. Then, there is a natural map $\varphi\colon B\mathbb{X} \to \mathcal{X}$ which fits in the following 2-cartesian diagram*

$$\begin{array}{ccc} E\mathbb{X} & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ B\mathbb{X} & \xrightarrow[\varphi]{} & \mathcal{X} \end{array}$$

Here, f is the map defined in the proof of Lemma 4.1. Furthermore, f is shrinkable. In particular, φ is locally shrinkable.

Proof. The $\mathbb{X}$-torsor $p\colon E\mathbb{X} \to B\mathbb{X}$ defined in Lemma 4.1 furnishes the map φ and the 2-cartesian diagram; see Lemma 3.2.

Let us show that f is shrinkable. Define the section $\sigma\colon X \to E\mathbb{X}$ by

$$x \mapsto (1\,\mathrm{id}_x, 0\,\mathrm{id}_x, \cdots, 0\,\mathrm{id}_x, \cdots).$$

This identifies X with a closed subspace of $E\mathbb{X}$. We define the desired strong deformation retraction $\Psi\colon [0,2] \times E\mathbb{X} \to E\mathbb{X}$ by juxtaposing the maps $\Psi_1\colon [0,1] \times E\mathbb{X} \to E\mathbb{X}$ and $\Psi_2\colon [1,2] \times E\mathbb{X} \to E\mathbb{X}$ which are defined as follows:

$$\Psi_1\colon \big(t, (t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)\big) \mapsto \big((t_0 - tt_0)\alpha_0, (t_1 - tt_1 + t)\alpha_1, \cdots, (t_n - tt_n)\alpha_n, \cdots\big),$$

and

$$\Psi_2\colon \big(t, (t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)\big) \mapsto \big((t-1)\,\mathrm{id}_x, (2-t)\alpha_1, 0\,\mathrm{id}_x, \cdots, 0\,\mathrm{id}_x, \cdots\big).$$

Here, x is the common source of the α_i. That φ is locally shrinkable follows from Lemma 5.6. □

Definition 6.2. Let $\mathcal{X}$ be a topological stack. By a **classifying space** for $\mathcal{X}$ we mean a topological space X and a map $\varphi\colon X \to \mathcal{X}$ such that φ is a universal weak equivalence (Definition 5.2). By abuse of noation, we often drop the φ from the notation and call X a classifying space for $\mathcal{X}$.

Let us rephrase the above theorem as a statement about existence of classifying spaces for topological stacks.

Theorem 6.3 (**Existence of a classifying space**). *Every topological stack $\mathcal{X}$ admits an atlas $\varphi\colon X \to \mathcal{X}$ which is locally shrinkable. In particular, (X, φ) is a classifying space for $\mathcal{X}$.*

Proof. Choose an arbitrary presentation $\mathbb{X}$ for $\mathcal{X}$. Then, the morphism $\varphi\colon B\mathbb{X} \to \mathcal{X}$ of Proposition 6.1 is the desired atlas; see Lemma 5.4. □

Corollary 6.4. *Every topological groupoid $[R \rightrightarrows X]$ is Morita equivalent (*[No1], *§8) to a topological groupoid $[R' \rightrightarrows X']$ in which the source and target maps are locally shrinkable (in particular, they are universal weak equivalences).*

Proof. The desired groupoid is $[B\mathbb{X} \times_{\varphi, \mathcal{X}, \varphi} B\mathbb{X} \rightrightarrows B\mathbb{X}]$, where $\mathbb{X} = [R \rightrightarrows X]$. □

Remark 6.5. We will see in §8 that the atlas $\varphi\colon X \to \mathcal{X}$ of Theorem 6.3 can be used to define a weak homotopy type for the stack $\mathcal{X}$. We saw in Remark 4.2 that the diagram

$$\begin{array}{ccc} B\mathbb{X} & \xrightarrow{q} & |\mathbb{X}| \\ & \searrow^{\varphi} & \vdots \\ & & [X/R] \end{array}$$

can not always be filled. It is, however, true in many cases that the map q is a weak-equivalence. In such cases, the geometric realization $|\mathbb{X}|$ of a groupoid presentation $\mathbb{X}$ for $\mathcal{X}$ can very well be used to define the weak homotopy type of $\mathbb{X}$. The question that remains to be answered is, under what condition is the map $q\colon B\mathbb{X} \to |\mathbb{X}|$ a weak equivalence?

7. Some category theoretic lemmas

In this section, we prove a few technical results of category theoretic nature. These results will be needed in our functorial description of the classifying space of a topological stack (§8). The reader who is not interested in category theoretical technicalities can proceed to the next section.

Throughout this section, the set up will be as follows. Let $\mathfrak{C}$ be a 2-category with fiber products. Assume all 2-morphisms in $\mathfrak{C}$ are invertible. Let $[\mathfrak{C}]$ be the category obtained by identifying 2-isomorphic 1-morphisms in $\mathfrak{C}$. Let B be a full subcategory of $\mathfrak{C}$ which is closed under fiber products. Assume that B is a 1-category, that is, there is at most one 2-morphism between every two morphisms in B. The example to keep in mind is where $\mathfrak{C} = \mathfrak{TopSt}$ is the 2-category of topological stacks and $\mathsf{B} = \mathsf{Top}$ is the category of topological spaces (realized as a subcategory of $\mathfrak{TopSt}$ via Yoneda embedding).

Let R be a class of morphisms in B which contains the identity morphisms and is closed under base extension and 2-isomorphism. We define $\tilde{R}$ to be the class of morphisms $f\colon y \to x$ in $\mathfrak{C}$ such that for every morphism $p\colon t \to x$, $t \in \mathsf{B}$, the base extension r of f along p belongs to R.

$$\begin{array}{ccc} t \times_x y & \longrightarrow & y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ t & \xrightarrow[p]{} & x \end{array}$$

In other words, $\tilde{R}$ is the smallest class of morphisms in $\mathfrak{C}$ which is invariant under base extension and contains R and all identity morphisms. The example to keep in mind is where R is the class of (weak) homotopy equivalences in Top. In this case, $\tilde{R}$ will be the class of representable universal (weak) equivalences in $\mathfrak{TopSt}$. Also see Proposition 8.1.

Lemma 7.1. *The set up being as above, assume that for every object x in $\mathfrak{C}$, there exists an object $\Theta(x)$ in B together with a morphism $\varphi_x \colon \Theta(x) \to x$ which belongs to $\tilde{R}$. Then, the inclusion functor $\mathsf{B} \to [\mathfrak{C}]$ induces a fully faithful functor $\iota \colon R^{-1}\mathsf{B} \to R^{-1}[\mathfrak{C}]$. Furthermore, Θ naturally extends to a functor $R^{-1}[\mathfrak{C}] \to R^{-1}\mathsf{B}$ that is a right adjoint to ι. Finally, Θ can be defined so that the counits of adjunction are the identity maps and the units of adjunction belong to $\tilde{R}$.*

Proof. In the proof we will make use of the calculus of right fractions for $\bar{R}$ (§13) to describe morphisms in the localized categories. Here, $\bar{R}$ is the closure of R under composition. (Notice that localization with respect to R and $\bar{R}$ yields the same result.)

First, let us explain how to extend Θ to a functor. We will assume $\Theta(x) = x$, whenever $x \in \mathsf{B}$.

Given a morphism $f \colon x \to y$ in $[\mathfrak{C}]$, set $z := \Theta(x) \times_y \Theta(y)$, as in the 2-cartesian diagram

$$\begin{array}{ccc} z & \xrightarrow{g} & \Theta(y) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \varphi_y} \\ \Theta(x) & \xrightarrow[f \circ \varphi_x]{} & y \end{array}$$

By hypothesis, we have $r \in R$. We define $\Theta(f) \colon \Theta(x) \to \Theta(y)$ to be the span (r, g). The proof that Θ is well-defined and respects composition is straightforward.

To prove that Θ is a right adjoint to ι, we show that composing with the morphism $\varphi_x \colon \Theta(x) \to x$ induces a bijection

$$\mathrm{Hom}_{R^{-1}\mathsf{B}}(t, \Theta(x)) \xrightarrow{\sim} \mathrm{Hom}_{R^{-1}[\mathfrak{C}]}(t, x)$$

for every $t \in \mathsf{B}$. With the notation of §13, we have to show that the map

$$P \colon \mathrm{Span}(t, \Theta(x)) \to \mathrm{Span}(t, x)$$

which sends a span (r, g) to $(r, \varphi_x \circ g)$ induces a bijection

$$\pi_0(P) \colon \pi_0 \mathrm{Span}(t, \Theta(x)) \to \pi_0 \mathrm{Span}(t, x).$$

We define a functor

$$Q \colon \mathrm{Span}(t, x) \to \mathrm{Span}(t, \Theta(x))$$

as follows. Let $(r,g) \in \mathrm{Span}(t,x)$. Then $Q(r,g)$ is defined to be $(r \circ \rho, g')$, as in the diagram

$$\begin{array}{ccccc} & & s \times_x \Theta(x) & \xrightarrow{g'} & \Theta(x) \\ & & \rho \downarrow & & \downarrow \varphi_x \\ t & \xleftarrow[r]{} & v & \xrightarrow[g]{} & x \end{array}$$

There are natural transformations of functors

$$\mathrm{id}_{\mathrm{Span}(t,\Theta(x))} \Rightarrow Q \circ P$$

and $P \circ Q \Rightarrow \mathrm{id}_{\mathrm{Span}(t,x)}$. This is enough to establish that $\pi_0(P)$ and $\pi_0(Q)$ induce inverse bijections between $\pi_0 \,\mathrm{Span}(t, \Theta(x))$ and $\pi_0 \,\mathrm{Span}(t,x)$. (For example, this can be seen by noticing that P and Q induce an equivalence of categories between the groupoids generated by inverting all arrows in $\mathrm{Span}(t,\Theta(x))$ and $\mathrm{Span}(t,x)$.)

Fully faithfulness of ι follows from the fact that the unit of adjunction $\iota\circ\Theta \Rightarrow \mathrm{id}_{\mathsf{B}}$ is an isomorphism. □

Corollary 7.2. *The functors ι and Θ induce an equivalence of categories $R^{-1}\mathsf{B} \cong \tilde{R}^{-1}[\mathfrak{C}]$. In fact, on the right hand side, instead of inverting $\tilde{R}$, it is enough to invert R together with all the morphisms $\varphi_x \colon \Theta(x) \to x$.*

Remark 7.3. It can be shown that the functors P and Q appearing in the proof of Lemma 7.1 are indeed adjoints

$$P\colon\ \mathrm{Span}(t,\Theta(x)) \rightleftharpoons \mathrm{Span}(t,x) : Q$$

Therefore, if we work with the 2-categorical enhancements of $R^{-1}\mathsf{B}$ and $R^{-1}[\mathfrak{C}]$ (see Remark 13.1), then the adjunction between ι and Θ can be enhanced to a 2-categorical adjunction.

Lemma 7.4. *Let $F\colon \mathsf{B} \rightleftharpoons \mathsf{C} : G$ be an adjunction between categories. Let $\mathsf{B}' \subset \mathsf{B}$ and $\mathsf{C}' \subset \mathsf{C}$ be full subcategories such that $F(\mathsf{B}') \subset \mathsf{C}'$ and $G(\mathsf{C}') \subset \mathsf{B}'$. Then, the restriction of F and G to these subcategories induces an adjunction $F'\colon \mathsf{B}' \rightleftharpoons \mathsf{C}' : G'$*

Proof. Obvious. □

Lemma 7.5. *Let $F\colon \mathsf{B} \rightleftharpoons \mathsf{C} : G$ be an adjunction between categories. Let $S \subset \mathsf{B}$ and $T \subset \mathsf{C}$ be classes of morphisms such that $F(S) \subset T$ and $G(T) \subset S$. Then, we have an induced adjunction*

$$\tilde{F}\colon S^{-1}\mathsf{B} \rightleftharpoons T^{-1}\mathsf{C} : \tilde{G}$$

between the localized categories. Furthermore, if F is fully faithful, then so is $\tilde{F}$.

Proof. We will use the 2-categorical formulation of adjunction ([Mac], IV.1, Theorem 2). A standard Yoneda type argument shows that to give an adjunction

$$F\colon \mathsf{B} \rightleftharpoons \mathsf{C} : G$$

is the same thing as giving adjunctions

$$\mathsf{K}^G\colon \mathsf{K}^{\mathsf{B}} \rightleftharpoons \mathsf{K}^{\mathsf{C}} : \mathsf{K}^F,$$

for every category K, which are functorial with respect to change of K. Here, K^{B} stands for the category of functors from B to K.

By the universal property of localization, $\mathsf{K}^{S^{-1}\mathsf{B}}$ is naturally identified with a full subcategory of K^{B}. Similarly, $\mathsf{K}^{S^{-1}\mathsf{C}}$ is naturally identified with a full subcategory of K^{C}. The assumption that F and G respect S and T implies that $\mathsf{K}^{G}(\mathsf{K}^{S^{-1}\mathsf{B}}) \subset \mathsf{K}^{S^{-1}\mathsf{C}}$ and $\mathsf{K}^{F}(\mathsf{K}^{S^{-1}\mathsf{C}}) \subset \mathsf{K}^{S^{-1}\mathsf{B}}$. So, we have an induced adjunction

$$\mathsf{K}^{G}\colon \mathsf{K}^{S^{-1}\mathsf{B}} \rightleftharpoons \mathsf{K}^{S^{-1}\mathsf{C}} : \mathsf{K}^{F}.$$

Since these adjunctions are functorial with respect to K, the Yoneda argument gives the desired adjunction $\tilde{F}\colon S^{-1}\mathsf{B} \rightleftharpoons T^{-1}\mathsf{C} : \tilde{G}$.

The statement about fully faithfulness follows from the fact that a left adjoint F is fully faithful if the unit of adjunction is an isomorphism of functors. ☐

7.1. **Lemma 7.1 for diagram categories.** We prove a version of Lemma 7.1 for diagrams. The set up will be as in Lemma 7.1. We will assume, in addition, that R is closed under fiber products. This means that, given two morphisms $Y \to X$ and $Z \to X$ in R, the fiber product $Z \times_X Y \to X$ is also in R. Since R is closed under base extension, this is automatic if R is closed under composition.

Suppose we are given a category D, which we think of as a diagram. We assume either of the following holds:[1]

- **A**. The category D has the property that every object d in D has finite "degree", that is, there are only finitely many arrows coming out of d; or,
- **B**. The class R is closed under arbitrary fiber products.

We denote the 2-category of lax functors from D to $\mathfrak{C}$ by $\mathfrak{C}^{\mathsf{D}}$. We think of objects in $\mathfrak{C}^{\mathsf{D}}$ as diagrams in $\mathfrak{C}$ indexed by D.

Lemma 7.6. *Let T (resp., $\tilde{T}$) be the class of all transformations τ in the diagram category $[\mathfrak{C}^{\mathsf{D}}]$ which have the property that for every $d \in \mathsf{D}$ the corresponding morphism τ_d in $\mathfrak{C}$ is in R (resp., $\tilde{R}$). Then, the inclusion functor $\mathsf{B}^{\mathsf{D}} \to [\mathfrak{C}^{\mathsf{D}}]$ induces a fully faithful functor $\iota\colon T^{-1}\mathsf{B}^{\mathsf{D}} \to T^{-1}[\mathfrak{C}^{\mathsf{D}}]$, and ι has a right adjoint Θ^{D}. Furthermore, Θ^{D} can be defined so that the counits of adjunction are the identity transformations and the units of adjunction are honest transformations in $\tilde{T}$.*

Proof. We use Lemma 7.1 with $\mathfrak{C}$ and B replaced by the corresponding diagram categories. We have to verify that for every functor $p\colon \mathsf{D} \to \mathfrak{C}$, there exists a functor $\Theta^{\mathsf{D}}(p)\colon \mathsf{D} \to \mathsf{B}$ together with a natural transformation of functors $\varphi_p\colon \Theta^{\mathsf{D}}(p) \Rightarrow p$ such that every morphism $\varphi_{p,d}$ in this transformation is in $\tilde{R}$. To show this, we will make use of the relative Kan extension of §14. Let us fix the set up first.

Let $\mathfrak{U}$ be the 2-category fibered over $\mathfrak{C}$ whose fiber $\mathfrak{U}(x)$ over an object $x \in \mathfrak{C}$ is the 2-category of morphisms $h\colon a \to x$ in $\tilde{R}$. More precisely, the objects in $\mathfrak{U}$ are morphisms $a \to x$ in $\tilde{R}$. The morphism in $\mathfrak{U}$ are 2-commutative squares

$$\begin{array}{ccc} b & \longrightarrow & a \\ \downarrow & \overset{\tau}{\nearrow\!\!\!\nearrow} & \downarrow \\ y & \longrightarrow & x \end{array}$$

[1]The common feature of these two conditions, which is all we need to prove our lemma, is that R is closed under products indexed by any set whose cardinality is less that or equal to the degree (i.e., the number of arrows coming out) of some object in D.

in $\mathfrak{C}$ whose vertical arrows are in $\tilde{R}$; such a morphism is defined to be cartesian if the square is 2-cartesian. The 2-morphisms in $\mathfrak{U}$ are defined in the obvious way.

It follows that morphisms in $\mathfrak{U}(x)$ are 2-commutative triangles in $\mathfrak{C}$. It is easy to see that $\mathfrak{U}(x)$ is indeed a 1-category and not just a 2-category. The functor $\pi\colon \mathfrak{U} \to \mathfrak{C}$ is the forgetful functor which sends $h\colon a \to x$ to x. The pull-back functor $f^{\Box}\colon \mathfrak{U}(y) \to \mathfrak{U}(x)$ for a morphism $f\colon x \to y$ is the base extension along f.

Let E be the discrete category with the same set of objects as D, and let $F\colon \mathsf{E} \to \mathsf{D}$ be the functor which sends an object to itself. Either of the conditions (**A**) or (**B**) above implies that $\pi\colon \mathfrak{U} \to \mathfrak{C}$ is F-complete (Definition 14.3) at every $p\colon \mathsf{D} \to \mathfrak{C}$.

We can now define $\Theta^{\mathsf{D}}(p)$ and φ_p as follows. Let $p\colon \mathsf{D} \to \mathfrak{C}$ be a diagram in $\mathfrak{C}$. For every object $d \in \mathsf{D}$, we denote $p(d)$ by x_d. For each d, choose a map $\varphi_d\colon \Theta(x_d) \to x_d$, with $\Theta(x_d)$ in B and φ_d in $\tilde{R}$. This gives a functor $P\colon \mathsf{E} \to \mathfrak{U}$, $d \mapsto \varphi_d$, which lifts p. By Proposition 14.6, we have a right Kan extension $RF(P)\colon \mathsf{D} \to \mathfrak{U}$. To give such a functor $RF(P)$ is the same thing as giving a functor $\Theta^{\mathsf{D}}(p)\colon \mathsf{D} \to \mathfrak{C}$ together with a natural transformation of functors $\varphi_p\colon \Theta^{\mathsf{D}}(p) \Rightarrow p$. More precisely, for every $d \in \mathsf{D}$, we define $\Theta^{\mathsf{D}}(p)(d)$ and $\varphi_{p,d}$ by

$$RF(P)_d = \quad \Theta^{\mathsf{D}}(p)(d) \stackrel{\varphi_{p,d}}{\longrightarrow} x_d \quad \in \mathfrak{U}(x_d).$$

All that is left to check is that $\Theta^{\mathsf{D}}(p)$ factors through B. To see this, note that, by the construction of the Kan extension (see proof of Proposition 14.6), $\Theta^{\mathsf{D}}(p)(d)$ is the product in $\mathfrak{U}(d)$ of a family of objects $\{y_0, y_1, \cdots\}$ in $\mathfrak{U}(d)$, one of which, say y_0, is $\Theta(x_d)$. (Note the abuse of notation: each y_i is actually a morphism $y_i \to x_d$.) Denote the product of the rest of the objects by y. So $\Theta^{\mathsf{D}}(p)(d) = \Theta(x_d) \times y$, the product being taken in $\mathfrak{U}(d)$. Note that product in the fiber category $\mathfrak{U}(d)$ is calculated by taking fiber product over x_d in $\mathfrak{C}$. Hence, the following diagram is 2-cartesian in $\mathfrak{C}$.

$$\begin{array}{ccc} \Theta^{\mathsf{D}}(p)(d) & \longrightarrow & y \\ \downarrow & & \downarrow{\scriptstyle h} \\ \Theta(x_d) & \xrightarrow[\varphi_d]{} & x_d \end{array}$$

Since $h\colon y \to x_d$ is in $\mathfrak{U}(d)$, its base extension along φ_d is in R. That is, $\Theta^{\mathsf{D}}(p)(d)$ is in B and $\Theta^{\mathsf{D}}(p)(d) \to \Theta(x_d)$ is a morphism in R. This shows that $\Theta^{\mathsf{D}}(p)(d)$ is in B, which is what we wanted to prove. □

While proving Lemma 7.6 we have also proved the following.

Corollary 7.7. *Let $\mathfrak{S}$ be a small sub 2-category of $\mathfrak{C}$, and denote the inclusion of $\mathfrak{S}$ in $\mathfrak{C}$ by i_{S}. Then, there is a functor $\Theta_{\mathfrak{S}}\colon \mathfrak{S} \to \mathsf{B}$ together with a natural transformation $\varphi_{\mathfrak{S}}\colon \Theta_{\mathfrak{S}} \Rightarrow i_{\mathfrak{S}}$ such that for every $x \in \mathfrak{C}$, $\varphi_{\mathfrak{S}}(x)\colon \Theta_{\mathfrak{S}}(x) \to x$ is in $\tilde{R}$.*

Proof. Let $\mathsf{D} = [\mathfrak{S}]$ and think of it as a diagram category. Think of $p = [i_{\mathfrak{S}}]\colon \mathsf{D} \to [\mathfrak{C}]$ as a D-diagram in $[\mathfrak{C}]$. Then, with the notation of Lemma 7.6, the sought after $\Theta_{\mathfrak{S}}$ and $\varphi_{\mathfrak{S}}$ are exactly $\Theta^{\mathsf{D}}([i_{\mathfrak{S}}])$ and $\varphi_{[i_{\mathfrak{S}}]}$ (precomposed with the projection $\mathfrak{S} \to [\mathfrak{S}]$). □

8. Functorial description of the classifying space

Theorem 6.3 is saying that every topological stack has a classifying space. In this section, we use the category theoretic lemmas of §7 to give a functorial formulation of this fact (Theorem 8.2 and Theorem 8.8).

Proposition 8.1. *Let* $R \subset \mathsf{Top}$ *be the class of locally shrinkable maps (Definition 5.1). Then, the inclusion functor* $\mathsf{Top} \to \mathfrak{TopSt}$ *induces a fully faithful functor* $\iota\colon R^{-1}\mathsf{Top} \to R^{-1}[\mathfrak{TopSt}]$*, and* ι *has a right adjoint* $\Theta\colon R^{-1}[\mathfrak{TopSt}] \to R^{-1}\mathsf{Top}$*. Furthermore,* Θ *can be defined so that the counits of adjunction are the identity maps and the units of adjunction are honest morphisms of topological stacks which are locally shrinkable.*

Proof. Apply Lemma 7.1 to the inclusion $\mathsf{Top} \to \mathfrak{TopSt}$ with R being the class of locally shrinkable maps of topological *spaces*. For every topological stack $\mathcal{X}$, the existence of a topological space $\Theta(\mathcal{X})$ which satisfies the requirement of Lemma 7.1 is guaranteed by Theorem 6.3. □

8.1. **Functorial description of the classifying space.** The following theorem implies that the classifying space of a topological stack $\mathcal{X}$ is functorial in $\mathcal{X}$. In particular, the classifying space of a topological stack can be used to define a weak homotopy type for it.

Theorem 8.2. *Let* $S_{w.e.}$ *be the class of weak equivalences in* Top*. Let* $\mathsf{Top}_{w.e.} := S_{w.e.}^{-1}\mathsf{Top}$ *be the category of weak homotopy types. Then, the inclusion functor* $\mathsf{Top} \to \mathfrak{TopSt}$ *induces a fully faithful functor* $\iota\colon \mathsf{Top}_{w.e.} \to S_{w.e.}^{-1}[\mathfrak{TopSt}]$*, and* ι *has a right adjoint* $\Theta\colon S_{w.e.}^{-1}[\mathfrak{TopSt}] \to \mathsf{Top}_{w.e.}$*. Furthermore,* Θ *can be defined so that the counits of adjunction are the identity maps and the units of adjunction are honest morphisms of topological stacks which are locally shrinkable.*

Proof. Consider the adjunction $\iota\colon R^{-1}\mathsf{Top} \rightleftharpoons R^{-1}[\mathfrak{TopSt}] : \Theta$ of Proposition 8.1. Let $S \subset R^{-1}\mathsf{Top}$ be the class of weak equivalences and set $T = \iota(R)$. It is easy to see that the conditions of Lemma 7.5 are satisfied. This gives us the desired adjunction $\iota\colon \mathsf{Top}_{w.e.} \rightleftharpoons S_{w.e.}^{-1}[\mathfrak{TopSt}] : \Theta$. (By abuse of notation, we have denoted the induced functors on localized categories again by ι and Θ.) □

The functor $\Theta\colon S_{w.e.}^{-1}[\mathfrak{TopSt}] \to \mathsf{Top}_{w.e.}$ should be thought of as a functor that associates to every topological stack its **weak homotopy type**.

We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks is a **weak homotopy equivalence**, if $\Theta(f)$ is so. Let $\mathfrak{TopSt}_{w.e.}$ be the localization of the category $[\mathfrak{TopSt}]$ of topological stacks with respect to weak equivalences.

Corollary 8.3. *The functors* ι *and* Θ *of Theorem 8.2 induce an equivalence of categories*

$$\mathsf{Top}_{w.e.} \cong \mathfrak{TopSt}_{w.e.}.$$

In fact, the category on the right can be obtained by inverting $S_{w.e.}$ *and all locally shrinkable morphisms of topological stacks.*

Proof. Immediate from Theorem 8.2 (also see Corollary 7.2). □

8.2. **The homotopy type of a hoparacompact topological stack.** For the class of *hoparacompact* topological stacks (Definition 8.4) we can strengthen Theorem 8.2 by showing that every such stack has a natural homotopy type (Theorem 8.8).

Definition 8.4. We say that a topological stack $\mathcal{X}$ is **hoparacompact** if there exists a parashrinkable map $\varphi\colon X \to \mathcal{X}$ (Definition 5.1) with X a paracompact topological space.

For instance, if the atlas $\varphi\colon X \to \mathcal{X}$ of Theorem 6.3 can be chosen so that X is paracompact, then $\mathcal{X}$ is hoparacompact.

Proposition 8.5. *Let $\mathcal{X}$ be the quotient stack of a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$. In each of the following cases, $B\mathbb{X}$ is paracompact, hence $\mathcal{X}$ is hoparacompact:*

1. *The spaces X_1 and X_0 are regular and Lindelöf. (A space X is* Lindelöf *if every open cover of X has a countable subcover. A space X is* regular *if every closed set can be separated from every point by open sets.)*
2. *The spaces X_1 and X_0 are metrizable.*
3. *The space X_1 and X_0 are paracompact, Hausdorff, and they admit a proper surjective map from a metric space.*

The above proposition was also (independently) observed by Johaness Ebert. We will omit the proof here. The key point is that, under the given assumptions, the multiple fiber products $X_n := X_1 \times_{X_0} \times \cdots \times_{\mathcal{X}_0} \times X_1$ are again paracompact. This is not necessarily true if X_1 and X_0 are only assumed to be paracompact, because the products of two paracompact spaces is not necessarily paracompact. For this reason, we have to replace the paracompactness requirement on X_1 and X_0 by something stronger which is closed under products.

Let Para be the category of paracompact topological spaces and $\mathfrak{HPTopSt}$ the category of hoparacompact topological stacks. Let $S_{h.e.} \subset \mathsf{Para}$ be the class of homotopy equivalence. Let $\mathsf{Para}_{h.e.} = S_{h.e.}^{-1}\mathsf{Para}$ be the category of paracompact homotopy types.

Remark 8.6. There is an alternative way of describing the categories $\mathsf{Para}_{h.e.}$ and $S_{h.e.}^{-1}[\mathfrak{HPTopSt}]$ which is perhaps more natural. Let us give this description in the case of $S_{h.e.}^{-1}[\mathfrak{HPTopSt}]$.

The objects of $S_{h.e.}^{-1}[\mathfrak{HPTopSt}]$ are the same as those of $[\mathfrak{HPTopSt}]$. The morphisms $\mathrm{Hom}_{S_{h.e.}^{-1}[\mathfrak{HPTopSt}]}(\mathcal{X},\mathcal{Y})$ are obtained from $\mathrm{Hom}_{\mathfrak{HPTopSt}}(\mathcal{X},\mathcal{Y})$ by passing to a certain equivalence relation. If $\mathcal{X}$ is a topological space, this relation is just the usual homotopy between maps (defined via cylinders). If $\mathcal{X}$ is not an honest topological space, then two morphism $f,g\colon \mathcal{X} \to \mathcal{Y}$ in $[\mathfrak{HPTopSt}]$ get identified in $\mathrm{Hom}_{S_{h.e.}^{-1}[\mathfrak{HPTopSt}]}(\mathcal{X},\mathcal{Y})$ if and only if for every map $h\colon T \to \mathcal{X}$ from a topological space T, the compositions $f \circ h$ and $g \circ h$ are equivalent.

It is easy to verify that this category satisfies the universal property of localization.

Lemma 8.7. *Let R be as in Proposition 8.1. Then, the inclusion functor* $\mathsf{Para} \to \mathsf{Top}$ *induces a fully faithful functor* $\mathsf{Para}_{h.e.} \to R^{-1}\mathsf{Top}$. *In the stack version, the functor $S_{h.e.}^{-1}[\mathfrak{HPTopSt}] \to R^{-1}[\mathfrak{TopSt}]$ may no longer be fully faithful, but it induces a bijection*

$$\mathrm{Hom}_{S_{h.e.}^{-1}[\mathfrak{HPTopSt}]}(T,\mathcal{Y}) \cong \mathrm{Hom}_{R^{-1}[\mathfrak{TopSt}]}(T,\mathcal{Y})$$

whenever T is a topological space.

Proof. We only prove the statement for $[\mathfrak{HPTopSt}]$. The case of Para is proved similarly.

Before proving the bijectivity, let us explain why we have an induced functor

$$S_{h.e.}^{-1}[\mathfrak{HPTopSt}] \to R^{-1}[\mathfrak{TopSt}]$$

in the first place. By the discussion of Remark 8.6, morphisms in $S_{h.e.}^{-1}[\mathfrak{HPTopSt}]$ are obtained from those of $[\mathfrak{HPTopSt}]$ by passing to a certain equivalence relation. It is easy to check that the localization functor $[\mathfrak{HPTopSt}] \to R^{-1}[\mathfrak{HPTopSt}]$ sends an entire equivalence class of morphisms to one morphism. (This follows from the fact that the projection map $X \times [0,1] \to X$ of a cylinder is in R.) Therefore, we have a functor $S_{h.e.}^{-1}[\mathfrak{HPTopSt}] \to R^{-1}[\mathfrak{TopSt}]$.

Now, let T be a paracompact topological space and $\mathcal{Y}$ a hoparacompact topological stack. We want to show that

$$\gamma\colon \operatorname{Hom}_{S_{h.e.}^{-1}[\mathfrak{HPTopSt}]}(T,\mathcal{Y}) \to \operatorname{Hom}_{R^{-1}[\mathfrak{TopSt}]}(T,\mathcal{Y})$$

is a bijection.

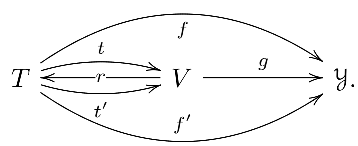
Let $\bar{R}$ be the class of morphisms in $\mathfrak{TopSt}$ which are compositions of finitely many locally shrinkable morphisms. By §13, morphisms in $R^{-1}[\mathfrak{TopSt}] = \bar{R}^{-1}[\mathfrak{TopSt}]$ can be calculated using a calculus of right fractions. By Lemma 13.2,

$$\operatorname{Hom}_{R^{-1}[\mathfrak{TopSt}]}(T,\mathcal{Y}) = \operatorname{Hom}_{[\mathfrak{TopSt}]}(T,\mathcal{Y})/_{\sim},$$

where $\sim$ is the equivalence relation generated by $\bar{R}$ -homotopy (§13). On the other hand

$$\operatorname{Hom}_{S_{h.e.}^{-1}[\mathfrak{HPTopSt}]}(T,\mathcal{Y}) = \operatorname{Hom}_{[\mathfrak{TopSt}]}(T,\mathcal{Y})/_{\sim'},$$

where $\sim'$ is the usual homotopy (Remark 8.6). To complete the proof, we show that $\sim$ and $\sim'$ are the same. That is, two honest morphisms $f, f'\colon T \to \mathcal{Y}$ are homotopic if and only if they are $\bar{R}$-homotopic. First assume that f and f' are $\bar{R}$-homotopic and consider an $\bar{R}$-homotopy (§13) between them, as in the diagram



Since r is a composition of locally shrinkable morphisms and T is paracompact, it follows from Lemma 5.5, together with the fact that $T \times [0,1]$ is paracompact, that t and t' are homotopic. This implies that f and f' are also homotopic. Conversely, assume f and f' are homotopic. Then, we can form an $\bar{R}$-homotopy diagram between them by taking $V = T \times [0,1]$. The proof is complete. □

The following theorem says that every hoparacompact topological stack has a natural homotopy type.

Theorem 8.8. *The inclusion functor* $\mathsf{Para} \to \mathfrak{HPTopSt}$ *induces a fully faithful functor* $\iota\colon \mathsf{Para}_{h.e.} \to S_{h.e.}^{-1}[\mathfrak{HPTopSt}]$, *and* ι *has a right adjoint* Θ. *Furthermore, the right adjoint* Θ *can be defined so that the counits of adjunction are the identity maps and the units of adjunction are honest morphisms of topological stacks which are locally shrinkable.*

Proof. Consider the adjunction $\iota\colon R^{-1}\mathsf{Top} \rightleftharpoons R^{-1}[\mathfrak{TopSt}] : \Theta$ of Proposition 8.1. We can arrange so that for every hoparacompact $\mathcal{X}$, $\Theta(\mathcal{X})$ is paracompact.

By Lemmas 7.4 and 8.7 , if in both sides of the adjunction we restrict to the paracompact objects, we obtain the adjunction

$$\iota\colon \mathsf{Para}_{h.e.} \rightleftharpoons S_{h.e.}^{-1}[\mathfrak{HPTopSt}] : \Theta,$$

which is what we were after. □

The functor $\Theta\colon S_{h.e.}^{-1}[\mathfrak{HPTopSt}] \to \mathsf{Para}_{h.e.}$ should be thought of as a functor that associates to every hoparacompact topological stack its **homotopy type**.

We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of hoparacompact topological stacks is a **homotopy equivalence**, if $\Theta(f)$ is so. Let $\mathfrak{HPTopSt}_{h.e.}$ be the localization of the category of hoparacompact stacks with respect to homotopy equivalences. We have the following.

Corollary 8.9. *The functors ι and Θ of Theorem 8.8 induce an equivalence of categories*

$$\mathsf{Para}_{h.e.} \cong \mathfrak{HPTopSt}_{h.e.}.$$

In fact, the category on the right can be obtained by inverting $S_{h.e.}$ and all locally shrinkable morphisms of hoparacompact topological stacks.

Proof. Immediate from Theorem 8.8 (also see Corollary 7.2). □

9. Homotopical stacks

In the previous sections, we proved the existence of a functorial classifying for an arbitrary topological stack. Although this may seem to be quite general, there are still some important classes of stacks to which this may not apply. The first example that comes to mind is the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ of two topological stacks. It can be shown (see [No2]) that if $\mathcal{Y} = [Y_0/Y_1]$, for a topological groupoid $[Y_1 \rightrightarrows Y_0]$ with Y_0 and Y_1 compact, then the mapping stack $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is again topological. The compactness condition on $\mathcal{Y}$, however, is quite restrictive and it seems that without it $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ may not be topological in general (but we do not have a counter example).

Nevertheless, we prove in [No2] that when Y_0 and Y_1 are locally compact and $\mathcal{X}$ is arbitrary, $\mathrm{Map}(\mathcal{Y},\mathcal{X})$ is not far from being topological. More precisely, it is *paratopological* in the sense of Definition 9.1 below. Every topological stack is, by definition, paratopological.

In this section, we show that our construction of the homotopy types for topological stacks can be extended to a larger class of stacks called *homotopical stacks* (Definition 9.1). Homotopical stacks include all paratopological stacks (hence, in particular, all topological stacks).

Definition 9.1. We say that a stack $\mathcal{X}$ is **paratopological** if it satisfies the following conditions:

- **A1**. Every map $T \to \mathcal{X}$ from a topological space T is representable (equivalently, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable);
- **A2**. There exists a morphism $X \to \mathcal{X}$ from a topological space X such that for every morphism $T \to \mathcal{X}$, with T a paracompact topological space, the base extension $T \times_{\mathcal{X}} X \to T$ is an epimorphism of topological spaces (i.e., admits local sections)

If (**A2**) is only satisfied with T a CW complex, then we say that $\mathcal{X}$ is **pseudotopological**. If in (**A2**) we require that the base extensions to be weak equivalences, we say that $\mathcal{X}$ is **homotopical**.[2] In this case, we say that X is a **classifying space** for $\mathcal{X}$.

Roughly speaking, a stack $\mathcal{X}$ being paratopological means that, in the eye of a paracompact topological space T, $\mathcal{X}$ is as good as a topological stack (Lemmas 9.2 and 9.3). Thus, it is not surprising that the homotopy theory of topological stacks can be extended to paratopological (or even pseudotopological) stacks.

Paratopological stacks form a full sub 2-category of the 2-category of stacks which we denote by $\mathfrak{ParSt}$. The 2-categories $\mathfrak{HoSt}$ and $\mathfrak{PsSt}$ of homotopical stacks and pseudotopological stack are defined similarly.

Lemma 9.2. *Let $\mathcal{X}$ be a stack over* Top *such that the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable. Then, $\mathcal{X}$ is paratopological (respectively, pseudotopological) if and only if there exists a topological stack $\bar{\mathcal{X}}$ and a morphism $p \colon \bar{\mathcal{X}} \to \mathcal{X}$ such that for every paracompact topological space T (respectively, every CW complex T), p induces an equivalence of groupoids $\bar{\mathcal{X}}(T) \to \mathcal{X}(T)$.*

Proof. We only prove the statement for paratopological stacks. The case of pseudotopological stacks is similar.

Suppose that such a map $\bar{\mathcal{X}} \to \mathcal{X}$ exists. Take an atlas $X \to \bar{\mathcal{X}}$ for $\bar{\mathcal{X}}$. It is clear that the composite map $X \to \mathcal{X}$ satisfies (**A2**) of Definition 9.1.

Conversely, assume $\mathcal{X}$ is paratopological and pick a map $X \to \mathcal{X}$ as in Definition 9.1, (**A2**). Set $X_0 := X$ and $X_1 := X \times_{\mathcal{X}} X$. It is easy to see that the quotient stack $\bar{\mathcal{X}} := [X_0/X_1]$ of the topological groupoid $[X_1 \rightrightarrows X_0]$ has the desired property. $\square$

Lemma 9.3. *Let $\mathcal{X}$ be a paratopological (resp., pseudotopological) stack. Let $p \colon \bar{\mathcal{X}} \to \mathcal{X}$ be as in Lemma 9.2. Then, for any map $T \to \mathcal{X}$, with T a paracompact topological space (resp., a CW complex), the base extension $p_T \colon T \times_{\mathcal{X}} \bar{\mathcal{X}} \to T$ is a homeomorphism.*

Proof. We only prove the statement for paratopological stacks. The case of pseudotopological stacks is similar.

Set $\mathcal{Y} := T \times_{\mathcal{X}} \bar{\mathcal{X}}$, and let $\mathcal{Y}_{mod}$ be its coarse moduli space ([No1], §4.3). By ([No1], Proposition 4.15.iii), we have a continuous map $f \colon \mathcal{Y}_{mod} \to T$. Since a point is paracompact, it follows from the definition that f is a bijection. On the other hand, since T is paracompact, f admits a section. Therefore, f is a homeomorphism.

Since a point is paracompact, it follows from the property of the map p that the inertia groups of $\mathcal{Y}$ are trivial. That is, $\mathcal{Y}$ is a quasitopological space in the sense of ([No1], §7, page 27). Since the coarse moduli map $\mathcal{Y} \to \mathcal{Y}_{mod} = T$ admits a section, it follows from ([No1], Proposition 7.9) that $\mathcal{Y} = \mathcal{Y}_{mod} = T$. $\square$

A slightly weaker version of Theorem 6.3 is true for paratopological and pseudotopological stacks.

Proposition 9.4. *Let $\mathcal{X}$ be a paratopological (resp., pseudotopological) stack. Then, there exists a parashrinkable (resp., a pseudoshrinkable) morphism $\varphi \colon X \to \mathcal{X}$ from a topological space X. In particular, $\mathcal{X}$ is a homotopical stack and (X, φ) is a classifying space for it (Definition 9.1).*

[2] Remark that if the latter condition is satisfied for all CW complexes T then it is satisfied for all topological spaces T.

Proof. Let $\mathcal{X}$ be a paratopological (resp., pseudotopological) stack, and let $p\colon \bar{\mathcal{X}} \to \mathcal{X}$ be an approximation for it by a topological stack $\bar{\mathcal{X}}$ as in Lemma 9.2. Choose an atlas $\bar{\varphi}\colon X \to \bar{\mathcal{X}}$ for it which is locally shrinkable (Theorem 6.3). Then, the composite $\varphi := p\circ\bar{\varphi}\colon X \to \mathcal{X}$ is parashrinkable (resp., pseudoshrinkable) by Lemma 9.3. □

Remark 9.5. Notice that, in contrast with Theorem 6.3, the map φ in Proposition 9.4 need not be an epimorphism.

Corollary 9.6. *We have the following full inclusions of 2-categories:*

$$\mathfrak{TopSt} \subset \mathfrak{ParSt} \subset \mathfrak{PsSt} \subset \mathfrak{HoSt}.$$

Proof. The first two inclusions are clear from the definition. The last inclusion follows from Proposition 9.4. □

Theorem 9.7. *Theorem 8.2 remains valid if we replace $\mathfrak{TopSt}$ with $\mathfrak{ParSt}$ (resp., $\mathfrak{PsSt}$ or $\mathfrak{HoSt}$). The last statement in Theorem 8.2 on the units and counits of the adjunction also remains valid provided that we replace locally shrinkable by parashrinkable (resp., pseudoshrinkable, universal weak equivalence).*

Proof. The same argument used in the proof of Theorem 8.2 carries over verbatim (we have to use Proposition 9.4 instead of Theorem 6.3). □

Definition 9.8. We say that a paratopological stack is **hoparacompact**, if there exists a parashrinkable morphism $\varphi\colon X \to \mathcal{X}$ such that X is a paracompact topological space (see Proposition 9.4). We denote the full subcategory of $\mathfrak{ParSt}$ consisting of hoparacompact paratopological stacks by $\mathfrak{HPParSt}$.

See Proposition 8.5 for examples of hoparacompact stacks.

The following theorem says that a hoparacompact paratopological stack has the homotopy type of a paracompact topological space.

Theorem 9.9. *Theorem 8.8 remains valid if we replace $\mathfrak{HPTopSt}$ by $\mathfrak{HPParSt}$. Furthermore, the last statement in Theorem 8.2 on the units and counits of the adjunction remains valid with locally shrinkable replaced by parashrinkable.*

Proof. The proof of Theorem 8.8 carries over. □

Remark 9.10. There is also a version of Theorem 9.9 for pseudotopological stacks. We define a pseudotopological stack to be *hoCW* if there exists a pseudoshrinkable morphism $\varphi\colon X \to \mathcal{X}$ such that X is a CW complex (see Proposition 9.4). We leave it to the reader to reformulate Theorem 9.9 accordingly. It follows that a hoCW pseudotopological stack has the homotopy type of a CW complex.

The following lemmas will be used later on when we discuss homotopy types of diagrams of stacks.

Lemma 9.11. *The 2-category of stacks $\mathcal{X}$ whose diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable is closed under arbitrary (2-categorical) limits.*

Proof. We prove a more general fact. Consider two diagrams $\mathbb{X} = \{\mathcal{X}_d\}$ and $\mathbb{Y} = \{\mathcal{Y}_d\}$ of stacks, where d ranges in some index category D. Let $\Delta\colon \mathbb{X} \Rightarrow \mathbb{Y}$ be a natural transformation such that for every d, $\Delta_d\colon \mathcal{X}_d \to \mathcal{Y}_d$ is representable. We claim that the induced morphism $\lim\Delta\colon \lim\mathbb{X} \to \lim\mathbb{Y}$ is also representable. (Applying this to the case where $\mathcal{Y}_d = \mathcal{X}_d \times \mathcal{X}_d$ and $\Delta_d \colon \mathcal{X}_d \to \mathcal{X}_d \times \mathcal{X}_d$ are the diagonal morphisms proves the lemma.)

To prove the claim, take an arbitrary map $f \colon T \to \lim \mathbb{Y}$ from a topological space T. Note that to give f is the same thing as to give a compatible family of maps $f_d \colon T \to \mathcal{Y}_d$. It follows easily from the universal property of limits that we have a natural isomorphism

$$\lim f_d^*(\mathcal{X}_d) \cong f^*(\lim \mathbb{X}).$$

(By f^* we mean pull-back along f, e.g., $f_d^*(\mathcal{X}_d) := T \times_{f_d, \mathcal{Y}_d, \Delta_d} \mathcal{X}_d$.) Since the diagram on the left is a diagram of topological spaces, it follows that $f^*(\lim \mathbb{X})$ is a topological space. □

Lemma 9.12. *Let $\mathcal{Y}$ be a stack whose diagonal $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable. Let $p_i \colon \mathcal{X}_i \to \mathcal{Y}$, $i \in I$, be a family of stacks over $\mathcal{Y}$. If every $\mathcal{X}_i$ is paratopological (resp., pseudotopoloical), then so is their fiber product $\prod_{\mathcal{Y}} \mathcal{X}_i$. If I is finite, the same statement is true for topological stacks and homotopical stacks.*

Proof. We prove the case of paratopological stacks. The other cases are proved similarly.

Condition (**A1**) of Definition 9.1 is satisfied by Lemma 9.11. Choose $\varphi_i \colon X_i \to \mathcal{X}_i$ as in Proposition 9.4. Set $X = \prod_{\mathcal{Y}} X_i$. We claim that the induced map $\varphi \colon X \to \prod_{\mathcal{Y}} \mathcal{X}_i$ is parashrinkable (hence, satisfies condition (**A2**) of Definition 9.1, **A2**). Let $g \colon T \to \prod_{\mathcal{Y}} \mathcal{X}_i$ be a map from a paracompact topological space T, and denote its i-th component by $g_i \colon T \to \mathcal{X}_i$. Let $f_i \colon T_i \to T$ be the base extension of φ_i along g_i, as in the 2-cartesian diagram

$$\begin{array}{ccc} T_i & \longrightarrow & X_i \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_i} \\ T & \xrightarrow[g_i]{} & \mathcal{X}_i \end{array}$$

We have a 2-cartesian diagram

$$\begin{array}{ccc} \prod_T T_i & \longrightarrow & \prod_{\mathcal{Y}} X_i \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ T & \xrightarrow[g]{} & \prod_{\mathcal{Y}} \mathcal{X}_i \end{array}$$

Since f_i is shrinkable for every i, the claim follows from Lemma 5.10. □

Lemma 9.13. *The 2-categories $\mathfrak{TopSt}$, $\mathfrak{ParSt}$, $\mathfrak{PsSt}$, and $\mathfrak{HoSt}$ are closed under finite limits.*

Proof. To show that these 2-categories are closed under finite limits, it is enough that 2-fiber products exist, which is the case by Lemma 9.12. The case of an arbitrary finite limit then follows from the general fact that a 2-category that has 2-fiber products and a final object is closed under arbitrary finite limits. □

10. Homotopy groups of homotopical stacks

By ([No1], §17) we know that if $(\mathcal{X}, x)$ is a pointed *Serre*[3] topological stack (Definition 10.1), the standard definition

$$\pi_n(\mathcal{X}, x) := [(S^n, \bullet), (\mathcal{X}, x)]$$

of homotopy groups in terms of homotopy classes of pointed maps gives rise to well-defined homotopy groups for $\mathcal{X}$ that enjoy the expected properties. Theorem 6.3 gives an alternative definition of higher homotopy groups that works for an arbitrary topological stack $\mathcal{X}$. (In fact, all we need for this definition to make sense is that $\mathcal{X}$ be a homotopical stack.) In this section we prove that these two definition are equivalent (Theorem 10.5).

Let us first recall the definition of a Serre topological stack.

Definition 10.1 ([No1], §17)**.** We say that a topological stack $\mathcal{X}$ is **Serre** if it is equivalent to the quotient stack of a topological groupoid $[s, t \colon R \rightrightarrows X]$ whose source map (hence, also its target map) is a local Serre fibrations. That is, for every $y \in R$, there exists an open neighborhood $U \subseteq R$ of y and $V \subseteq X$ of $f(y)$ such that the restriction of $s|_U \colon U \to V$is a Serre fibration.

Lemma 10.2. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of homotopical stacks (resp., paratopological stacks). Then, there is a 2-commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

where X and Y are topological spaces (resp. paracompact topological spaces) and φ and ψ are universal weak equivalences (resp., parashrinkable morphisms).

Proof. We only prove the case of homotopical stacks. By Theorem 6.3, we can choose universal weak equivalences $\psi \colon Y \to \mathcal{Y}$ and $h \colon X \to \mathcal{X} \times_{\mathcal{Y}} Y$. (Notice that, by Lemma 9.13, $\mathcal{X} \times_{\mathcal{Y}} Y$ is homotopical.) Set $\varphi = \mathrm{pr}_1 \circ h$ and $g = \mathrm{pr}_1 \circ h$. ☐

The following lemmas were proved implicitly in the course of proof of Lemma 7.1. We state them separately for future reference.

Lemma 10.3. *Let $\varphi \colon X \to \mathcal{X}$ be a universal weak equivalence with X a topological space. Let $f_1, f_2 \colon Y \to X$ be continuous maps of topological spaces such that $\varphi \circ f_1$ and $\varphi \circ f_2 \colon Y \to \mathcal{X}$ are 2-isomorphic. Then, f_1 and f_2 are equal in the weak homotopy category $\mathsf{Top}_{w.e.}$ of topological spaces.*

Proof. Let $g = \varphi \circ f_1$, and consider the 2-cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[g]{} & \mathcal{X} \end{array}$$

[3] In [ibid.] we call these *topological stacks.*

where $Z = Y \times_{\mathcal{X}} X$. The maps f_1 and f_2 correspond to section $s_1, s_2 \colon Y \to Z$ of ψ. Since ψ is a weak equivalence, s_1 and s_2 are equal in $\mathsf{Top}_{w.e.}$. Therefore, $f_1 = h \circ s_1$ and $f_2 = h \circ s_2$ are also equal in $\mathsf{Top}_{w.e.}$. □

Lemma 10.4. *Let $\varphi \colon X \to \mathcal{X}$ be a parashrinkable (resp., pseudoshrinkable) morphism (see Definition 5.2) with X a paracompact topological space (resp. a CW complex). Let $f_1, f_2 \colon Y \to X$ be continuous maps of topological spaces such that $\varphi \circ f_1$ and $\varphi \circ f_2 \colon Y \to \mathcal{X}$ are 2-isomorphic. Then, f_1 and f_2 are homotopic.*

Proof. Copy the proof of Lemma 10.3. □

Theorem 10.5. *Let $(\mathcal{X}, x)$ be a pointed homotopical stack. Then, one can define homotopy groups $\pi_n(\mathcal{X}, x)$ that are functorial with respect to pointed morphisms of stacks. When $\mathcal{X}$ is a Serre topological stack, these homotopy groups are naturally isomorphic to the ones defined in (*[No1]*, §17). That is, $\pi_n(\mathcal{X}, x) \cong [(S^n, \bullet), (\mathcal{X}, x)]$.*

Proof. Let $(\mathcal{X}, x)$ be a pointed homotopical stack. Choose a universal weak equivalence $\varphi \colon X \to \mathcal{X}$. Pick a point $\tilde{x} \in X$ sitting above x. (This means, a map $\tilde{x} \colon \bullet \to X$, together with a 2-morphism $\alpha \colon x \Rightarrow p \circ \tilde{x}$, which we usually suppress from the notation for convenience.) For $n \geq 0$, we define $\pi_n(\mathcal{X}, x) := \pi_n(X, \tilde{x})$.

Let us see why this definition is independent of the choice of $\tilde{x}$. Let $\tilde{x}' \in X$ be another point above x. Let $F = \bullet \times_{\mathcal{X}} X$ be the fiber of φ over x. The map $F \to \bullet$, being the base extension of φ, is a weak homotopy equivalence. This means that F is a weakly contractible topological space. The lifts $\tilde{x}$ and $\tilde{x}'$ of x correspond to points $\bar{x}$ and $\bar{x}'$ in F. Since F is weakly contractible, there is a path γ, unique up to homotopy, joining $\bar{x}$ and $\bar{x}$. Taking the image of γ in X we find a natural path joining $\tilde{x}$ and $\tilde{x}'$. This path defines a natural isomorphism $\pi_n(X, \tilde{x}) \xrightarrow{\sim} \pi_n(X, \tilde{x}')$.

We will leave it to the reader to verify that $\pi_n(\mathcal{X}, x)$ is also independent of the chart $\varphi \colon X \to \mathcal{X}$ and that it is functorial. The proof makes use of the Lemmas 10.2 and 10.3.

In the case where $\mathcal{X}$ is Serre topological, it follows from Corollary 5.9 that the homotopy groups defined above are naturally isomorphic to the ones defined in ([No1], §17). □

11. (Co)homology theories for homotopical stacks

By virtue of Theorem 9.7 we can do algebraic topology on homotopical stacks by transporting things back and forth between a stack and its classifying space. In this section we use this idea to show how we can extend generalized (co)homology theories to stacks.

Fact. Let h be a (co)homology theory on the category of topological spaces which is invariant under weak equivalences. Then h extends naturally to the category of homotopical stacks. If h is only invariant under homotopy equivalences,[4] then h extends naturally to the category of hoparacompact paratopological stacks (Definition 9.8).

Let us show, for example, how to define $h^*(\mathcal{X}, \mathcal{A})$ for a pair $(\mathcal{X}, \mathcal{A})$ of homotopical stacks. (In §12 we will discuss in detail how to define homotopy types of small

[4]Usually, (co)homology theories of Čech type, or certain sheaf cohomologies, are only invariant under homotopy equivalences.

diagrams of stacks.) From now on, we will assume that h is contravariant, and denote it by h^*. Everything we say will be valid for a homology theory as well.

Pick a universal weak equivalence $\varphi\colon X \to \mathfrak{X}$, and set $A := \varphi^{-1}\mathcal{A} \subseteq X$. Define $h^*(\mathfrak{X}, \mathcal{A}) := h^*(X, A)$. This definition is independent, up to a natural isomorphism, of the choice of φ. To see this, let $\varphi'\colon X' \to \mathfrak{X}$ be another universal weak equivalence, and form the fiber product $\varphi''\colon X'' \to \mathfrak{X}$.

$$\begin{array}{ccc} (X'', A'') & \xrightarrow{p} & (X, A) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (X', A') & \xrightarrow[\varphi']{} & (\mathfrak{X}, \mathcal{A}) \end{array}$$

Since p and q are weak equivalences of pairs, it follows that there are natural isomorphisms $h^*(X', A') \cong h^*(X'', A'') \cong h^*(X, A)$.

Covariance of h^* with respect to morphisms of pairs of homotopical stacks follows from Lemmas 10.2 and 10.3.

For more or less trivial reasons, the resulting cohomology theory on the category of homotopical stacks will maintain all the reasonable (read functorial) properties/structures that it has with topological spaces (e.g., excision, long exact sequence for pairs, Mayor-Vietoris, products, etc.). Homotopic morphisms (in particular, 2-isomorphic morphisms) induce the same map on cohomology groups.

The proof of all of these follows by same line of argument: choose a universal weak equivalence $\varphi\colon X \to \mathfrak{X}$, verify the desired property/structure on X, and then use Lemmas 10.2 and 10.3 to show that the resulting property/structure on $h^*(\mathfrak{X})$ is independent of the choice of φ and is functorial.

Remark 11.1. In the case where h is singular (co)homolgoy, what we constructed above coincides with the one constructed by Behrend in [Be] (we will not give the proof of this here). When $\mathfrak{X} = [X/G]$ is the quotient stack of a topological group action, the discussion of §4.3 shows the cohomology theories defined above coincide with the corresponding G-equivariant theories constructed via the Borel construction.

11.1. **(Co)homology theories that are only homotopy invariant.** There are certain (co)homology theories that are only invariant under homotopy equivalences. Among these are certain sheaf cohomology theories or cohomology theories defined via a Čech procedure. Such (co)homology theories do not, a priori, extend to topological stacks because they are not invariant under weak equivalences. They do, however, extend to hoparacompact paratopological stacks $\mathfrak{X}$.

The same argument that we used in the previous subsection applies here, more or less word by word, as long as we choose the classifying space X of $\mathfrak{X}$ to be paracompact and $\varphi\colon X \to \mathfrak{X}$ to be parashrinkable. For instance, the reader can easily verify that, under these assumptions, the morphisms p and q in the commutative square of the previous subsection will be homotopy equivalences. This guarantees that $h^*(\mathfrak{X}, \mathcal{A})$ is well-defined. To prove functoriality one makes use of Lemmas 10.2 and 10.4.

Remark 11.2. The above discussion remains true if we replace the category of hoparacompact paratopological stacks by the category of *hoCW pseudotopological stacks* (see Remark 9.10).

11.2. **A remark on supports.** The notion of *supports* for a (co)homology theory can sometimes be extended to the stack setting. The following result will not be used elsewhere in the paper, but we include it to illustrate the idea.

Let us say that a homology theory h on topological spaces is (para)compactly supported if for every topological space X the map

$$\varinjlim_{K \to X} h_*(K) \to h_*(X)$$

is an isomorphism. Here, the limit is taken over all maps $K \to X$ with K (para)compact. For example, singular homology is compactly supported.

Proposition 11.3. *Let h be a (para)compactly supported homology theory. Then, for every paratopological stack $\mathcal{X}$, we have a natural isomorphism*

$$\varinjlim_{K \to \mathcal{X}} h_*(K) \xrightarrow{\sim} h_*(\mathcal{X}),$$

where the limit is taken over all maps $K \to \mathcal{X}$ with K a (para)compact topological space.

Proof. Choose a parashrinkable morphism $\varphi\colon X \to \mathcal{X}$ and use the fact that every morphism $K \to \mathcal{X}$ from a paracompact topological space K has a lift, unique up to homotopy, to X; see Lemma 5.5. ☐

12. Classifying spaces for diagrams of topological stacks

In this section, we prove a diagram version of Theorem 9.7. We show (Theorem 12.1) that to every small diagram of topological stacks (with a certain condition on the shape of the diagram) one can associate a diagram of classifying topological spaces which is well-defined up to (objectwise) weak equivalence of diagrams. Theorem 12.1 is actually formulated in a way that it implies versions of the above statement for topological, homotopical, paratopological, and pseudotopological stacks.

Let D be a category (which we will think of as a diagram). In what follows, we will assume that C and R are any of the following pairs:

1) $\mathfrak{C}$ is $\mathfrak{TopSt}$ and R is the class of locally shrinkable maps of topological spaces;
2) $\mathfrak{C}$ is $\mathfrak{HoSt}$ and R is the class of universal weak equivalences of topological spaces;
3) $\mathfrak{C}$ is $\mathfrak{ParSt}$ and R is the class of parashrinkable maps of topological spaces;
4) $\mathfrak{C}$ is $\mathfrak{PsSt}$ and R is the class of pseudoshrinkable maps of topological spaces.

In the first two cases, assume in addition that the category D has the property that for every object d in D there are only finitely many arrows coming out of d. Lemma 5.10 now guarantees that in all four cases at least one of the two conditions (**A**) or (**B**) of §7.1 is satisfied.

By the notation of §7.1, $\tilde{R}$ stands for the class of representable morphisms of stacks which are locally shrinkable, universal weak equivalence, parashrinkable, or pseudoshrinkable (depending on which pair **1-4** we are considering).

Recall that $\mathfrak{C}^{\mathsf{D}}$ stands for the category of lax functors $\mathsf{D} \to \mathsf{C}$. A morphism τ in $\mathfrak{C}^{\mathsf{D}}$ is a natural transformation of functors.

Theorem 12.1. *Let D be a small category, and let $\mathfrak{C}$ and R be as above. Let T (resp., $\tilde{T}$) be the class of all transformations τ in $\mathfrak{C}^{\mathsf{D}}$ which have the property that for every $d \in \mathsf{D}$ the corresponding morphism τ_d in $\mathfrak{C}$ is in R (resp., $\tilde{R}$). Then,* *the inclusion functor* $\mathsf{Top}^{\mathsf{D}} \to \mathfrak{C}^{\mathsf{D}}$ *induces a fully faithful functor* $\iota^{\mathsf{D}} \colon T^{-1}\mathsf{Top}^{\mathsf{D}} \to T^{-1}[\mathfrak{C}^{\mathsf{D}}]$*, and* ι^{D} *has a right adjoint* Θ^{D}*. Furthermore,* Θ^{D} *can be defined so that the counits of adjunction are the identity transformations and the units of adjunction are honest transformations in* $\tilde{T}$*.*

Proof. Let $\mathsf{B} = \mathsf{Top}$ and use Lemma 7.6. □

Let $P \colon \mathsf{D} \to \mathfrak{C}$ be a diagram of stacks in $\mathfrak{C}$. The diagram $\Theta^{\mathsf{D}}(P) \colon \mathsf{D} \to \mathsf{Top}$ should be regraded as the **weak homotopy type** of P. The transformation $\varphi \colon \Theta^{\mathsf{D}}(P) \Rightarrow P$ allows one to relate the homotopical information in the diagram P to the homotopical information in its homotopy type $\Theta^{\mathsf{D}}(P)$. Notice that φ is an objectwise universal weak equivalence.

The following propositions say that the classifying space functor $\Theta \colon R^{-1}\mathfrak{PsSt} \to R^{-1}\mathsf{Top}$ of Theorem 9.7, can be lifted to a functor to Top if we restrict it to a small sub 2-category $\mathfrak{S}$.

Proposition 12.2. *Let* $\mathfrak{S}$ *be a small sub 2-category of the 2-category* $\mathfrak{PsSt}$ *of pseudotopological stacks, and denote the inclusion functor by* $i_{\mathfrak{S}}$*. Identify* Top *with a subcategory of* $\mathfrak{PsSt}$*. Then, there is a functor* $\Theta_{\mathfrak{S}} \colon \mathfrak{S} \to \mathsf{Top}$ *and a transformation* $\varphi_{\mathfrak{S}} \colon \Theta_{\mathfrak{S}} \Rightarrow i_{\mathfrak{S}}$ *such that for every* $\mathcal{X}$ *in* $\mathfrak{S}$*,* $\varphi_{\mathfrak{S}}(\mathcal{X}) \colon \Theta_{\mathfrak{S}}(\mathcal{X}) \to \mathcal{X}$ *is pseudoshrinkable (in particular, a universal weak equivalence). In the case where* $\mathfrak{S}$ *sits inside* $\mathfrak{ParSt}$*,* $\Theta_{\mathfrak{S}}$ *and* $\varphi_{\mathfrak{S}}$ *can be chosen so that* $\varphi_{\mathfrak{S}}(\mathcal{X})$ *are parashrinkable.*

Proof. Follows from Corollary 7.7. □

Proposition 12.3. *Let* $\mathfrak{S}$ *be a small sub 2-category of the 2-category* $\mathfrak{HoSt}$ *of homotopical stacks. Identify* Top *with a subcategory of* $\mathfrak{HoSt}$*. Assume that* $\mathfrak{S}$ *has the property that for every stack* $\mathcal{X}$ *in* $\mathfrak{S}$ *there are only finitely many* $\mathcal{Y}$ *in* $\mathfrak{S}$ *to which there is a morphism from* $\mathcal{X}$*. Then, there is a functor* $\Theta_{\mathfrak{S}} \colon \mathfrak{S} \to \mathsf{Top}$ *and a transformation* $\varphi_{\mathfrak{S}} \colon \Theta_{\mathfrak{S}} \Rightarrow \mathrm{id}_{\mathfrak{S}}$ *such that for every* $\mathcal{X}$ *in* $\mathfrak{S}$*,* $\varphi_{\mathfrak{S}}(\mathcal{X}) \colon \Theta_{\mathfrak{S}}(\mathcal{X}) \to \mathcal{X}$ *is an atlas for* $\mathcal{X}$ *which is a universal weak equivalence. In the case where* $\mathfrak{S}$ *sits inside* $\mathfrak{TopSt}$*,* $\Theta_{\mathfrak{S}}$ *and* $\varphi_{\mathfrak{S}}$ *can be chosen so that* $\varphi_{\mathfrak{S}}(\mathcal{X})$ *are locally shrinkable.*

Proof. Follows from Corollary 7.7. □

12.1. **Homotopy types of special diagrams.** The weak homotopy type of a diagram $\{\mathcal{X}_d\}$ of stacks can be constructed more easily if we assume that: 1) our diagram category D has a final object $\star$, 2) the morphisms in the diagram are representable. For this, choose a locally shrinkable (parashrinkable, pseudoshrinkable, or a universal equivalence, depending on which class of stacks we are working with) map $X_\star \to \mathcal{X}_\star$, and define X_d, $d \in \mathsf{D}$, simply by base extending $X_\star$ along the the morphism $\mathcal{X}_d \to \mathcal{X}_\star$, as in the diagram

$$\begin{array}{ccc} X_d & \to & X_\star \\ \downarrow & & \downarrow \\ \mathcal{X}_d & \to & \mathcal{X}_* \end{array}$$

This construction of the weak homotopy type of a diagram has certain advantages over the general construction of the previous subsection. Suppose that every morphism f in D is labeled by a property $\mathbf{P}_f$ of morphisms of topological spaces which is invariant under base change. (Note that such property can then be extended to representable morphisms of stacks.) Then, it is obvious that if the morphisms f in a diagram $\{\mathcal{X}_d\}$ have the properties $\mathbf{P}_f$, then so will the corresponding morphisms in the diagram $\{X_d\}$.

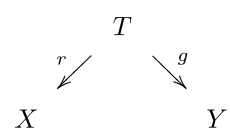
Example 12.4. Let $\mathsf{D} = \{1 \to 2\}$, and assume the label assigned to the unique morphism in D is 'closed immersion'. Then, it follows that every closed pair $(\mathcal{X}, \mathcal{A})$ of topological stacks has the weak homotopy type of a closed pair (X, A) of topological spaces. Furthermore, there is a morphism of pairs $\varphi \colon (X, A) \to (\mathcal{X}, \mathcal{A})$ which is a universal weak equivalence on both terms. This is essentially what we discussed in §11.

In the case where $\mathcal{A}$ is a point, (X, A) will be a pair with A weakly contractible. Therefore, we can define $\pi_n(\mathcal{X}, x) := \pi_n(X, A)$. This is exactly what we discussed in §10.

13. Appendix I: Calculus of right fractions

Let C be a category. Let R be a class of morphisms in C which contains all identity morphisms and is closed under composition and base extension. The localized category $R^{-1}\mathsf{C}$ can be calculated using a *calculus of right fractions*, as we see shortly. Our setting is slightly different from that of Gabriel-Zisman ([GaZi], §2.2) in that their condition (d) may not be satisfied in our case. We are, however, making a stronger assumption that R is closed under base extension.

Let $R^{-1}\mathsf{C}$ be a category with the same set of objects as C. The morphisms in $R^{-1}\mathsf{C}$ are defined as follows. A morphism from X to Y is presented by a span (r, g)



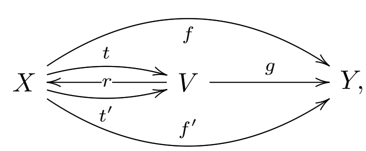
where r is in R and g is a morphism in C. For fixed X and Y, the spans between them form a category $\mathrm{Span}(X, Y)$. The morphisms in $\mathrm{Span}(X, Y)$ are morphisms $T' \to T$ in C which respect the two legs of the spans. By definition, two spans in $\mathrm{Span}(X, Y)$ give rise to the same morphism in $\mathrm{Hom}_{R^{-1}\mathsf{C}}(X, Y)$ if and only if they are in the same connected component of $\mathrm{Span}(X, Y)$. That is, if they are connected by a zig-zag of morphisms in $\mathrm{Span}(X, Y)$. In other words,

$$\mathrm{Hom}_{R^{-1}\mathsf{C}}(X, Y) := \pi_0 \,\mathrm{Span}(X, Y).$$

The composition of spans is defined in the obvious way. It is easy to see that $R^{-1}\mathsf{C}$ satisfies the universal property of localization.

Remark 13.1. We can enhance $R^{-1}\mathsf{C}$ to a bicategory by defining the the hom-category between X and Y to be $\mathrm{Span}(X, Y)$. The localized category $R^{-1}\mathsf{C}$ is recovered from this bicategory by declaring all 2-cells to be equalities. That is, by replacing the hom-categories $\mathrm{Span}(X, Y)$ with the set $\pi_0 \,\mathrm{Span}(X, Y)$.

Let $f, f' \colon X \to Y$ be morphisms in C. We say that f, f' are *R-homotopic* if there is a commutative diagram



where r is in R. Let $\sim$ be the equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ generated by R-homotopy. We have a natural map

$$\eta\colon \mathrm{Hom}_{\mathsf{C}}(X,Y)/_{\sim} \to \mathrm{Hom}_{R^{-1}\mathsf{C}}(X,Y)$$
$$f \mapsto (\mathrm{id}, f).$$

Lemma 13.2. *Assume that $X \in \mathsf{C}$ has the property that every morphism $r\colon V \to X$ in R admits a section. Then, for every Y in C, the natural map*

$$\eta\colon \mathrm{Hom}_{\mathsf{C}}(X,Y)/_{\sim} \to \mathrm{Hom}_{R^{-1}\mathsf{C}}(X,Y)$$

is a bijection.

Proof. Given a span (r,g) from X to Y, choose a section s for r. Then, $g \circ s$, or rather the span $(\mathrm{id}, g\circ s)$, represents the same morphism in $R^{-1}\mathsf{C}$ as (r,g). This shows that η is surjective. To prove injectivity, consider $f, f' \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$. It is easy to see that there is a morphism in $\mathrm{Span}(X,Y)$ between (id, f) and (id, f') if and only if f and f' are R-homotopic. Therefore, the R-homotopy classes of morphisms in $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ correspond precisely to the connected components of $\mathrm{Span}(X,Y)$. This proves injectivity. ☐

14. Appendix II: Relative Kan extensions

We introduce a right Kan extension construction in the setting of fibered 2-categories $\pi\colon \mathfrak{U} \to \mathfrak{B}$. In the case where the base 2-category $\mathfrak{B}$ is just a point and $\mathfrak{U}$ is a category, this reduces to the usual right Kan extension as defined in ([Mac], §X).

Let $\mathfrak{B}$ and $\mathfrak{U}$ be 2-categories, and let $\pi\colon \mathfrak{U} \to \mathfrak{B}$ be a fibered 2-category (not necessarily in 2-groupoids). It is sometimes more convenient to think of this fibered 2-category as the contravariant 2-category-valued lax functor $\mathfrak{B} \to 2\mathfrak{Cat}$ which assigns to an object b in $\mathfrak{B}$ the fiber $\mathfrak{U}(b)$ of $\mathfrak{U}$ over b. (In our application (§7.1), π is fibered in 1-categories, so the corresponding lax functor takes values in $\mathfrak{Cat}$.) For every morphism $f\colon a \to b$ in $\mathfrak{B}$, we have the *pull-back* functor $f^{\square}\colon \mathfrak{U}(b) \to \mathfrak{U}(a)$. (To define $f^{\square}$ we need to make some choices, but the resulting functor $f^{\square}$ will be unique up to higher coherences.) The laxness of our 2-category-valued functor means that, for every pair of composable morphisms f and g in $\mathfrak{B}$, we have a natural transformation $g^{\square} \circ f^{\square} \Rightarrow (f \circ g)^{\square}$, and that these transformations satisfy the usual coherence conditions. The fibered 2-category $\mathfrak{U}$ can be recovered from this lax functor by applying the Grothendieck construction.

Let $\pi\colon \mathfrak{U} \to \mathfrak{B}$ be a fibered 2-category, and Γ a 2-category. Let b be an object in $\mathfrak{B}$. Consider a diagram $P\colon \Gamma \to \mathfrak{U}(b)$, that is, a lax functor from Γ to $\mathfrak{U}(b)$. Assume that P has a limit $\lim P$ in $\mathfrak{U}(b)$. Let $\underline{\lim P}\colon \Gamma \to \mathfrak{U}(b)$ denote the constant functor with value $\lim P$, and let $\Upsilon_P\colon \underline{\lim P} \Rightarrow P$ be the universal transformation.

Definition 14.1. The notation being as above, we say that the limit $\lim P$ of P in $\mathfrak{U}(b)$ is **global**, if for every morphism $f\colon a \to b$ in $\mathfrak{B}$, and every object $k \in \mathfrak{U}(a)$, the functor

$$\mathrm{Hom}_{\mathfrak{U},f}(k, \lim P) \to \mathrm{Trans}_f(\underline{k}, P)$$
$$\tilde{f} \mapsto \Upsilon_P \circ \tilde{f}$$

is an equivalence of categories. Here, $\mathrm{Hom}_{\mathfrak{U},f}$ means those morphisms in $\mathfrak{U}$ which map to f under π. Similarly, Trans_f stands for those transformations Φ (of functors $\Gamma \to \mathfrak{U}$) such that, for every $d \in \mathsf{D}$, the image of the morphism $\Phi(d)$ under π is equal to f. (Note that both sides are 1-categories. In the case where $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is fibered in 1-categories, they are actually equivalent to sets.)

Remark 14.2. The limit $\lim P$ being global is equivalent to requiring the pullback $f^{\Box}(\lim P) \in \mathfrak{U}(a)$ to be the limit of the pullback diagram $f^{\Box} \circ P\colon \Gamma \to \mathfrak{U}(a)$, for every $f\colon a \to b$.

Definition 14.3. Let $\pi\colon \mathfrak{U} \to \mathfrak{B}$ be a fibered 2-category, and Γ a 2-category. Let b be an object in $\mathfrak{B}$. We say that $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is **Γ-complete** at b, if every diagram $P\colon \Gamma \to \mathfrak{U}(b)$ has a limit and the limit is global (Definition 14.1). More generally, let D and E be 2-categories,[5] and $F\colon \mathsf{E} \to \mathsf{D}$ and $p\colon \mathsf{D} \to \mathfrak{B}$ functors. We say that $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is **F-complete** at p if it is $(d\downarrow \mathsf{E})$-complete at $p(d)$ for every $d \in \mathsf{D}$.

The comma 2-category $(d \downarrow \mathsf{E})$ appearing in the above definition is defined as follows. The objects are pairs (e, α), where $e \in \operatorname{Ob} \mathsf{E}$ and $\alpha\colon d \to F(e)$ is a morphism in D. A morphism $(e, \alpha) \to (e', \alpha')$ in $(d\downarrow \mathsf{E})$ is a morphism $\gamma\colon e \to e'$ in E, together with a 2-morphism $\tau\colon F(\gamma) \circ \alpha \Rightarrow \alpha'$. A 2-morphism from (γ, τ) to (γ', τ') is a 2-morphism $\epsilon\colon \gamma \Rightarrow \gamma'$ in E which makes the 2-cell in D consisting of τ, τ', and $F(\epsilon)$ commute. (Note that in the case where D and E are 1-categories, $(d\downarrow \mathsf{E})$ is also a 1-category and it coincides with the one defined in [Mac], II.6. If, furthermore, E is discrete, then $(d\downarrow \mathsf{E})$ is also discrete.)

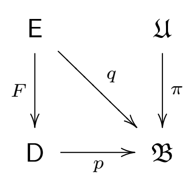
Definition 14.4. Let $\pi\colon \mathfrak{U} \to \mathfrak{B}$ be a fibered 2-category, and I an index set. We say that $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is **I-complete** (or it **has global I-products**) if it is I-complete at every $b \in \mathfrak{B}$. Here, we think of I as the discrete 2-category with objects I and no nontrivial morphisms or 2-morphisms.

The following lemma shows that completeness with respect to a functor is invariant under base change of fibered categories.

Lemma 14.5. *Let $\pi\colon \mathfrak{U} \to \mathfrak{B}$ be a fibered 2-category, and $F\colon \mathsf{E} \to \mathsf{D}$ a functor of 2-categories. Let $\mathfrak{B}' \to \mathfrak{B}$ be a functor, and let $\pi'\colon \mathfrak{U}' \to \mathfrak{B}'$ be the pullback fibered 2-category. Let $p'\colon \mathsf{D} \to \mathfrak{B}'$ be a functor and $p\colon \mathsf{D} \to \mathfrak{B}$ the composite functor. Suppose that $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is F-complete at p. Then, $\pi'\colon \mathfrak{U}' \to \mathfrak{B}'$ is F-complete at p'*

Proof. Straightforward. $\Box$

Let D and E be 2-categories, and fix a "base" functor $p\colon \mathsf{D} \to \mathfrak{B}$. Let $F\colon \mathsf{E} \to \mathsf{D}$ be a functor and denote $p \circ F$ by q.



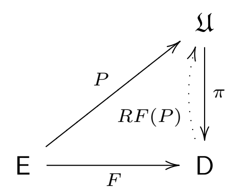
Let $\mathfrak{U}_p^{\mathsf{D}}$ be the 2-category of strict lifts of p to $\mathfrak{U}$. That is, an object in $\mathfrak{U}_p^{\mathsf{D}}$ is a functor $P\colon \mathsf{D} \to \mathfrak{U}$ such that $\pi \circ P = p$. (The latter is an equality, not a natural transformation of functors.) Define $\mathfrak{U}_q^{\mathsf{E}}$ similarly. Note that in the case where $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is fibered in 1-categories $\mathfrak{U}_p^{\mathsf{D}}$ and $\mathfrak{U}_q^{\mathsf{E}}$ are 1-categories.

[5] We make an exception to our notational convention (§2) that Sans Serif symbols stand for 1-categories, because in our application (§7.1) D and E will be 1-categories.

Proposition 14.6 (Relative right Kan extension)**.** *Notation being as in the previous paragraph, suppose that* $\pi\colon \mathfrak{U} \to \mathfrak{B}$ *is* F*-complete at* p *(Definition 14.3). Then, the functor* $F^*\colon \mathfrak{U}_p^{\mathsf{D}} \to \mathfrak{U}_q^{\mathsf{E}}$ *obtained by precomposing with* F *admits a right adjoint* $RF\colon \mathfrak{U}_q^{\mathsf{E}} \to \mathfrak{U}_p^{\mathsf{D}}$.

Proof. Observe that in the case where $\mathfrak{B}$ is the trivial 2-category with one object and $\mathfrak{U}$ is a 1-category, the proposition reduces to the existence of the usual right Kan extension. In fact, the construction of RF is simply the imitation of the one ([Mac], §X). We briefly outline how it is done.

By base extending $\mathfrak{U}$ along p, we may assume that $\mathsf{D} = \mathfrak{B}$, $p = \mathrm{id}$, $q = F$. Fix a functor $P\colon \mathsf{E} \to \mathfrak{U}$ such that $P \circ \pi = F$. The desired right Kan extension $RF(P)\colon \mathsf{D} \to \mathfrak{U}$ of P will then be a section to the projection $\pi\colon \mathfrak{U} \to \mathsf{D}$.



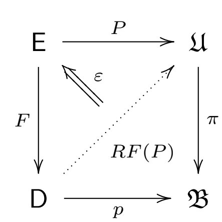
For an object $d \in \mathsf{D}$, define a functor $\Psi_d\colon (d\downarrow \mathsf{E}) \to \mathfrak{U}(d)$ by the rule $\Psi_d(e,\alpha) := \alpha^{\square}(P(e))$. Observe two things: 1) by assumption, $P(e)$ sits above $F(e)$, so it makes sense to pull it back along α; 2) for every (e,α) there is a natural (cartesian) morphism $\eta_{(e,\alpha)}\colon \alpha^{\square}(P(e)) \to P(e)$ in $\mathfrak{U}$ over α. (If you want, this is the definition of the pullback $\alpha^{\square}(P(e))$.)

Define the functor $RF(P)\colon \mathsf{D} \to \mathfrak{U}$ by the rule $d \mapsto \lim \Psi_d$. Note that, by definition, $\lim \Psi_d$ is global (Definition 14.1).

Given a morphism $f\colon a \to b$ in D, the morphism $RF(P)(f)$ in $\mathfrak{U}$ is defined as follows. Let $\underline{\lim \Psi_a}\colon (b \downarrow \mathsf{E}) \to \mathfrak{U}(a)$ be the constant functor with value $\lim \Psi_a$. There is a natural transformation of functors $\underline{\lim \Psi_a} \Rightarrow \Psi_b$ over f induced by the morphisms $\eta_{(e,\alpha)}$ discussed two paragraphs above. Since $\lim \Psi_b$ is a global limit, this transformation induces a natural morphism $\lim \Psi_a \to \lim \Psi_b$ over f. We define $RF(P)(f)$ to be this morphism. It is readily verified that $RF(P)$ is a functor (and, obviously, $RF(P) \circ \pi = \mathrm{id}_{\mathfrak{B}}$).

We leave it to the reader to verify that $RF(P)$ is the desired right Kan extension. □

The right Kan extension $RF(P)$ can be illustrated by the following diagram.



The lower triangle and the big square in this diagram are strictly commutative. The natural transformation ε in the upper triangle is the counit of adjunction.

The following corollary is useful in applications to liftings diagrams of topological stacks.

Corollary 14.7. *Let* $\pi\colon \mathfrak{U} \to \mathfrak{B}$ *be a fibered 2-category,* D *a 2-category, and* $F\colon \mathsf{E} \to \mathsf{D}$ *a functor with* E *discrete (i.e.,* E *has no nontrivial morphisms and 2-morphisms). Suppose that any of the following conditions is satisfied:*

(i) *for every object* b *in* $\mathfrak{B}$*,* $\mathfrak{U}(b)$ *has finite global limits (Definition 14.1), and for every* $d \in \mathsf{D}$ *there are only finitely many morphisms emanating from* d *whose target is in the image of* F*, and there are only finitely many 2-morphisms between such morphisms;*

(ii) *for every object* b *in* $\mathfrak{B}$*,* $\mathfrak{U}(b)$ *has finite global products,* D *is a 1-category, and for every* $d \in \mathsf{D}$ *there are only finitely many morphisms emanating from* d *whose target is in the image of* F*;*

(iii) D *is a 1-category and* $\mathfrak{U}(b)$ *has global products for every object* b *in* $\mathfrak{B}$*.*

Then, for every functor $p\colon \mathsf{D} \to \mathfrak{B}$*, the functor* $F^*\colon \mathfrak{U}_p^{\mathsf{D}} \to \mathfrak{U}_q^{\mathsf{E}}$ *obtained by precomposing by* F *admits a right adjoint* $RF\colon \mathfrak{U}_q^{\mathsf{E}} \to \mathfrak{U}_p^{\mathsf{D}}$*.*

Proof. It is obvious that $\pi\colon \mathfrak{U} \to \mathfrak{B}$ is F-complete for every $p\colon \mathsf{D} \to \mathfrak{B}$. The result follows from Proposition 14.6. □

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