Statistical Physics Approach to Risk in Networked Systems

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NETADIS Scientific Kick-Off Meeting - Torino, Feb 3–6, 2013
Fundamental Problem of Risk Analysis

- **Estimation of risk**
  - **Market:** potential negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices)
  - **Credit:** potential change of credit quality, including default (asset values of firms, ratings, stock-prices)
  - **Operational:** potential losses incurred by process failures (human errors, hardware/software-failures, lack of communication, fraud, external catastrophes)
  - **Liquidity:** potential losses incurred by rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity
Task

- Estimate PDF of
  - Portfolio value at time $T$, $PV(T)$, given $PV(0)$
  - Market value of assets of obligors at time $T$, $A_i(T)$, given $A_i(0)$
  - Losses $L(T)$ due to process failures incurred during risk horizon $T$
  - Losses $L(T)$ incurred during risk horizon $T$ by need to maintain liquidity in situations of stress

- Quantity of interest: Value at Risk

$$\text{VaR}_{q,T} = (Q_q[L(T)] - EL) e^{-rT} \quad \text{Prob}(L(T) \leq Q_q[L(T)]) = q$$
• Quantity of interest: Value at Risk

\[ \text{VaR}_{q,T} = (Q_q[L(T)] - EL) \ e^{-rT} \quad \text{Prob}(L(T) \leq Q_q[L(T)]) = q \]

• Requires good knowledge of tails of loss distributions.
Assessment of PDFs (tails, VaR)

- adjustment of business model, (re)design of processes
  - charged fees, interest rates, rating of clients
  - activities on derivative markets (hedging)
  - insurance policies

- Proper risk control and management is
  - demanded by international banking supervision (BASEL)
  - recognised by rating agencies, analysts
Our Main Interest and Concern: Interactions

- Traditional approaches treat risk elements as independent or at best statistically correlated.

- Misses functional & dynamic nature of relations: terminal–mainframe/input errors–results/manufacturer–supplier relations . . . ⇒ heterogeneous networks of dependencies

- Effect of interactions between risk elements
  - Possibility of avalanches of risk events (process failures, defaults)
  - Fat tails in loss distributions
  - Volatility clustering in markets (intermittency)
The Case of Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes

- Two state model: processes either up and running \((n_i = 0)\) or down \((n_i = 1)\)

- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes (quenched disorder); connectivity functionally defined
  \[\Rightarrow\] (lattice gas) model defined on random graph

- Losses determined (randomly) each time a process goes down (annealed disorder)
Dynamics

- Processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)

- $h_{it}$ total support received by process $i$ at time $t$

$$h_{it} = \vartheta_i - \sum_{j} J_{ij} n_{jt} - \eta_{it}$$

with $\eta_{it}$ random (e.g. Gaussian white noise).

- Process $i$ will fail, if the total support for it falls below a critical threshold

$$n_{it+\Delta t} = \Theta \left( \sum_{j} J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

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Meaning of Parameters

- Probability of failure in given ‘situation’

\[
\text{Prob} \left( n_{it+\Delta t} = 1 \mid n(t) \right) = \Phi \left( \sum_j J_{ij} n_{jt} - \vartheta_i \right)
\]

with

\[
\Phi(x) = \frac{1}{2} \left[ 1 + \text{erf}(x/\sqrt{2}) \right]
\]

- unconditional and conditional probability of failure

\[
\begin{align*}
p_i &= \Phi(-\vartheta_i) \\
p_{i|j} &= \Phi(J_{ij} - \vartheta_i) \\
\Rightarrow \quad \vartheta_i &= -\Phi^{-1}(p_i), \quad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)
\end{align*}
\]
Analysis for a Stochastic Setting

- Interactions on a random graph

\[ J_{ij} = c_{ij} \left( \frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij} \right) \]

with

\[ P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left( 1 - \frac{c}{N} \right) \delta_{c_{ij},0} , \]

symmetric \((c_{ij} = c_{ji})\), and

\[ \overline{x}_{ij} = 0 , \quad \overline{x_{ij}^2} = 1 , \quad \overline{x_{ij}x_{ji}} = \alpha . \]

Study the limit

\[ c \gg 1 , \quad N \gg 1 . \]

Processes/organizations: small \( J \)!
Solution at $\alpha = 0$

- With asymmetric interactions as specified,

$$J_{ij} = c_{ij} \left( \frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij} \right)$$

get large-$c$ Erdős-Renyi random graph ...

- Contributions to interaction-term in dynamic evolution sufficiently weakly correlated s.t. — by LLN and CLT —

$$\sum_j J_{ij} n_{jt} = J_0 m_t + \zeta_t$$

where

$$m_t = \frac{1}{N} \sum_i \langle n_{it} \rangle$$

is the fraction of defaulted firms, and

$$\zeta \sim \mathcal{N}(0, \sigma_t) \quad \text{with} \quad \sigma_t^2 = J^2 m_t$$
Results OR — Stationary Solution at $\alpha = 0$

- $\alpha = 0$ (asymmetric network), no memory effects

$$m_t = \frac{1}{N} \sum_i n_{it}$$

sufficient to describe the dynamics

- Get closed form expression for macroscopic dynamics

$$m_{t+1} = \left\langle \Phi \left( \frac{J_0 m_t - \vartheta}{\sqrt{1 + J^2 m_t}} \right) \right\rangle \vartheta$$

(K Anand and RK, Phys Rev E75 (2007))

Stationary fraction of down-processes and phase diagram for $p_i = 0.001$, $\alpha = 0$, $J = 0.2$
Results OR — Stationary Solution at $\alpha \neq 0$

- For $\alpha \neq 0 \iff$ (partially) symmetric couplings: have to work much harder (Generating functional techniques)

Stationary fraction of down-processes for $\alpha = 0.5$ (left) $\alpha = 1.0$ (right) at $J = 0.2$

(K Anand and RK, Phys Rev E75 (2007))
Key Features

• For sufficiently strong cooperative interactions, get first order phase transition: coexistence of ‘functioning state’ and state of ‘catastrophic breakdown’ (dominoes)

• Critical point at sufficiently high $p_i$.

• Resilience to (external) stress can be tested.

• No precursors of phase-transition; no signatures of entering metastable regime.
The Case of Credit Risks — Interacting Companies

- Risk arising from the possibility of obligors going bankrupt or from changes in ‘credit quality’ (⇒ credit trading)

- Here only influence of defaults

- Two state model: company up and running ($n_i = 0$) or down ($n_i = 1$)

- Probabilities of default and mutual impacts of defaults heterogeneous across the set of companies (quenched disorder); connectivity functionally defined

  ⇒ (lattice gas) model defined on random graph

- Losses determined (randomly: recovery process) when a company defaults (annealed disorder)
Dynamics

- Companies need “orders” (support, cash inflow) to maintain wealth and avoid default

- $h_{it}$ total wealth of company $i$ at time $t$, 
  $$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$
  with $\eta_{it}$ random (e.g. Gaussian white noise).

- Company $i$ defaults, if the total wealth falls below zero
  $$n_{it+1} = n_{it} + (1 - n_{it}) \Theta \left( \sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

- No recovery within ‘risk horizon’ $T$: $n_i = 1$ is absorbing state. Time unit: 1 month; $T = 12 \Leftrightarrow 1$ year.
Macroscopic Dynamics

- Random network setting as for OR

- Decompose stochastic force (minimal Basel II)
  \[ \eta_{it} = \sqrt{\rho} \eta_0 + \sqrt{1 - \rho} \xi_{it} \]
  Global component \( \eta_0 \) slowly varying (e.g. fixed over risk horizon)

- CR \( (n_{it} = 1 \text{ absorbing}) \Rightarrow \text{no memory effects even at } \alpha \neq 0 \) &
  closed form evolution equations for defaulted fraction \( m_t \).

\[
\langle n_{t+1} \rangle = \langle n_t \rangle + (1 - \langle n_t \rangle) \Phi \left( \frac{J_0 m_t + \sqrt{\rho} \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right)
\]

\[
m_{t+1} = m_t + \left( (1 - \langle n_t \rangle) \Phi \left( \frac{J_0 m_t + \sqrt{\rho} \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right) \right) \vartheta
\]
Results CR — Defaulted Fraction and Loss Distribution

- Loss distribution driven by macro-economic noise $\eta_0$

\begin{align*}
\text{Fraction of defaulted firms for neutral macro-economic conditions } &\eta_0 = 0 \text{ at } (J_0, J) = (0, 0), (1, 0), (0, 1) \\
\text{and } (1, 1) \text{ (left, bottom to top); Loss distribution for a system with } & (J_0, J) = (0, 0) \text{ and } (1, 1) \text{ and } \\
\ell(\psi) = 1/(\epsilon + p_d(\psi)) \text{ (right)}
\end{align*}


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Results CR — Value at Risk

![Graphs showing ratios of value at risk and average losses for systems with and without functional interaction.](image)


Ratios of value at risk (upper curves) and average losses (lower curves) for systems with and without functional interaction evaluated along straight lines in the $J_0$-$J$ plane, with $R = \sqrt{J_0^2 + J^2}$. Left: $J_0 = 0$; right: $J = 0$; lower: $J_0/J = 1$. 

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Key Features

- Interpretation of parameters $\vartheta_i$ and $J_{ij}$ in terms of unconditional and conditional default probabilities

- For sufficiently strong interactions — in particular cooperative interactions — get possibility of collective acceleration of default rates in times of economic stress

- Typical behaviour less sensitive to interactions than rare events $\Rightarrow$ fat tails in loss distribution; important for risk analysis where rare event asymptotics is relevant.

- Due to initial conditions $n_{i0} \equiv 0$ and absorbing state: $\Rightarrow$ no equilibrium dynamics
Including CDS

- Credit defaults clustered around times of economic stress
  - dependency on macro-economic factors
  - credit contagion (see credit risk model above)

- Contagion dynamics radically changed in the last decade through Credit Default Swaps (CDS); AIG, tipping pint of past financial crisis

- Yet role of CDS in contagion in the past only looked at in the context of pricing individual products

- CDS introduce new 3 particle interactions, and can contribute to systemic instability ⇒ Talk Pierre Paga
Functional Dependencies in Market Risk

• Standard approach: Geometric Brownian motion (GBM) for Log-returns of risk elements $i$ (stocks, bonds . . .)

\[
\frac{1}{S_i(t)} \frac{dS_i(t)}{dt} = \mu_i + \sigma_i \eta_i(t)
\]

• Statistical dependencies via correlated Gaussian noises

\[
\langle \eta_i(t) \eta_j(t') \rangle = \rho_{ij} \delta(t - t')
\]

• ⇒ no functional dependencies, no collective behaviour, no market bubbles, crashes
Functional Dependencies in Market Risk — iGBM

- Functional dependencies between risk elements e.g. via recommendations of analysts, economic dependencies.

- GBM in terms of $h_i(t) = \log(S_i(t)/S_{i0})$

  \[ \frac{dh_i(t)}{dt} = \mu_i - \frac{1}{2}\sigma_i^2 + \sigma_i\eta_i(t) \]

- Minimal interacting generalisation (iGBM): stabilisation and non-linear feedback (e.g. $g(x) = \tanh(x)$)

  \[ \frac{dh_i(t)}{dt} = -\kappa h_i(t) + \mu_i - \frac{1}{2}\sigma_i^2 + \sum_j J_{ij} g(h_j(t)) + \sigma_i\eta_i(t) \]

- $\Leftrightarrow$ dynamics of graded-response neurons
  - many meta-stable states; transitions between them $\Rightarrow$ intermittent dynamics.
Functional Dependencies in Market Risk – iGBM

- Solve in stochastic setting using GFA as for OR/CR
- Here MC study; trigger transitions by ‘unexpected news’

Change of index $I(t) = N^{-1}\sum_i S_i(t)$ over time increment $\tau = 25$ (left) and normalised distribution of log-returns for $\tau = 25$, and $\tau = 50$ (right).

(RK and P Neu, J Phys A41 (2008))
Summary

- Standard risk models based on statistically correlated risk elements miss dynamically generated functional correlations

- Interacting processes capture functional dependencies in OR.

- Similarly: economic interactions $\Rightarrow$ credit contagion in CR

- Physics analogy: lattice gas model on (random) graph.

- Describes bursts and avalanches of risk events

- Important: coexistence with phases of catastrophic breakdown — no noticeable precursors!

- Analogue Neuron model for MR — intermittent dynamics, fat tails.
Dynamical Mean-field Analysis — Generating Functions

- Generating function for correlation functions

\[ Z[\psi] = \langle e^{-i \sum_{t} \psi_t n_{it}} \rangle = \sum_{n_0, \ldots, n_T} P[n_0, \ldots, n_T] e^{-i \sum_{t} \psi_t n_{it}} \]

- Correlation functions for typical disorder (\(\Leftrightarrow\) averaged correlation functions)

\[ \langle n_{it} \rangle = i \frac{\partial Z[\psi]}{\partial \psi_{it}} \bigg|_{\psi \equiv 0} , \quad \langle n_{is} n_{jt} \rangle = i^2 \frac{\partial^2 Z[\psi]}{\partial \psi_{is} \partial \psi_{jt}} \bigg|_{\psi \equiv 0} \]

- Credit risk: decompose stochastic force

\[ \eta_{it} = \sqrt{\rho} \ \eta_0 + \sqrt{1 - \rho} \ \xi_{it} \]

Global component \(\eta_0\) slowly varying (e.g. fixed over risk horizon of one year); OR: \(\rho = 0\)
Generating Functions — Solution

- System represented by ensemble of effective single site processes parameterised by $\vartheta$

  - OR ($\rho = 0$)
    \[ n_{t+1} = \Theta(J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s - \vartheta + \phi_t) \]

  - CR ($\rho \neq 0$, $n_{it} = 1$ absorbing)
    \[ n_{t+1} = n_t + (1 - n_t) \Theta(J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s + \sqrt{\rho} \eta_0 - \vartheta + \phi_t) \]

- Single-site processes exhibit

  coloured noise \( \{\phi_t\} \), and memory \( G_{ts} \)
Generating Functions — Self-Consistency

• Self-consistency equations

\[
\langle \phi_s \phi_t \rangle = (1 - \rho) \delta_{st} + J^2 q_{st}
\]
\[
m_t = \langle \langle n_t \rangle \rangle
\]
\[
q_{st} = \langle \langle n_s n_t \rangle \rangle
\]
\[
G_{ts} = \langle \frac{d \langle n_t \rangle}{dh_s} \rangle
\]

• Interpretation of variables

\[
m_t = \frac{1}{N} \sum_i \langle n_{it} \rangle
\]
\[
q_{st} = \frac{1}{N} \sum_i \langle n_{is} n_{it} \rangle, \quad G_{ts} = \frac{1}{N} \sum_i \frac{d}{dh_s} \langle n_{it} \rangle
\]
Results CR — Defaulted Fraction and Loss Distribution

- Recall effective single node process:

\[ n_{t+1} = n_t + (1 - n_t) \Theta \left( J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s + \sqrt{\rho \eta_0} - \vartheta + \phi_t \right) \]

- Memory term vanishes as long as \( n_t = 0 \), and becomes irrelevant, once \( n_t = 1 \).

- Can use \( \alpha = 0 \) theory:

\[ m_{t+1} = m_t + \left\langle \left( 1 - \langle n_t \rangle \right) \Phi \left( \frac{J_0 m_t + \sqrt{\rho \eta_0} - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right) \right\rangle_{\vartheta} \]