Systemic Risk and the Mathematics of Falling Dominoes

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Outline

1. The Laws of Falling Dominoes
2. Risk and Falling Dominoes
3. Fundamental Problems of Risk Analysis
   - Main Interest and Concern: Interactions
4. Operational Risks — Interacting Processes
   - Dynamics – Mathematics of Falling Dominoes
   - A Simple Homogeneous Process Network
5. Summary
The Laws of Falling Dominoes

- A domino falls, if kicked sufficiently vigorously.
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○ A domino can be toppled by another domino.
The Laws of Falling Dominoes

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- A domino can be toppled by another domino.
- Avalanches can occur, if dominoes are set too closely.
Risk and Falling Dominoes

Operational Risk

Domino Theory & Spread of Communism

Blackouts in Power Grids

Financial Crisis
Fundamental Problem of Risk Analysis

Estimate likelihood and potential losses due to

- negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices) ↔ market risk
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- rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity \(\leftrightarrow\) **liquidity risk**
Main Interest and Concern: Interactions

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Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes
- Two state model: processes either up and running \((n_i = 0)\) or down \((n_i = 1)\)
- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes; connectivity & concept of neighbourhood functionally defined

\[ \Rightarrow \text{model defined on random graph} \]

- Losses determined (randomly) each time a process goes down
Dynamics – Mathematics of Falling Dominoes

- Processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)

- \( h_{it} \) total support received by process \( i \) at time \( t \)

\[
h_{it} = h_i^* - \sum_j J_{ij} n_{jt} + x_{it}
\]

- \( h_i^* \) support in fully functional environment
- \( J_{ij} \) support to process \( i \) provided by process \( j \)
- \( x_{it} \) random (e.g. Gaussian white noise).

- Process \( i \) will fail, if the total support for it falls below a critical threshold (if \( h_{it} \leq 0 \) – domino falls, if kicked too strongly)

\[
n_{it+\Delta t} = \Theta \left( \sum_j J_{ij} n_{jt} - h_i^* - x_{it} \right)
\]
Probability that a Domino Falls

- Probability of failure/probability of domino falling

\[
\text{Prob}(n_{i(t+\Delta t)} = 1|\mathbf{n}(t)) = \int_{-\infty}^{\sum_j J_{ij} n_{jt} - h_i^*} dx \, p(x) \equiv \Phi\left(\sum_j J_{ij} n_{jt} - h_i^*\right)
\]

- unconditional and conditional probability of failure

\[
p_i = \Phi(-h_i^*)
\]
\[
p_i|j = \Phi\left(J_{ij} - h_i^*\right)
\]
A Simple Homogeneous Process Network

- Large system $1 \leq i \leq N$, $(N \gg 1)$, with all-to-all couplings, and $h_i^* = h^*$ independent of $i$.

$$J_{ij} = \frac{J_0}{N}, \quad \forall (i, j) \Rightarrow \sum_j J_{ij} n_{jt} = \frac{J_0}{N} \sum_j n_{jt} = J_0 m_t$$

- Dynamics

$$n_{it + \Delta t} = \Theta \left( \sum_j J_{ij} n_{jt} - h_i^* - x_{it} \right) = \Theta \left( J_0 m_t - h^* - x_{it} \right).$$

Thus by Law of Large Numbers (LLN)

$$m_{t + \Delta t} = \frac{1}{N} \sum_{i=1}^{N} \Theta \left( J_0 m_t - h^* - x_{it} \right) \simeq \Phi \left( J_0 m_t - h^* \right)$$
Analysis of the Dynamics

- Iterated function dynamics

\[ m_{t+\Delta t} = \Phi(J_0 m_t - h^*) \]
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- For not too small values of \( h^* \) can vary between lower and upper curve in above diagram (i.e. between system with only low-\( m \), via system with coexisting low-\( m \) and high-\( m \) states, to system with only high-\( m \) states) by increasing \( J_0 \). For small \( h^* \) have only high-\( m \) state.
Summary

- We found that networks can be destabilized by large degrees of interdependency (large $J_0$) even if all processes are very reliable (with large $h^*$).
- For intermediate levels of dependency (intermediate $J_0$), functioning and dysfunctional states of the system coexist.
- (Not shown): In systems with finite $N$, a functioning state can spontaneously switch to the dysfunctional state (without an apparent 'big' perturbation.)
- Results qualitatively unchanged for heterogeneous networks (not all-to-all interactions, heterogeneous levels of reliability, heterogeneous mutual dependency)
- Similar methods for credit risk.