DIRECT SIMILARITIES

Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = az$, where a is a non-zero constant. Let $a = re^{i\theta}$, the polar form. Two special cases arise, when $r = 1$ and when $\theta = 0$:

1. If $a = e^{i\theta}$, then f is rotation about 0 through the angle θ.

2. If $a = r$, then f is enlargement about 0, with scale-factor r.

In the general case, f is the composite of a rotation and an enlargement about 0, and such a map is called a *spiral similarity* about 0.

More generally, a spiral similarity f about w is given by

$$f(z) - w = a(z - w),$$

and such a map is of the form $f(z) = az + b$.

Conversely, the map $f(z) = az + b$ (where a, $b \in \mathbb{C}$, $a \neq 0$) is a translation if $a = 1$, and if $a \neq 1$ then it has a unique fixed point w given by solving

$$w = aw + b,$$

so that

$$w = \frac{b}{1 - a},$$

and then

$$f(z) - w = a(z - w),$$

so we have a spiral similarity about w.

Now suppose that $f(z) = az + b$ and $g(z) = cz + d$ are two spiral similarities. We have

$$\begin{aligned} fg(z) &= a(cz+d)+b = (ac)z + (ad+b), \quad \text{and} \\ gf(z) &= c(az+b)+d = (ca)z + (cb+d). \end{aligned}$$

From this, $fg(z) - gf(z)$ is independent of z, so that if $fg(z_0) = gf(z_0)$ for **some** $z_0 \in \mathbb{C}$, then $fg(z) = gf(z)$ for **all** $z \in \mathbb{C}$, that is, $fg = gf$.

Suppose now that $f(w) = w$ and $g(w) = w'$, and $fg = gf$. Then we claim $w = w'$, that is, $g(w) = w$, so that f and g have the same fixed point. For

$$f(w') = fg(w) = gf(w) = g(w) = w';$$

but w is the *unique* fixed point of f, so $w' = w$, as claimed.

Now suppose we have a quadrilateral $ABCD$; if this is not a parallelogram, then there are spiral similarities f, g, with

$$f(A) = B, \quad f(D) = C, \quad g(A) = D, \quad \text{and} \quad g(B) = C.$$

But now $fg(A) = f(D) = C$ and $gf(A) = g(B) = C$, so that $fg = gf$, and therefore f and g have the same fixed point, O.

Where is O? If one of the lines AB, BC, CD, DA is omitted, the remaining three form a triangle, and it has a circumcircle. By doing this all four possible ways, we obtain four circles, and it is not hard to show that these meet at O, the *Miquel point* of $ABCD$.

For suppose AD, BC meet at P, and suppose $f(z) = re^{i\theta}z + b$.

Since $f(O) = O$ and $f(A) = B$, we have $\angle AOB = \theta$.

Then f sends the line AD to the line BC, whence $\angle APB = \theta$ also.

We deduce that $ABPO$ is cyclic, that is, O lies on the circumcircle of $\triangle ABP$.

Similarly, since $f(D) = C$, O lies on the circumcircle of $\triangle CDP$.

And, if AB and CD meet at Q, then a similar argument applied to g shows that O lies on the circumcircles of $\triangle ADQ$ and of $\triangle BCQ$.