SEMIREGULAR TILINGS

The external angles of a polygon add up to 360°.

So each external angle of a regular n-gon is $\frac{360}{n}$, and thus each internal angle is

$$f(n) = 180 - \frac{360}{n} = 360\left(\frac{1}{2} - \frac{1}{n}\right).$$

Here are the first few values of f:

$n =$	3	4	5	6	7	8	9	10
$f(n) =$	60°	90°	108°	120°	$128\frac{4}{7}^\circ$	135°	140°	144°

If a regular p-gon, a q-gon, an r-gon, ... (and so on) fit together at a point, we must have

$$f(p) + f(q) + f(r) + \ldots = 360.$$

We may as well assume $p \leq q \leq r \leq \ldots$, and also $3 \leq p$, so

$$60 \leq f(p) \leq f(q) \leq f(r) \leq \ldots.$$

If there are n terms (that is, n polygons fitting together), then

$$60n \leq f(p) + f(q) + f(r) + \ldots = 360,$$

so $n \leq \frac{360}{60} = 6$.

Further, if $n = 6$, then all six of the angles must be 60°, and all six of the polygons must be equilateral triangles. This is our first answer: the regular tiling (3,3,3,3,3,3).

Suppose $n = 5$. So we have $p \leq q \leq r \leq s \leq t$, and

$$f(p) + f(q) + f(r) + f(s) + f(t) = 360.$$

Now $\frac{360}{5} = 72 < 90 = f(4)$, so $p < 4$, and we must have $p = 3$.

Then $360 - f(p) = 300$, and $\frac{300}{4} = 75 < 90$, so likewise $q = 3$.

Next, $300 - f(q) = 240$, and $\frac{240}{3} = 80 < 90$, so $r = 3$ as well.

Finally, $240 - f(r) = 180$, and $\frac{180}{2} = 90$, so $s = 3$ or 4, giving $180 - f(s) = 120$ or 90, and so $t = 6$ or 4, respectively.

We thus obtain two answers when $n = 5$, namely (3,3,3,3,6) and (3,3,3,4,4).

We are left to deal with $n = 4$ and $n = 3$. We'll do $n = 3$ first. So we have $p \le q \le r$ and $f(p) + f(q) + f(r) = 360$, that is,

$$360\left(\frac{1}{2} - \frac{1}{p}\right) + 360\left(\frac{1}{2} - \frac{1}{q}\right) + 360\left(\frac{1}{2} - \frac{1}{r}\right) = 360.$$

Dividing by 360 and rearranging,

$$\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = 1,$$

or

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Now $\frac{360}{3} = 120 = f(6)$, so $p \le 6$. We shall deal with the cases $p = 3$, 4, 5, 6 separately.

First, $p = 3$. So $\frac{1}{2} = \frac{1}{3} + \frac{1}{q} + \frac{1}{r}$. Now $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $\frac{1}{6} = \frac{1}{q} + \frac{1}{r}$. Multiply each side by 6qr to get

$$qr = 6q + 6r, \quad \text{or} \quad qr - 6q - 6r = 0.$$

Adding 36 to each side and factorising, we have

$$(q - 6)(r - 6) = 36.$$

Noting that the bracketed terms are whole numbers, and that $q - 6 \leq r - 6$, we obtain

$q - 6$:	1	2	3	4	6
$r - 6$:	36	18	12	9	6
q :	7	8	9	10	12
r :	42	24	18	15	12

So this yields five more answers:

$$(3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), \text{ and } (3, 12, 12).$$

Next, $p = 4$. So $\frac{1}{2} = \frac{1}{4} + \frac{1}{q} + \frac{1}{r}$. Now $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $\frac{1}{4} = \frac{1}{q} + \frac{1}{r}$. Multiply each side by 4qr to get

$$qr = 4q + 4r, \quad \text{or} \quad qr - 4q - 4r = 0.$$

Adding 16 to each side and factorising, we have

$$(q - 4)(r - 4) = 16.$$

Noting that the bracketed terms are whole numbers, and that $q - 4 \leq r - 4$, we obtain

$q - 4$:	1	2	4
$r - 4$:	16	8	4
q :	5	6	8
r :	20	12	8

So this yields another three answers:

$$(4, 5, 20), (4, 6, 12), \text{ and } (4, 8, 8).$$

The case $p = 5$ is a little trickier. We have $\frac{1}{2} = \frac{1}{5} + \frac{1}{q} + \frac{1}{r}$. Then $\frac{1}{2} - \frac{1}{5} = \frac{3}{10}$, so $\frac{3}{10} = \frac{1}{q} + \frac{1}{r}$. Multiply each side by $10qr$ to get

$$3qr = 10q + 10r, \quad \text{or} \quad 3qr - 10q - 10r = 0.$$

To make this factorise nicely, we must multiply by another 3 and then add 100 to each side, giving

$$(3q - 10)(3r - 10) = 100.$$

$3q - 10$:	1	2	4	5	10
$3r - 10$:	100	50	25	20	10
$3q$:	11	12	14	15	20
$3r$:	110	60	35	30	20
q :	–	4	–	5	–
r :	–	20	–	10	–

Discarding the fractional answers, and remembering also that $q \geq p \geq 5$, we obtain only one new answer: (5,5,10).

Finally (for $n = 3$) the case $p = 6$. We have $\frac{1}{2} = \frac{1}{6} + \frac{1}{q} + \frac{1}{r}$. Now $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$, so $\frac{1}{3} = \frac{1}{q} + \frac{1}{r}$. Multiply each side by 3qr to get

$$qr = 3q + 3r, \quad \text{or} \quad qr - 3q - 3r = 0.$$

Adding 9 to each side and factorising, we have

$$(q - 3)(r - 3) = 9.$$

Noting that the bracketed terms are whole numbers, and that $q - 3 \leq r - 3$, we obtain

$q - 4$:	1	3
$r - 4$:	9	3
q :	4	6
r :	12	6

Remembering that $q \geq p \geq 6$, we get only one new answer:

$$(6, 6, 6).$$

We have now dealt with $n = 3$, 5 and 6, so we finish by dealing with $n = 4$. So $p \le q \le r \le s$ and $f(p) + f(q) + f(r) + f(s) = 360$, that is,

$$360\left(\frac{1}{2} - \frac{1}{p}\right) + 360\left(\frac{1}{2} - \frac{1}{q}\right) + 360\left(\frac{1}{2} - \frac{1}{r}\right) + 360\left(\frac{1}{2} - \frac{1}{s}\right) = 360.$$

Dividing by 360 and rearranging,

$$\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}\right) = 1,$$

or

$$1 = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}.$$

Since $\frac{360}{4} = 90 = f(4)$, we have $p \le 4$, that is, $p = 3$ or 4.

First take $p = 3$. Then $360 - f(p) = 300$, and $\frac{300}{3} = 100 < f(5)$, so $q < 5$, that is, $q = 3$ or 4.

If $q = 3$, then $1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{r} + \frac{1}{s}$. Now $1 - \frac{2}{3} = \frac{1}{3}$, so $\frac{1}{3} = \frac{1}{r} + \frac{1}{s}$. Thus $rs - 3r - 3s = 0$, or $(r - 3)(s - 3) = 9 = 1 \times 9$ or 3×3, giving the answers (3,3,4,12) and (3,3,6,6) respectively.

If $q = 4$, then $1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{r} + \frac{1}{s}$. Now $1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$, so $\frac{5}{12} = \frac{1}{r} + \frac{1}{s}$. Thus $5rs - 12r - 12s = 0$, or $(5r - 12)(5s - 12) = 144$. This means $5r - 12 = 1$, 2, 3, 4, 6, 8, or 12, so $5r = 13$, 14, 15, 16, 20 or 24. Since $r \geq q$, the only new answer we find is (3,4,4,6).

Finally, if $p = 4$, then $360 = f(p) + f(q) + f(r) + f(s) \geq 4f(p) = 360$, so $p = q = r = s = 4$ and we obtain the answer (4,4,4,4).

OK?

We have found 17 answers. Here is a list to show which actually give rise to tilings:

(3,7,42):	No	(6,6,6):	Yes
(3,8,24):	No	(3,3,4,12):	No, and (3,4,3,12): No
(3,9,18):	No	(3,3,6,6):	No, but (3,6,3,6): Yes
(3,10,15):	No	(3,4,4,6):	No, but (3,4,6,4): Yes
(3,12,12):	Yes	(4,4,4,4):	Yes
(4,5,20):	No	(3,3,3,3,6):	Yes (RH and LH versions)
(4,6,12):	Yes	(3,3,3,4,4):	Yes, and also (3,3,4,3,4): Yes
(4,8,8):	Yes	(3,3,3,3,3,3):	Yes
(5,5,10):	No		

(Note that the answer is "No" whenever any of the numbers is odd, and with that number removed, what remains contains no repetitions. Exercise: explain why this must be so.)

To sum up:

We have three regular tilings: (6,6,6), (4,4,4,4), and (3,3,3,3,3,3).

We also have eight semi-regular tilings: (3,12,12), (4,6,12), (4,8,8), (3,6,3,6), (3,4,6,4), (3,3,3,3,6), (3,3,3,4,4), and (3,3,4,3,4).

An alternative approach is to use a computer to search for solutions to the equation

$$f(p) + f(q) + f(r) + \ldots = 360. \qquad (1)$$

In order to complete the search in a reasonable time, we need to work out suitable ranges of values of p, q, ... So first we assume $3 \leq p \leq q \leq r \leq \ldots$, as before, and note (as before) that since the smallest value of f is 60° and $\frac{360}{60} = 6$, we need only look at cases of (1) with up to six terms on the left. Now recall that $f(p) = 360(\frac{1}{2} - \frac{1}{p})$ and notice that $f(2) = 0$, that is, the vertex angle of a regular 2-gon is 0°! It makes the search tidier if we always take *precisely* six terms,

$$f(p) + f(q) + f(r) + f(s) + f(t) + f(u) = 360,$$

but with $2 \leq p \leq q \leq r \leq s \leq t \leq u$. So, for example, the previous solution (3,12,12) will now appear as (2,2,2,3,12,12).

We need upper bounds for p, q, $\ldots$. Since $\frac{360}{6} = 60 = f(3)$, we must have $p \leq 3$.

For the largest value of q, we take $p = 2$ and note that $\frac{360}{5} = 72 < f(4)$, so $q < 4$. Thus $q \leq 3$.

Next, for the largest r, we take $p = q = 2$ and observe that $\frac{360}{4} = 90 = f(4)$, from which $r \leq 4$.

Then (similarly) $\frac{360}{3} = 120 = f(6)$, so $s \leq 6$.

This argument won't work for t, because $\frac{360}{2} = 180 > f(t)$ for all t. We must look instead for the smallest *non-zero* value of $f(p) + f(q) + f(r) + f(s)$, and this is given by $(p, q, r, s) = (2, 2, 2, 3)$. We then have $f(t) + f(u) = 360 - 60 = 300$, so $f(t) \leq \frac{300}{2} = 150 = f(12)$, and thus $t \leq 12$.

Finally, we need an upper bound for u. We have

$$f(u) = 360 - f(p) - f(q) - f(r) - f(s) - f(t),$$

so we need the *least* value of $f(p) + f(q) + f(r) + f(s) + f(t)$ that *exceeds* 180°. (If it didn't exceed 180°, it wouldn't place any limit on u.)

Clearly we take $p = q = r = 2$. (OK?) Then $f(3) + f(6) = 180$ and also $f(4) + f(4) = 180$, so we look at $f(3) + f(7)$, $f(4)+f(5)$, and no more. Now $f(3)+f(7) = 60+128\frac{4}{7} = 188\frac{4}{7}$, and $f(4)+f(5) = 90+108 = 198$, so we take the former. Finally, $360 - 188\frac{4}{7} = 171\frac{3}{7} = f(42)$ (or, if you prefer, $\frac{1}{2} - \frac{1}{3} - \frac{1}{7} = \frac{1}{42}$), so we deduce that $u \leq 42$.