Painting by Numbers
(or: *Polyominoes Revisited*)

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In his book *Polyominoes* Solomon Golomb gives a colouring proof that it is impossible to tile a chess board with twenty-one 3×1 rectangles and one 1×1 square unless the 1×1 square is in one of four particular positions. In note 73.28 (Mathematical Gazette 73, No. 465, 1989) Nick MacKinnon gives a proof of the same result by an algebraic argument, involving writing a complex number in each square of the chess board and solving a pair of simultaneous congruences. Here we give an argument that combines features of both the above proofs, is shorter and more transparent than either, and involves nothing beyond integer arithmetic. It also extends to some more general problems of tiling with 3×1 rectangles.

We shall follow Golomb and refer to a 3×1 rectangle as a *straight tromino*, though no ambiguity will be caused here by omitting the word *straight*; and a 1×1 square is a *monomino*. The problem, then, is to cover an 8×8 board with twenty-one trominoes and one monomino, and ask where we can place the monomino. Golomb does this by a "patriotic" colouring of the chess board, in which each of the 64 small squares is coloured red, white or blue, in such a way that each tromino, no matter where it is placed, must cover one square of each colour. (See figure 1, where, for reasons that will emerge, I have not quite followed Golomb's colouring.)

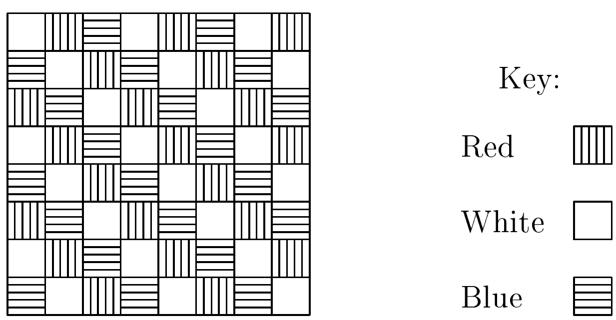

Figure 1: Patriotic colouring

Golomb now counts the colours: in my version of the diagram, there are 21 red, 22 white and 21 blue squares, so the monomino must be on a white square. (It is not necessary actually to count the squares: just observe that the last six columns obviously contain equal numbers of red, white and blue squares, and likewise the last six rows of the first two columns. This is because each of these regions can be tiled completely with trominoes. So now concentrate on the remaining top left-hand 2×2 square, which contains one red square, two white squares and one blue square.) The argument then proceeds as follows: if the monomino were on the white square in the top left-hand corner, then a reflection or rotation of the tiling used could put it at the top right-hand corner. But that corner is not white, a contradiction. Similarly for most of the other white squares, *except* the four whose symmetrically related positions are also white. This argument can be shortened considerably, and made more vivid, by taking the reflected colouring (figure 2) and placing it on top of the first colouring, as in figure 3. Only the four squares that are white in both patterns can be monomino positions in our tiling.

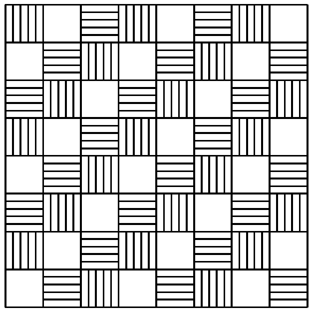

Figure 2: Reflection

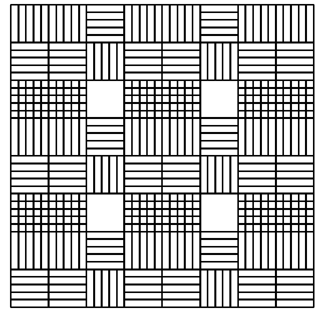

Figure 3: Superposition

The result is most dramatically demonstrated by making two OHP slides of figure 1, placing one on top of the other, and then rotating one of them through a right angle. (To complete Golomb's theorem, one now has to find tilings that place a monomino in each of the four special squares, but that is an easy exercise for the reader.)

The algebraic method used by Nick MacKinnon puts complex numbers in the squares. So, in each white square in figure 1, put the number 1; in each red square put the number $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$; and in each blue square put the number $\omega^2 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$. Since $1+\omega+\omega^2 = 0$, each tromino, regardless of where it is placed, covers numbers whose sum is zero. It is easy to see that the total of all the numbers is 1. MacKinnon has a rather involved argument with polynomials in two variables for this, but all that is needed is to observe, much as before, that the last six columns clearly contain numbers with zero sum, as do the last six rows of the first two columns; as before, this is because each of these regions can be tiled completely with trominoes. So now we need only sum the numbers in the top left hand 2×2 square, giving $1 + \omega + \omega^2 + 1 = 1$. Thus the monomino must cover a 1. If we use matrix notation $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ for the numbers in the squares in fig. 1, fig. 2 respectively, then $\mathbf{A}$, $\mathbf{B}$ are, respectively,

$$\begin{pmatrix} 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\ \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\ \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\ \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 \end{pmatrix}.$$

So $a_{ij} = 1$ if and only if $i - j \equiv 0 \pmod 3$, and $b_{ij} = 1$ if and only if $i + j \equiv 0 \pmod 3$. These simultaneous congruences immediately give $i \equiv j \equiv 0 \pmod 3$, so that the monomino must be in position $(3,3)$, $(3,6)$, $(6,3)$ or $(6,6)$, cf. fig. 3. (MacKinnon numbers his rows and columns 0–7 rather than 1–8, so he has slightly harder congruences to solve.)

The above method is based on three numbers 1, ω, ω^2 whose sum is zero, and which are then put in the squares of the chess board so that wherever a tromino is placed it covers numbers with zero sum. In fact, any three numbers with zero sum will do equally well, so for example if we use 0, 1, -1 in place of 1, ω, ω^2 we get

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \end{pmatrix}$$

which has a total sum of zero, so that the monomino must cover a zero. To complete the argument this way, one would also need the corresponding matrix **B**, but with more cunning choice of entries this can be avoided. So instead consider the matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -3 & 2 & 1 & -3 & 2 & 1 & -3 \\ -2 & 2 & 0 & -2 & 2 & 0 & -2 & 2 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -3 & 2 & 1 & -3 & 2 & 1 & -3 \\ -2 & 2 & 0 & -2 & 2 & 0 & -2 & 2 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -3 & 2 & 1 & -3 & 2 & 1 & -3 \end{pmatrix}.$$

Here again each tromino covers numbers with zero sum, and the total sum is zero. Thus the monomino must cover a zero, but this time there are *precisely* four such: $c_{ij} = 0$ if and only if $i \equiv j \equiv 0 \pmod 3$. It is hard to see that this argument could be further shortened!

(There *is* a short direct proof. Notate the square in row i and column j by the pair (i, j), and observe that if we add up all 64 pairs (i, j) we get $(288, 288)$. On the other hand, if a tromino has its middle square at (i, j), then the three squares it occupies have pairs adding up to $(3i, 3j)$. So if we add up the pairs for all 63 squares covered by trominoes, we must get $(3r, 3s)$ for some r and s, and this means that the monomino must be in position $(288 - 3r, 288 - 3s)$; but $288 - 3r$ and $288 - 3s$ are both divisible by 3.)

Generalisation is now easy. The pattern of **C** applies equally well to a square of size $(3n+2) \times (3n+2)$, or indeed to a rectangle of size $(3n+2) \times (3m+2)$, any $n \geq 1$, $m \geq 1$, and shows that the (single) monomino is at the (i, j) position where $i \equiv j \equiv 0 \pmod 3$. For a rectangle (possibly a square) of size $(3n+1) \times (3m+1)$, any $n \geq 1$, $m \geq 1$, we use the same pattern but starting in a different place. For example, for the 7×7 case we use

$$\mathbf{D} = \begin{pmatrix} 0 & -2 & 2 & 0 & -2 & 2 & 0 \\ -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 2 & 1 & -3 & 2 & 1 & -3 & 2 \\ 0 & -2 & 2 & 0 & -2 & 2 & 0 \\ -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 2 & 1 & -3 & 2 & 1 & -3 & 2 \\ 0 & -2 & 2 & 0 & -2 & 2 & 0 \end{pmatrix},$$

which again has total sum zero. So the monomino is at (i, j) where $i \equiv j \equiv 1 \pmod 3$.

Of course, a rectangle (or square) where either side is of length a multiple of 3 can be covered completely by trominoes. This leaves one case unanswered: that of a rectangle of size $(3n + 1) \times (3m + 2)$, for some $n \geq 1$, $m \geq 1$. Here the number of squares to be covered is $(3n+1)(3m+2) = 3(3nm+2n+m)+2$, so we need $3nm+2n+m$ trominoes, and *two* squares will be then left uncovered. We shall cover these two squares with a single *domino*, a 2×1 rectangle. One case will suffice: the 7×8 rectangle. The matrix to use is

$$\mathbf{E} = \begin{pmatrix} -2 & 2 & 0 & -2 & 2 & 0 & -2 & 2 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -3 & 2 & 1 & -3 & 2 & 1 & -3 \\ -2 & 2 & 0 & -2 & 2 & 0 & -2 & 2 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -3 & 2 & 1 & -3 & 2 & 1 & -3 \\ -2 & 2 & 0 & -2 & 2 & 0 & -2 & 2 \end{pmatrix}.$$

Again, the total sum is zero, so the domino has to cover two adjacent squares with sum zero. This means it covers a 2 and a -2, so it occupies either (i, j) and $(i, j + 1)$ with $i \equiv j \equiv 1 \pmod 3$, or else $(i - 1, j)$ and (i, j) with $i \equiv j \equiv 0 \pmod 3$. We leave the reader to show that all these solutions correspond to actual tilings.

The moral is: choose the right matrix, and the rest is easy!

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