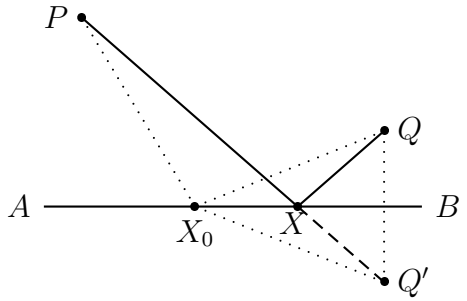
All done with mirrors

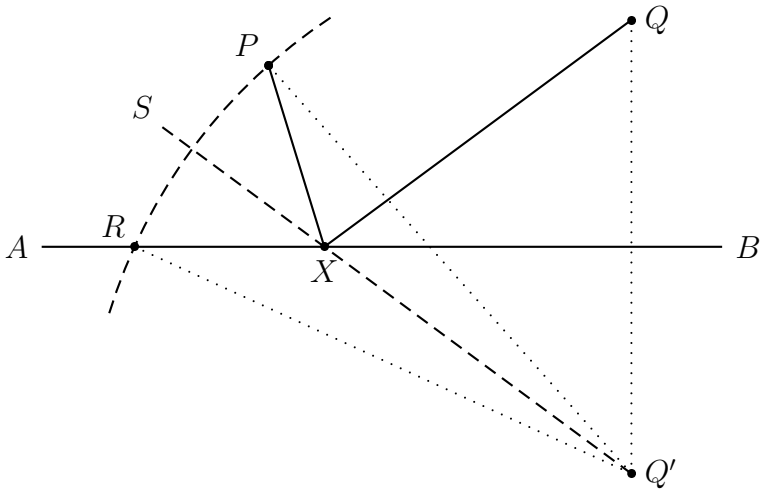
A well-known geometrical puzzle runs thus: given a line AB, and two points P, Q not on AB, but on the same side of it, find a point X on AB so that $|PX| + |XQ|$ is least possible.



Solution: let Q' be the reflection of Q in AB. For any X_0 on AB, $|X_0Q| = |X_0Q'|$ (by the reflection), so $|PX_0| + |X_0Q| = |PX_0| + |X_0Q'|$, and this is least when PX_0Q' is a straight line. So join PQ', and X is where this line meets AB.

An extension of the problem starts with *two* lines, AB and AC, and two points P, Q not on either line, and on the same side of each of them. Now find X on AB and Y on AC so that $|PX| + |XY| + |YQ|$ is least possible. To solve this, reflect P in AB and Q in AC to get P' and Q', and then join $P'Q'$. (Details are left to the reader; it is easy unless $P'Q'$ goes the "wrong" side of A, and then it gets a bit nasty.) Now you are ready to tackle the further extension to reflections in *three* lines, or *four*, or ...

For variety, the original problem can also be set in terms of angles: given AB, and P, Q, as before, find X on AB such that $\angle PXA = \angle QXB$. (The argument is then somewhat different, though the construction, and the point X, are the same.) Here is a hard variation on this: find, instead, X on AB with $\angle PXA = 2\angle QXB$.



The solution is very pretty. Let Q' be the reflection of Q in AB, as before, and draw an arc of a circle, centre Q', through P, to cut AB in R. Join P and R to Q', and construct the bisector $Q'S$ of $\angle PQ'R$. Then X is where the line $Q'S$ meets AB. To see that this works, note first that reflection in $Q'S$ swaps P with R. Then, $\angle QXB = \angle Q'XB$ (reflection in AB), which is equal to $\angle RXS$ (vertically opposite), and this is in turn equal to $\angle PXS$ (reflection in $Q'S$). So $\angle PXA = \angle PXS + \angle RXS = 2\angle QXB$. Done!

[Something to worry about: that circle, centre Q', must cut AB in *two* points, one of which we chose as R. What is the significance of the other one?]