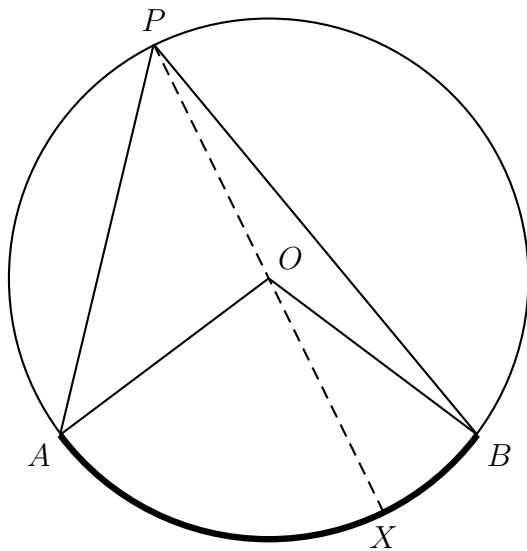


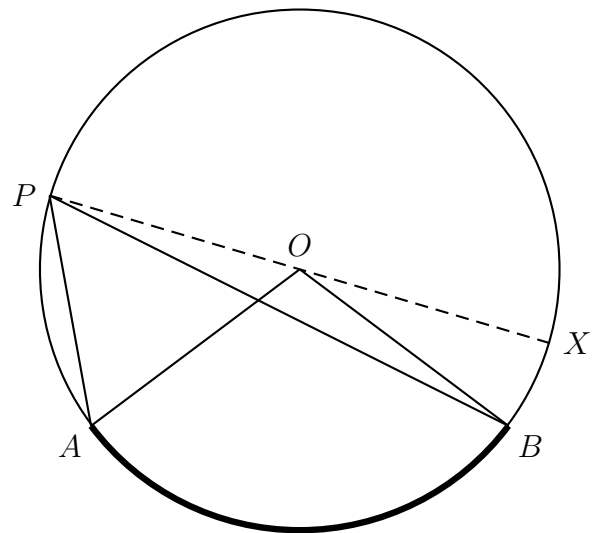
Keep it simple!

The article [1] in last November's *Mathematics in School* gives a very long proof of the theorem (mis)quoted from Key Stage 4, Higher Tier, that '*the angle subtended by an arc at the centre of a circle is twice the angle subtended by any point on the circumference*'. (It should, of course, be '*... subtended at any point...*'. The statement in the National Curriculum would be clearer if it included '*... any point on **the remaining part of the circumference.***') After the observation that many books give only one diagram for this theorem, the article then consists of lengthy proofs of three cases (mysteriously called *stages*), one of which is only there due to a misunderstanding of the meaning of the word *subtend*. As we show below, the other two cases have proofs which are nearly identical—indeed, with a little extra sophistication, they can be made identical—but the article unfortunately obscures what they have in common, and needlessly makes them seem as different (and, in the second case, as difficult) as possible.

There is one preliminary result needed for our proof, namely the theorem that, if one side of a triangle is produced, the external angle thus formed is equal to the sum of the opposite internal angles. The particular case of this that we need is for an isosceles triangle PQR (with $|PQ| = |PR|$), when it says that the external angle at P is equal to twice either of the opposite internal angles. We use this, twice, in the proof below.



Case 1.



Case 2.

THEOREM. Given a circle, centre O , and an arc AB of this circle (marked bold, above), let P be on the remaining part of the circumference. Then $\angle AOB = 2\angle APB$.

PROOF. Join PO and produce to X . We have $|OA| = |OP|$, so $\triangle OAP$ is isosceles, and by the preliminary remarks, $\angle AOX = 2\angle APO$. Similarly, $\angle XOB = 2\angle OPB$.

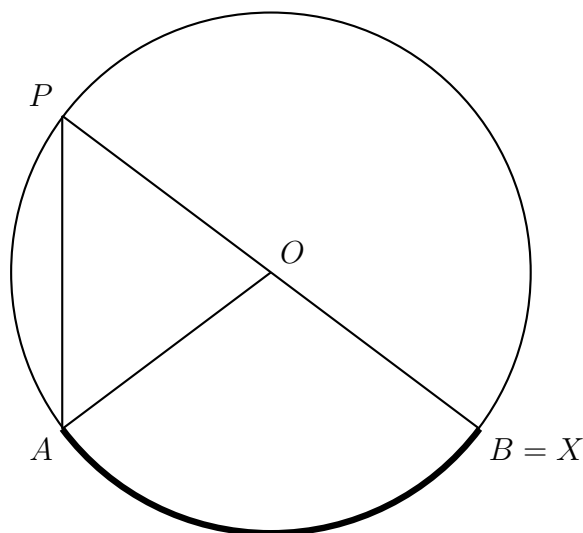
Case 1. Adding, we have $\angle AOB = 2\angle APB$.

Case 2. Subtracting, we have $\angle AOB = 2\angle APB$.

(End of proof.)

The two cases are distinguished by whether or not the line OP meets the interior of $\triangle APB$. (Note that Case 2 only occurs if $\angle AOB < \pi$.) We have been very careful, in Case 1, to name all the angles in the same sense, i.e., counterclockwise. The extra level of sophistication referred to above is to agree to work with *signed*, or *directed* angles, so that in Case 1, all the angles are positive, but in Case 2, $\angle XO B$ and $\angle OPB$ are negative. With this convention, we simply add the two equations in both Case 1 and Case 2, and let the signs take care of the distinction: the proofs are now identical. (This is particularly nice to demonstrate in a dynamic geometry program such as *Geometer's Sketchpad*, which allows one to specify that angles be measured in directed degrees. Moving P around the circle clearly shows one or other of $\angle AOX$, $\angle XO B$ changing sign at the critical positions, and likewise $\angle APO$, $\angle OPB$; but the sums remain constant throughout.)

A word about “Stage” 3 in [1]. Here P is put on the banned arc of the circle and it is pointed out that you now have to use the reflex angle at O , and there is then yet another proof. This is unnecessary, being covered by Case 1, above, on allowing the banned arc to increase in size until it is more than half of the circumference (when the angle subtended at O is *automatically* reflex), and leaving P where it should be, on the remaining part of the circumference.



Case 3.

There is perhaps a genuine Case 3 at the transition from Case 1 to Case 2, namely when BP passes through O . (As with Case 2, this cannot happen unless $\angle AOB < \pi$.) In Case 3, $\angle XO B$ and $\angle OPB$ are both zero, so we don't care whether we add or subtract our two equations.

It is also remarked in [1] that a proof of the above theorem via angles in the same segment is “not appropriate”, as that theorem is “based upon the result we are currently attempting to prove”. Whilst it is undoubtedly true that this is the easiest and shortest proof of this result, it is nonetheless perfectly possible to give an independent proof of the same segment theorem. Indeed, advocates of proof by transformation may well prefer this alternative approach; see [2], p.179, Proposition 7.14.

References

- [1] “A Mathematician’s Excuse Me (Er ... don’t get caught short!)”, Paul Wakefield, Mathematics in School, November 2001, 25–27.
- [2] “Geometry Ancient and Modern”, John R. Silvester, Oxford 2001.

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