Decimal déjà vu

We start with a quotation from our esteemed ex-president[1]:

> “Decimals are just a way of making all numbers appear equally boring”
>
> *(Tony Gardiner)*

Well, let’s see. The decimal expansion of a rational number (the ratio of two whole numbers), if it does not terminate, will eventually recur. Thus, for example,

$$\frac{989}{138875} = 0{\cdot}007\,1215\,1215\,1215\,\ldots \tag{1}$$

has a sequence of three digits (James Bond, 007) that does not recur, followed by a sequence of four digits (Magna Carta, 1215) that recurs. In this article, we investigate how the lengths of these sequences depend on the prime factorisation of the denominator of our rational number, and we obtain a proof of Fermat’s little theorem as a by-product.

1 Reciprocals of natural numbers

Here is a table showing the decimal expansion of $1/n$ for $2 \leq n \leq 50$. The expansions are truncated at 30 places. The column r shows the length of the non-recurring part of the expansion, and s shows the length of the recurring part. A dash means that the corresponding value is zero, and a question mark means that 30 places are not enough to determine s.

n	$1/n$ (to 30 decimal places)	r	s
2	0·500000000000000000000000000000	1	-
3	0·333333333333333333333333333333	-	1
4	0·250000000000000000000000000000	2	-
5	0·200000000000000000000000000000	1	-
6	0·166666666666666666666666666666	1	1
7	0·142857142857142857142857142857	-	6
8	0·125000000000000000000000000000	3	-
9	0·111111111111111111111111111111	-	1
10	0·100000000000000000000000000000	1	-
11	0·090909090909090909090909090909	-	2
12	0·083333333333333333333333333333	2	1
13	0·076923076923076923076923076923	-	6
14	0·071428571428571428571428571428	1	6
15	0·066666666666666666666666666666	1	1
16	0·062500000000000000000000000000	4	-
17	0·058823529411764705882352941176	-	16
18	0·055555555555555555555555555555	1	1
19	0·052631578947368421052631578947	-	18
20	0·050000000000000000000000000000	2	-

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[1] Out of context, of course. Sorry, Tony.

n	$1/n$ (to 30 decimal places)	r	s
21	0·047619047619047619047619047619	-	6
22	0·045454545454545454545454545454	1	2
23	0·043478260869565217391304347826	-	22
24	0·041666666666666666666666666666	3	1
25	0·040000000000000000000000000000	2	-
26	0·038461538461538461538461538461	1	6
27	0·037037037037037037037037037037	-	3
28	0·035714285714285714285714285714	2	6
29	0·034482758620689655172413793103	-	28
30	0·033333333333333333333333333333	1	1
31	0·032258064516129032258064516129	-	15
32	0·031250000000000000000000000000	5	-
33	0·030303030303030303030303030303	-	2
34	0·029411764705882352941176470588	1	16
35	0·028571428571428571428571428571	1	6
36	0·027777777777777777777777777777	2	1
37	0·027027027027027027027027027027	-	3
38	0·026315789473684210526315789473	1	18
39	0·025641025641025641025641025641	-	6
40	0·025000000000000000000000000000	3	-
41	0·024390243902439024390243902439	-	5
42	0·023809523809523809523809523809	1	6
43	0·023255813953488372093023255813	-	21
44	0·022727272727272727272727272727	2	2
45	0·022222222222222222222222222222	1	1
46	0·021739130434782608695652173913	1	22
47	0·021276595744680851063829787234	-	?
48	0·020833333333333333333333333333	4	1
49	0·020408163265306122448979591836	-	?
50	0·020000000000000000000000000000	2	-

1.1 Why do they recur?

An easy example: the decimal expansion of $\frac{1}{7}$. This is done by dividing 7 into 1, and we set out the calculation as a long division:

$$
\begin{array}{l}
\,0\cdot 14285714\ldots \\
7\overline{)1\cdot 00000000\ldots} \\
\,\underline{7} \\
\,30 \\
\,\underline{28} \\
\,20 \\
\,\underline{14} \\
\,60 \\
\,\underline{56} \\
\,40 \\
\,\underline{35} \\
\,50 \\
\,\underline{49} \\
\,10 \\
\,\underline{7} \\
\,30 \\
\,\underline{28} \\
\,2\ldots\text{ déjà vu?}
\end{array}
$$

This can be restated as:

$$
\begin{array}{rrcll}
 & 10 \div 7 & = & 1, & \text{remainder } 3; \\
\text{then} & 30 \div 7 & = & 4, & \text{remainder } 2, \\
\text{then} & 20 \div 7 & = & 2, & \text{remainder } 6, \\
\text{then} & 60 \div 7 & = & 8, & \text{remainder } 4, \quad \text{and so on.}
\end{array}
$$

Since we are dividing by 7, the remainder at each step must be one of 0, 1, 2, 3, 4, 5, 6, and so if the expansion does not terminate we *must* eventually get a remainder we have had before, and from that point onwards the expansion recurs.

Likewise for $\dfrac{a}{b}$ (where a, b are natural numbers), the remainder at each step of the division must be one of 0, 1, 2, …, $b-1$, and so if the expansion does not terminate, it must eventually recur.

Converse: A decimal that terminates or recurs represents a rational number. For example, if

$$x = 0{\cdot}19\,19\,19\,\ldots,$$

then, shifting the decimal point two places to the right,

$$100x = 19{\cdot}19\,19\,19\,\ldots.$$

Subtracting the first of these two equations from the second,

$$100x - x = 19{\cdot}19\,19\,19\,\ldots - 0{\cdot}19\,19\,19\,\ldots$$

which is equal to 19 exactly, as the parts after the decimal points cancel. So $99x = 19$, and $x = 19/99$, a rational number. Again, suppose

$$x = 0{\cdot}007\,1215\,1215\,1215\,\ldots.$$

Then first shift the decimal point by three places, to give

$$10^3x = 7{\cdot}1215\,1215\,1215\,\ldots.$$

Now shift it a further four places:

$$10^7x = 71215{\cdot}1215\,1215\,1215\,\ldots.$$

Subtracting,

$$(10^7 - 10^3)x = 71215{\cdot}1215\,1215\,1215\,\ldots - 7{\cdot}1215\,1215\,1215\,\ldots = 71215 - 7 = 71208.$$

Since $10^7 - 10^3 = 9999000$, we obtain $x = 71208/9999000$, another rational number. (A factor of 72 will now cancel from numerator and denominator to give equation (1) again.)

In general, if the decimal expansion of x has a non-recurring part of length r and a recurring part of length s, then we can shift the decimal point by r places, and then a further s places, to obtain two numbers 10^rx and $10^{r+s}x$ which differ by an integer, since their decimal expansions agree after the decimal point. So $(10^{r+s} - 10^r)x = c$ (an integer), say, and

$$x = \frac{c}{10^{r+s} - 10^r} = \frac{c}{10^r(10^s - 1)},$$

a rational number. Notice that $10^s - 1 = 99\ldots9$ (s digits), so that the denominator (before cancellation of any common factors) consists of s nines followed by r zeros.

1.2 What can we say about r and s?

Let $x = \dfrac{a}{b}$ be a rational number in its lowest terms, so that a and b are *coprime*, that is, their highest common factor is 1. From section 1.1, we know the decimal expansion of x will terminate or recur, and indeed

$$\frac{a}{b} = x = \frac{c}{10^r(10^s - 1)}$$

where r is the length of the non-recurring part and s is the length of the recurring part. But now, cross-multiplying,

$$bc = 10^r(10^s - 1)a$$

so that

$$b \mid 10^r(10^s - 1)a.$$

(Read this as "b divides $10^r(10^s - 1)a$".) Since a and b are coprime,

$$b \mid 10^r(10^s - 1),$$

that is, $\dfrac{10^r(10^s-1)}{b}$ is an integer. In fact r and s are the *least* natural numbers such that $\dfrac{10^r(10^s-1)}{b}$ is an integer. For if r, s are chosen in this way, consider the number $10^{r+s}x - 10^r x$, constructed by moving the decimal point (in the expansion of x) first r places and then a further s places, and subtracting the results. Since $x = a/b$, we have

$$10^{r+s}x - 10^r x = 10^r(10^s-1)x = 10^r(10^s-1)\frac{a}{b} = \left(\frac{10^r(10^s-1)}{b}\right)a.$$

But this is an integer. It follows that the decimal expansions of $10^{r+s}x$ and of $10^r x$ must agree after the decimal point, which means that they recur with period s. This in turn means that the decimal expansion of x, after the first r digits, recurs with period s.

Important observation: The values of r and s for the fraction $\dfrac{a}{b}$ (where a and b are coprime) **do not depend on the value of** a, **but only on the value of** b. We shall use this fact, later.

Note also that, if we take out any factors of 2 or 5 from b, writing $b = 2^u 5^v b'$ where b' is not divisible by 2 or 5, then $2^u 5^v \mid 10^r$ and $b' \mid 10^s - 1$. So $r = \max\{u, v\}$; and in particular, if b is not divisible by 2 or 5, then $r = 0$. For example, for $45 \le b \le 50$ we obtain

b :	45	46	47	48	49	50
u :	0	1	0	4	0	1
v :	1	0	0	0	0	2
r :	1	1	0	4	0	2

which agrees with the last six rows of the table in section 1.

1.3 Factors of $10^s - 1$

We have seen that $b \mid 10^r(10^s-1)$ for some r and s; and if b is not divisible by 2 or 5, then $b \mid 10^s - 1$ for some s. Applying this to $9b$ in place of b, we see that $9b \mid 10^s - 1$ for some (different) s, that is,

$$\begin{array}{rcll} 9b & \mid & 99\ldots9 & (s \text{ digits}), \\ \text{so} \quad b & \mid & 11\ldots1 & (s \text{ digits}). \end{array}$$

So the numbers 1, 11, 111, 1111,... include among their factors *every* natural number that is not divisible by 2 or 5, that is, every natural number that ends in 1, 3, 7 or 9. In particular, if we list the *prime* factors of 11, 111, 1111, ...:

$$\begin{array}{rcl} 11 & = & (11) \\ 111 & = & (3)(37) \\ 1111 & = & (11)(101) \\ 11111 & = & (41)(271) \end{array}$$

$$\begin{aligned} 111111 &= (3)(7)(11)(13)(37) \\ 1111111 &= (239)(4649) \\ &\ \ \dots \end{aligned}$$

we shall eventually pick up, in this list, *every* prime number except 2 and 5.

Notice that the list has repetitions. For instance,

$$11 \mid 1111 \quad \text{and} \quad 11 \mid 111111$$

(in fact, $1111 = 11 \times 101$ and $111111 = 11 \times 10101$), and

$$111 \mid 111111$$

(in fact, $111111 = 111 \times 1001$). In general, let $u = 11\dots1$ (s digits) and $v = 11\dots1$ (t digits). We assert that, if $s \mid t$, then $u \mid v$. This is because v, written out, consists of t/s blocks of 1's, each looking just like u. So the division of v by u looks like this:

$$\begin{array}{r} 1\;00\dots01\;00\dots01\quad\dots\quad00\dots01 \\ 11\dots11\big)\overline{11\dots11\;11\dots11\;11\dots11\quad\dots\quad11\dots11} \end{array}$$

The converse is an exercise for the reader: if $u \mid v$, then $s \mid t$.

Now $9u = 99\dots9$ (s digits), that is, $9u = 10^s - 1$, and similarly $9v = 10^t - 1$. Since $u \mid v$ iff $9u \mid 9v$, we conclude:

$$10^s - 1 \mid 10^t - 1 \quad \text{iff} \quad s \mid t.$$

2 Reciprocals of prime numbers

Now let us look at the expansion of $1/p$ when p is a prime number (not 2 or 5).

p	$1/p$ (to 30 decimal places)	r	s
3	0·333333333333333333333333333333	-	1
7	0·142857142857142857142857142857	-	6
11	0·090909090909090909090909090909	-	2
13	0·076923076923076923076923076923	-	6
17	0·058823529411764705882352941176	-	16
19	0·052631578947368421052631578947	-	18
23	0·043478260869565217391304347826	-	22
29	0·034482758620689655172413793103	-	28
31	0·032258064516129032258064516129	-	15
37	0·027027027027027027027027027027	-	3
41	0·024390243902439024390243902439	-	5
43	0·023255813953488372093023255813	-	21
47	0·021276595744680851063829787234	-	?

Of course, $r = 0$ in all these cases.

Notice that in all the above examples, we have $s \mid p-1$: so we can make an intelligent guess that when $p = 47$, we have $s \mid 46$; and since $s \neq 2$ or 23, then $s = 46$. In fact, $\frac{1}{47}$, to 50 decimal places, is $0{\cdot}02127659574468085106382978723404255319148936170212$, so the guess is correct. (But see Exercise 6, below.)

2.1 Cyclic permutations, and Fermat's little theorem

An example:

$$\begin{aligned}
\frac{1}{7} &= 0{\cdot}142857\,142857\ldots \\
\frac{2}{7} &= 0{\cdot}285714\,285714\ldots \\
\frac{3}{7} &= 0{\cdot}428571\,428571\ldots \\
\frac{4}{7} &= 0{\cdot}571428\,571428\ldots \\
\frac{5}{7} &= 0{\cdot}714285\,714285\ldots \\
\frac{6}{7} &= 0{\cdot}857142\,857142\ldots
\end{aligned}$$

Notice that the recurring parts are cyclic permutations of each other. To see why this is, note that, since

$$\begin{aligned}
\frac{1}{7} &= 0{\cdot}142857\,142857\ldots, \\
\text{we have}\quad \frac{10}{7} &= 1{\cdot}428571\,428571\ldots, \\
\text{so that}\quad \frac{10}{7} - 1 &= 0{\cdot}428571\,428571\ldots, \\
\text{or}\quad \frac{3}{7} &= 0{\cdot}428571\,428571\ldots
\end{aligned}$$

The same trick can now be repeated. Multiply each side by 10, and subtract 4 from each side to get another cyclic permutation:

$$\begin{aligned}
\frac{2}{7} = \frac{30}{7} - 4 &= 10 \times (0{\cdot}428571\,428571\ldots) - 4 \\
&= 0{\cdot}285714\,285714\ldots
\end{aligned}$$

The general case: A cyclic permutation of the recurring part of $\frac{1}{p}$ gives $\frac{a}{p}$ for some $a < p$. The s different permutations (including the trivial one) obtained by repeated cyclic permutation give $\frac{a}{p}$ for s different values of $a \in \{1, 2, \ldots, p-1\}$. Recall from section 1.2 that the expansions of $\dfrac{a}{p}$ for $a \in \{1, 2, \ldots, p-1\}$ all have recurring part of the same length s, independent of a. (s is the least positive integer such that $p \mid 10^s - 1$.) So this set of $p-1$ decimal expansions falls into disjoint subsets of size s, expansions in the same subset being related by repeated cyclic permutation.

We deduce that $s \mid p-1$. From section 1.3, $10^s - 1 \mid 10^{p-1} - 1$, and from section 1.2, $p \mid 10^s - 1$, so it follows that $p \mid 10^{p-1} - 1$. This is the case $n = 10$ of **Fermat's little theorem**, which says that if p is prime and does not divide n, then $p \mid n^{p-1} - 1$. The general case is left to the reader: see Exercise 4, below.

2.2 A topical example

$$\frac{1}{1999} = 0{\cdot}00050025012506253126563281640 8\ldots$$

When does this recur?, i.e., what is s?

Now $p = 1999$ is a prime number, and $p - 1 = 1998 = (2)(3^3)(37)$; here 2, 3 and 37 are prime numbers. Since $s \mid p-1$, we must have $s = (2^a)(3^b)(37^c)$, where $a = 0$ or 1; $b = 0$, 1, 2, or 3; and $c = 0$ or 1. Since there are 2 possible values of a, 4 possible values of b, and 2 possible values of c, there are $2 \times 4 \times 2 = 16$ possible values of s. Explicitly, these are

$$1,\ 2,\ 3,\ 6,\ 9,\ 18,\ 27,\ 37,\ 54,\ 74,\ 111,\ 222,\ 333,\ 666,\ 999,\ 1998,$$

and s is the first number in this list for which $1999 \mid 10^s - 1$. Now we know from Fermat's theorem that $1999 \mid 10^{1998} - 1$, so there is no need to check that. Instead, we check whether or not 1999 divides $10^{999} - 1$. Of course, this is easy if you have access to a computer running Maple, or a similar mathematical package, but with care the calculation can be done on a pocket calculator. We shall work *modulo* 1999, so $n \equiv m$ will mean $1999 \mid n - m$, or that n and m leave the same remainder on dividing by 1999. We have[2]

$$\begin{aligned}
10^9 &= 1000000000 = (500250)(1999) + 250 \equiv 250\\
10^{18} &\equiv 250^2 = 62500 = (31)(1999) + 531 \equiv 531\\
10^{36} &\equiv 531^2 = 281961 = (141)(1999) + 102 \equiv 102\\
10^{54} &\equiv (531)(102) = 54162 = (27)(1999) + 189 \equiv 189\\
10^{108} &\equiv 189^2 = 35721 = (17)(1999) + 1738 \equiv 1738\\
10^{111} &\equiv 1738000 = (869)(1999) + 869 \equiv 869\\
10^{222} &\equiv 869^2 = 755161 = (377)(1999) + 1538 \equiv 1538\\
10^{333} &\equiv (869)(1538) = 1336522 = (668)(1999) + 1190 \equiv 1190\\
10^{666} &\equiv 1190^2 = 1416100 = (708)(1999) + 808 \equiv 808\\
10^{999} &\equiv (1190)(808) = 961520 = (481)(1999) + 1 \equiv 1
\end{aligned}$$

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[2] The calculation that follows uses repeatedly the fact that, if $a \equiv b$ and $c \equiv d$, then $ac \equiv bd$, and also $a^2 \equiv b^2$. Prove these statements.

So $1999 \mid 10^{999} - 1$, and therefore $s \mid 999$. It follows that s is odd, and $s = (3^b)(37^c)$ with $0 \le b \le 3$ and $0 \le c \le 1$. Now if $b \le 2$, then $s \mid (3^2)(37) = 333$; but the above calulation shows $10^{333} \equiv 1190$, not 1, so we must in fact have $b = 3$. If $c = 0$, then $s = 3^3 = 27$; but then $s \mid 54$, and the above calculation shows $10^{54} \equiv 189$, not 1. So we must have $c = 1$, and we conclude that $s = (3^3)(37) = 999$.

In fact 1/1999 is (to 1200 places of decimals, at 60 digits per line):

$$\begin{array}{rl}
0{\cdot} & 000500250125062531265632816408204102051025512756378189094547 \\
 & 273636818409204602301150575287643821910955477738869434717358 \\
 & 679339669834917458729364682341170585292646323161580790395197 \\
 & 598799399699849924962481240620310155077538769384692346173086 \\
 & 543271635817908954477238619309654827413706853426713356678339 \\
 & 169584792396198099049524762381190595297648824412206103051525 \\
 & 762881440720360180090045022511255627813906953476738369184592 \\
 & 296148074037018509254627313656828414207103551775887943971985 \\
 & 992996498249124562281140570285142571285642821410705352676338 \\
 & 169084542271135567783891945972986493246623311655827913956978 \\
 & 489244622311155577788894447223611805902951475737868934467233 \\
 & 616808404202101050525262631315657828914457228614307153576788 \\
 & 394197098549274637318659329664832416208104052026013006503251 \\
 & 625812906453226613306653326663331665832916458229114557278639 \\
 & 319659829914957478739369684842421210605302651325662831415707 \\
 & 853926963481740870435217608804402201100550275137568784392196 \\
 & 098049024512256128064032016008004002001000500250125062531265 \\
 & 632816408204102051025512756378189094547273636818409204602301 \\
 & 150575287643821910955477738869434717358679339669834917458729 \\
 & 364682341170585292646323161580790395197598799399699849924962
\end{array}$$

The recurrence begins in the fourth line from the bottom, 21 digits from the end of the line.

3 Exercises

1. We have found n for which the recurring part of $1/n$ has length $s = 1$, 2, 3, 5, 6, 8. So find the least n for which $s = 4$, 7, 9, 10, 11, 12.
2. Investigate s when n is composite. For example, when $n = 11$, $s = 2$, and when $n = 37$, $s = 3$. So what is s when $n = 407$?
3. Can you explain why $s = 42$ when $n = 49$?
4. We showed that $p \mid 10^{p-1} - 1$ when p is prime and does not divide 10. Show that $p \mid n^{p-1} - 1$ when p is prime and does not divide n. (Fermat's little theorem.)
5. Find r, s for $n = 1997$, 1998, 2000, 2001, 2002, 2003.

6. How long a sequence of digits do we need to see repeated to know the value of s? For example, $17/19 = 0{\cdot}8947368\ldots$, but the first occurrence of a second '8' does *not* allow us to deduce $s = 6$. In fact $17/19 = 0{\cdot}894736842105263157 89\ldots$, and the first occurrence of a second '89' *does* give the right value of s, namely $s = 18$. So what is the rule?
7. If $1/1999$ is expanded as a *binary* "decimal" (i.e., base 2), what is the value of s?
8. At the start of the (base 10) decimal expansion of $1/1999$, successive blocks of 4 digits are 0005, 0025, 0125, 0625; and just before the recurrence begins, successive blocks of 3 digits are 512, 256, 128, 064, 032, 016, 008, 004, 002, 001. Can you explain this?
9. Again, the decimal expansion of $1/1999$ contains the digit sequence 0880440220110, but *not* the sequence 888444222111. Why?
10. Is there an integer a such that the *binary* expansion of $a/1999$ contains the digit sequence (i) 10101010101; (ii) 10011100101 ?

Answers

1.

s:	1	2	3	4	5	6	7	8	9	10	11	12
Least n:	3	11	27	101	41	7	239	73	81	451	21649	707

The answer for $s = 7$, for example, comes from the prime factorization $10^7 - 1 = 3^2 \times 239 \times 4649$, and the answer for $s = 11$ from $10^{11} - 1 = 3^2 \times 21649 \times 513239$.

2. When $n = 11$, $s = 2$, and when $n = 37$, $s = 3$. So when $n = 11 \times 37 = 407$, then s is the least common multiple of 2 and 3, that is, $s = 6$.

3. The number of a with $1 \leq a \leq 49$ such that $\dfrac{a}{49}$ is already in its lowest terms is $7^2 - 7 = 49 - 7 = 42$.

4. Just work in base n instead of base 10.

5.

n:	1997	1998	2000	2001	2002	2003
r:	-	1	4	-	1	-
s:	998	3	-	308	6	1001

6. For an m-digit denominator, it is sufficient to see a sequence of m digits occur for a second time. We deal with the expansion of a/b where $0 < a < b$, leaving other cases to the reader. If b is an m-digit number, then $b < 10^m$. Suppose $\dfrac{a}{b} = 0{\cdot}y_1y_2\ldots$, where the y's are the decimal digits, and suppose $y_i = y_{s+i}$ for $r + 1 \leq i \leq r + m$. Then

$$\begin{aligned} 10^r\frac{a}{b} &= y_1\ldots y_r{\cdot}y_{r+1}y_{r+2}\cdots &= n_1 + \varepsilon_1, \\ \text{and}\quad 10^{r+s}\frac{a}{b} &= y_1\ldots y_{r+s}{\cdot}y_{r+s+1}y_{r+s+2}\cdots &= n_2 + \varepsilon_2, \end{aligned}$$

say, where n_1, n_2 are integers; and $\varepsilon_1 = 0{\cdot}y_{r+1}y_{r+2}\ldots$ and $\varepsilon_2 = 0{\cdot}y_{r+s+1}y_{r+s+2}\ldots$ are decimals that agree for the first m decimal places. So $|\varepsilon_2 - \varepsilon_1| \le \dfrac{1}{10^m} < \dfrac{1}{b}$, and thus $\left|10^r(10^s-1)\dfrac{a}{b} - (n_2 - n_1)\right| < \dfrac{1}{b}$. But this means that the expression on the left is zero, so that $b \mid 10^r(10^s-1)$, and $\varepsilon_1 = \varepsilon_2$, whence $y_i = y_{s+i}$ for all $i \ge r+1$.

7. We must find the least $s \ge 1$ such that $1999 \mid 2^s - 1$, or $2^s \equiv 1$ modulo 1999. We know from Fermat's theorem that $s \mid 1998$. In fact, $s = 333$. (Check that, working modulo 1999, we have $2^9 = 512$, $2^{18} \equiv 275$, $2^{36} \equiv 1662$, $2^{54} \equiv 1278$, $2^{108} \equiv 101$, $2^{111} \equiv 808$, $2^{222} \equiv 1190$, and finally $2^{333} \equiv 1$. Is that enough?) The expansion (to 360 places, at 60 digits per line) is

$$\begin{array}{l}
0{\cdot}000000000010000011001000110011011110110101001101011110101000\\
\phantom{0{\cdot}}111010101001110100000001101010100011001001110101000011001110\\
\phantom{0{\cdot}}111100111001001111101001111110010001010110100100100011111111\\
\phantom{0{\cdot}}000110101000001001011110100000101110000110100110000110010101\\
\phantom{0{\cdot}}101101001111010001011000100111101100110010100101011101010110\\
\phantom{0{\cdot}}111101001001101000110000011010001000000000010000011001000110
\end{array}$$

The recurrence begins in the last line, 27 digits from the end. (By Exercise 6, we need to see a sequence of eleven digits repeated to find the value of s just by inspection. Is this OK? Why *eleven*?)

8. Let $x = 1/1999$. Thus

$$x = \sum_{n=1}^{\infty} \frac{1}{2000^n} = \sum_{n=1}^{4} \frac{1}{2000^n} + \sum_{n=5}^{\infty} \frac{1}{2000^n} = \sum_{n=1}^{4} \frac{1}{2000^n} + \frac{x}{2000^4},$$

$$\begin{aligned}
\text{so that}\quad x &= 0{\cdot}0005\\
&+\ 0{\cdot}0000\,0025\\
&+\ 0{\cdot}0000\,0000\,0125\\
&+\ 0{\cdot}0000\,0000\,0000\,0625 + \varepsilon,
\end{aligned}$$

where $\varepsilon = x/2000^4 < 10^{-16}$. (OK?) The decimal expansion of ε therefore starts with at least 16 zeros, and thus $x = 0{\cdot}0005\,0025\,0125\,0625\ldots$.

Next, let $y = 2^{10}x = 1024/1999$. Then

$$10^{30}y = 2000^{10}x = 2000^{10}\sum_{n=1}^{\infty}\frac{1}{2000^n} = m + x, \quad \text{where}$$

$$m = \sum_{n=0}^{9} 2000^n = \sum_{n=0}^{9} 2^n 10^{3n} = 512\,256\,128\,064\,032\,016\,008\,004\,002\,001.$$

It follows that $y = 0{\cdot}512\,256\,128\,064\,032\,016\,008\,004\,002\,001\ldots$, and that the recurring part of the decimal expansion of y is obtained from that of x by moving everything 30 places to the right, and bringing the last 30 digits to the front. Thus the last 30 digits in the recurring part of the decimal expansion of x are 512 256 128 064 032 016 008 004 002 001.

9. We have just seen that the decimal expansion of 1/1999 contains the sequence 00800400200100, and so the expansion of 10/1999 contains the same sequence, one place further to the left; and the expansion of 100/1999 contains the same sequence one place *still further* to the left. So, adding up, the expansion of 11/1999 must contain the sequence 0880440220110, and the expansion of 111/1999 must contain the sequence 888444222111. We can see by inspection that the former sequence does occur and the latter does not. If $10^r \equiv a$ (modulo 1999), where $1 \le a \le 1998$, then the decimal expansion of $a/1999$ is obtained from that of 1/1999 by shifting the point r places to the right, and then removing the part before the point. So it must be the case that $10^r \equiv 11$ has a solution, but $10^r \equiv 111$ has *no* solution. The reader might like to check that $10^{937} \equiv 11$. (I found that just by staring at the expansion of 1/1999. How?) If $10^r \equiv 111$, then $111^{999} \equiv 10^{999r} \equiv 1^r = 1$, since $10^{999} \equiv 1$. But again, the reader might like to check that $111^{999} \equiv 1998$, so that no such r can exist.

   Alternatively, $10^r \equiv 1998$ has no solution, since it is equivalent to $10^r \equiv -1$, which would yield $1 = 1^r \equiv 10^{999r} \equiv (-1)^{999} = -1$, impossible. So 1998/1999 has an expansion the recurring part of which is *not* obtainable from that of 1/1999 by repeated cyclic permutation. Since by inspection the expansion of 1/1999 contains the sequence 1115557778889, then $1998/1999 = 1 - 1/1999$ must contain the sequence 888444222111. No sequence of as many as 4 successive digits can appear in both expansions, by Exercise 6, whence the result.

10. (i) If there is, then there will be an integer a such that the binary expansion of $a/1999$ begins $0{\cdot}10101010101\ldots$. The binary number 10101010101, in base 10, is 1365, so we need a such that

$$\frac{1365}{2^{11}} \le \frac{a}{1999} < \frac{1366}{2^{11}}, \quad \text{or} \quad \frac{2728635}{2048} \le a < \frac{2730634}{2048}.$$

   Since $2728635/2048 \approx 1332{\cdot}3$ and $2730634/2048 \approx 1333{\cdot}3$, we deduce $a = 1333$. (Check that this works.) (ii) The binary number 10011100101, in base 10, is 1253, so this time we need an integer a with

$$\frac{1253}{2^{11}} \le \frac{a}{1999} < \frac{1254}{2^{11}}, \quad \text{or} \quad \frac{2504747}{2048} \le a < \frac{2506746}{2048}.$$

   But $2504747/2048 \approx 1223{\cdot}02$ and $2506746/2048 \approx 1223{\cdot}997$, so no such a exists.

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