**Circle theorems and a property of the (2,3,4) triangle**

Let us say that $\triangle ABC$ has the *twice complimentary exterior angle property* (or is a 'TCEA triangle') if $\angle BCA$ is obtuse and also $\angle CAB = 2\angle CBP$, where $P$ is the foot of the perpendicular from $B$ to $AC$ produced (Figure 1). Note that the order of naming vertices in $\triangle ABC$ is critical for this definition. We shall therefore denote the triangle by $\triangle A^{e}BC^{o}$, where the superscript $o$ indicates that the obtuse angle is at $C$, and the superscript $e$ indicates that it is the angle at $A$ which is twice the complimentary exterior angle.

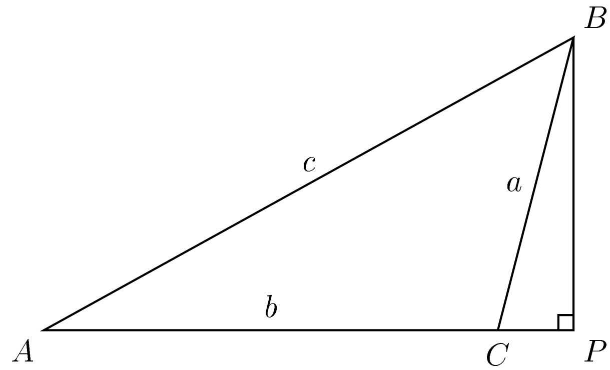


Figure 1

Using a GCSE question as a starting point, Nick Lord (Notes, 82.11) showed, via the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$, that

$\triangle A^{e}BC^{o}$ is a TCEA triangle if and only if $c^2 = a^2 + bc$; moreover

when $\triangle A^{e}BC^{o}$ is a TCEA triangle, then $\sin(\angle CBP) = \dfrac{a}{2c}$.

(The GCSE question asked for $\angle CBP$ in the case when $\triangle ABC$ was a (2,3,4) triangle, with $(a, b, c) = (20, 30, 40)$. This satisfies the above condition, and so the answer was $\sin^{-1}\frac{1}{4}$.)

Nick Lord also pointed out that the condition $c^2 = a^2 + bc$ is equivalent to $(b, 2a, 2c - b)$ being a Pythagorean triple. The purpose of this note is to locate the above within the theory of angle and rectangle properties of circles. In addition to keeping the discussion within the bounds of GCSE, from where it originated, and not involving the above trigonometric identities, this approach also enables a geometric illumination of the correspondence between 'TCEA triples' and Pythagorean triples.

First note that if in $\triangle ABC$ we have $c^2 = a^2 + bc$, then $c^2 > bc$, whence $c > b$ and so $bc > b^2$. Thus $c^2 > a^2 + b^2$, and $\angle BCA$ is obtuse.

Given $\triangle ABC$ with $\angle BCA$ obtuse, then, let $P$ be the foot of the perpendicular from $B$ onto $AC$ produced. Now whenever you have a quadratic relationship, it is a good idea to look for a circle for which one of the circle theorems might underlie the relationship. One such circle here (and there are also others which do the job) is that with centre $A$, through $B$. Let $AP$ produced meet this circle at $D$, and let $BP$ produced meet the circle at $B'$, so that $B'$ is the reflection of $B$ in $AD$. Join $BD$ and $AB'$ (Figure 2).

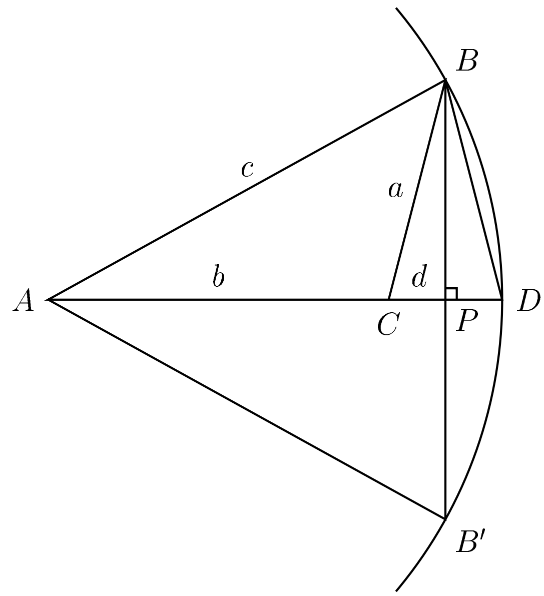


Figure 2

We have

$$\begin{aligned} \angle CAB &= \angle DAB \\ &= \angle DAB' \quad \text{(reflecting in } AD\text{)} \\ &= 2\angle DBB' \quad \text{(angle at centre is twice angle at circumference)} \\ &= 2\angle DBP. \end{aligned}$$

It follows that the TCEA condition $\angle CAB = 2\angle CBP$ is equivalent to the condition $\angle CBP = \angle DBP$, and hence to $CP = DP$. Writing $d = CP$, this can be expressed as $c = b + 2d$. Applying Pythagoras' theorem to $\triangle CPB$ and $\triangle APB$, we have $a^2 - d^2 = BP^2 = c^2 - (b+d)^2$, so that $c^2 = a^2 + b(b+2d)$. It follows immediately that the TCEA condition is equivalent to $c^2 = a^2 + bc$.

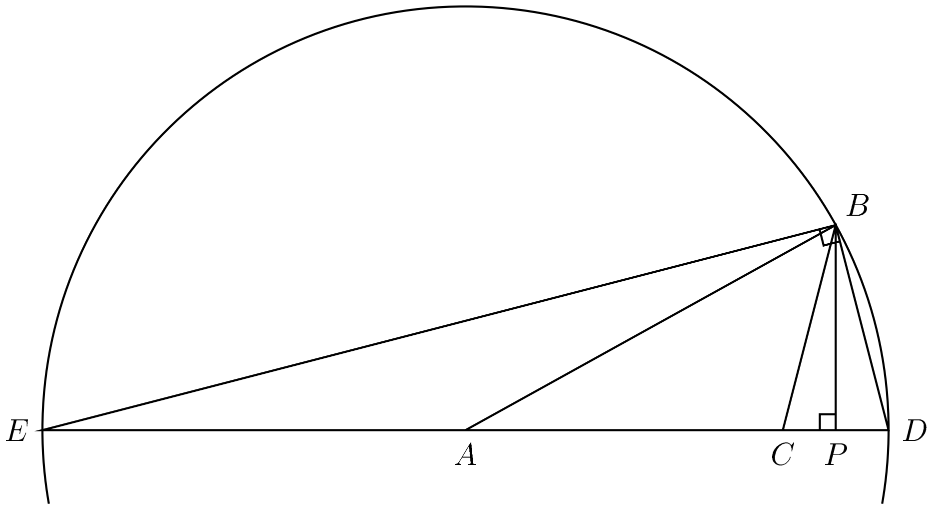


Figure 3

To prove that the TCEA condition implies that $\sin(\angle CBP) = \dfrac{a}{2c}$, let $DA$ produced meet the circle again at $E$, so that $DE = 2c$; and join $EB$ (Figure 3). By the above, $CP = DP$, so that $CB = DB$, or $DB = a$. Then $\angle EBD$, being the angle in a semicircle, is a right-angle, so that $\sin(\angle DEB) = \dfrac{DB}{DE} = \dfrac{a}{2c}$. But

$$\begin{aligned} 2\angle DEB &= \angle DAB \quad \text{(angle at centre is twice angle at circumference)} \\ &= \angle CAB \\ &= 2\angle CBP \quad (\triangle A^{e}BC^{o} \text{ has the TCEA property}). \end{aligned}$$

Hence $\sin(\angle CBP) = \dfrac{a}{2c}$.

We do not claim that the above proof of the quadratic criterion for a TCEA triangle is the shortest or most elementary. A shorter proof would be to take $F$ on $AB$ such that $AF = AC$ and note, since $\triangle BCF$ and $\triangle BAC$ share an angle at $B$, that each of the conditions is equivalent to $\triangle BCF$ and $\triangle BAC$ being similar. This too can be seen in terms of circle theorems, for if $\angle BCA$ is obtuse, then (Figure 4)

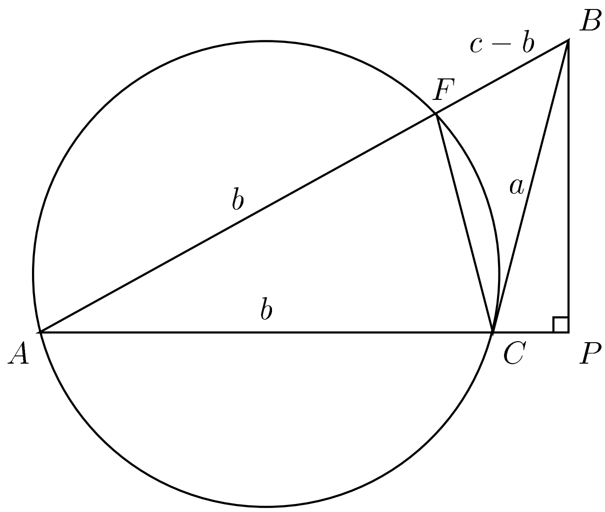


Figure 4

$\triangle A^{e}BC^{o}$ is a TCEA triangle

$\Leftrightarrow$ $\angle BCF = \angle CAF$

$\Leftrightarrow$ $BC$ is tangential to the circumcircle of $\triangle AFC$ (alternate segment theorem)

$\Leftrightarrow$ $BC^2 = BA.BF$ (tangent-chord rectangular property)

$\Leftrightarrow$ $a^2 = c(c-b)$.

Turning to the equivalence of the condition $c^2 = a^2 + bc$ with $(b, 2a, 2c-b)$ being a Pythagorean triple, write $\mathbf{P}$ for the invertible matrix $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}$, and $\mathbf{x}$, $\mathbf{a}$ for the column vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, respectively. Then the invertible linear transformation on $\mathbb{R}^3$ given by $\mathbf{x} = \mathbf{Pa}$ restricts to a bijection between triples of positive reals $(a, b, c)$ satisfying $c^2 = a^2 + bc$, and triples of positive reals $(x, y, z)$ satisfying $x^2 + y^2 = z^2$. (In the language of linear algebra, the quadratic forms $a^2 + bc - c^2$ and $x^2 + y^2 - z^2$ are congruent, with transforming matrix $\frac{1}{2}\mathbf{P}$.) For example, the Pythagorean triple corresponding to the TCEA triple (2,3,4) is (3,4,5).

The corresponding bijection between TCEA triangles and right-angled triangles can be seen geometrically as follows. Given $\triangle ABC$ with $\angle BCA$ obtuse, put $D$ on $AC$ produced (as before) so that $AD = AB$. Now let the circle on $AC$ as diameter have centre $G$ (the mid-point of $AC$), and let $H$ be on this circle such that $DH$ is tangential to the circle. Join $GH$, and note that $\angle GHD$ is a right-angle (Figure 5).

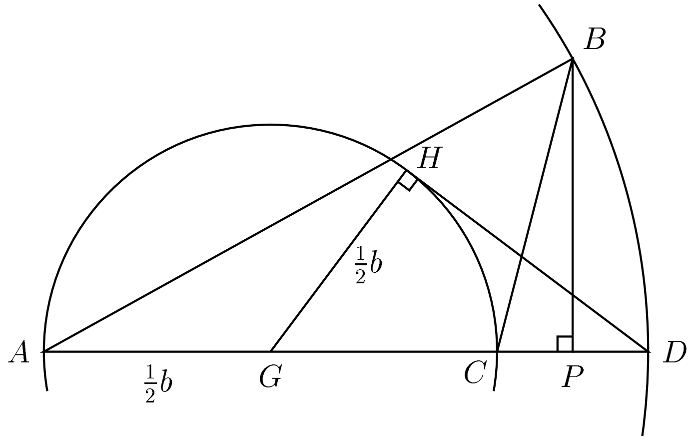


Figure 5

We shall show that $\triangle GHD$ gives the Pythagorean triple we seek.

Now by the tangent-chord rectangular property, $DH^2 = DA.DC = c(c-b)$, and by Pythagoras' theorem in $\triangle GHD$, $DH^2 = DG^2 - GH^2 = (c - \frac{1}{2}b)^2 - (\frac{1}{2}b)^2$. So we conclude

$$a^2 = c(c-b) \quad \Leftrightarrow \quad DH = a \quad \Leftrightarrow \quad a^2 = (c-\tfrac{1}{2}b)^2 - (\tfrac{1}{2}b)^2 \quad \Leftrightarrow \quad (2a)^2 = (2c-b)^2 - b^2,$$

from which $c^2 = a^2 + bc$ if and only if $(b, 2a, 2c-b)$ is a Pythagorean triple.

*Tony Barnard and John Silvester,*
*Department of Mathematics,*
*King's College London,*
*Strand WC2R 2LS*
*(tony.barnard@kcl.ac.uk, jrs@kcl.ac.uk)*