Ceva = (Menelaus)^2

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The famous "twin" theorems of Ceva and Menelaus can be stated together as follows: given $\triangle ABC$, with points D, E, F on BC, CA, AB respectively (and distinct from A, B, C), then (i) (**Ceva**, Figure 1) if the lines AD, BE, CF are concurrent, we have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1, \tag{1}$$

and (ii) (**Menelaus**, Figure 2) if the points D, E, F are collinear, we have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1. \tag{2}$$

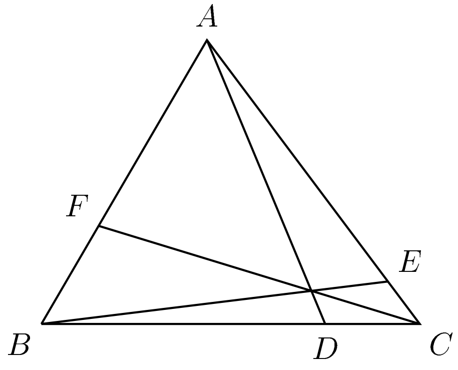

Figure 1: Ceva's theorem

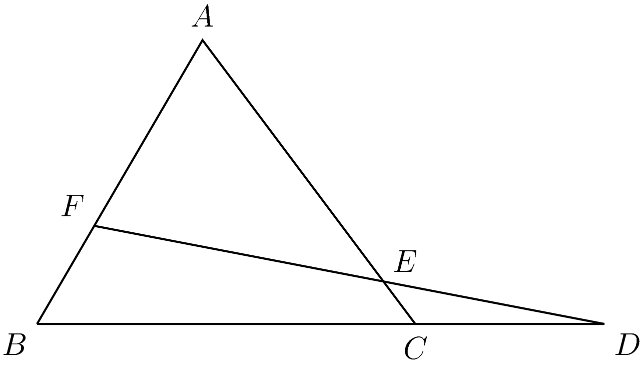

Figure 2: Menelaus' theorem

The ratios of lengths here are *signed*, so that (for example) $\dfrac{BD}{DC}$ is positive if D is between B and C, and negative otherwise. (The converse theorems are also true, so that "we have" can be replaced by "if and only if", except that in the case of Ceva's theorem we then have to allow the possibility that AD, BE, CF are concurrent *or* all parallel. But here, to simplify matters, we shall concentrate mostly on the versions stated above.)

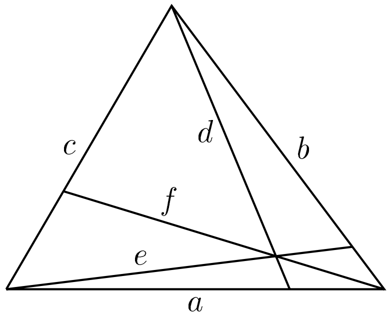

Figure 3: Dual of Menelaus' theorem

Now the diagram for Ceva's theorem is the dual of the diagram for Menelaus' theorem: instead of (Figure 2) three points A, B, C forming a triangle, and three collinear points

D, E, F on the lines BC, CA, AB, respectively, we have (Figure 3, cf. Figure 1) three lines a, b, c forming the sides of a triangle, and three concurrent lines d, e, f through the meets bc, ca, ab, respectively. The theorems as stated above are not duals, as there is no concept dual to the ratio of two lengths. However, if we take barycentric coordinates (x, y, z) with respect to $\triangle ABC$ as triangle of reference, and write λ, μ, ν for the ratios $\frac{BD}{DC}$, $\frac{CE}{EA}$, $\frac{AF}{FB}$, respectively, then $D = (0, 1, \lambda)$, $E = (\mu, 0, 1)$, and $F = (1, \nu, 0)$. The condition for these to be collinear is

$$\begin{vmatrix} 0 & 1 & \lambda \\ \mu & 0 & 1 \\ 1 & \nu & 0 \end{vmatrix} = 0, \tag{3}$$

or $\lambda\mu\nu = -1$, whence Menelaus' theorem, (2). The dual statement is that the *lines* with (line) coordinates $(0, 1, \lambda)$, $(\mu, 0, 1)$, and $(1, \nu, 0)$ (that is, the lines $y + \lambda z = 0$, $\mu x + z = 0$, $x + \nu y = 0$) are concurrent (or all parallel) iff (3) holds, or $\lambda\mu\nu = -1$, *again*. Comparing with (1), it looks as if a sign has gone wrong, but of course $y + \lambda z = 0$ meets $x = 0$ at $(0, \lambda, -1)$, which we now call D, so that here $\frac{BD}{DC} = -\frac{1}{\lambda}$, not λ, and similarly for E and F. So we just need to rewrite our equation $\lambda\mu\nu = -1$ as

$$\left(-\frac{1}{\lambda}\right)\left(-\frac{1}{\mu}\right)\left(-\frac{1}{\nu}\right) = 1,$$

and this is now Ceva's theorem, (1).

Since the left hand sides of (1) and (2) are the same, one suspects that there may be a more elementary route from one of the two theorems to the other, and indeed it is very easy to obtain Ceva's theorem as a *corollary* of Menelaus' theorem.

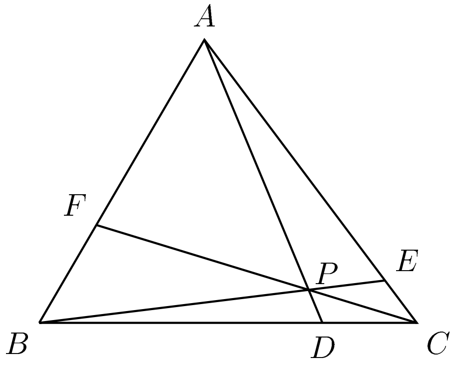

Figure 4: Ceva from Menelaus

For if AD, BE, CF meet at P (Figure 4), then we apply Menelaus' theorem first to $\triangle BDA$ and the collinear points C, P, F to obtain

$$\frac{BC}{CD} \cdot \frac{DP}{PA} \cdot \frac{AF}{FB} = -1, \tag{4}$$

and secondly to $\triangle DCA$ and the collinear points B, E, P to obtain

$$\frac{DB}{BC} \cdot \frac{CE}{EA} \cdot \frac{AP}{PD} = -1. \tag{5}$$

We now multiply the left hand sides of (4) and (5) together, noting that $\frac{DP}{PA}\cdot\frac{AP}{PD} = 1$ and $\frac{BC}{CD}\cdot\frac{DB}{BC} = \frac{BD}{DC}$. So

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = (-1)^2 = 1,$$

which is Ceva's theorem.

The question now arises: can Menelaus' theorem be obtained as a corollary of Ceva's theorem? The answer is yes, but we have to work much harder. To get Ceva's theorem from Menelaus, we just applied Menelaus' theorem twice to the Ceva diagram. To reverse the process and deduce Menelaus' theorem from Ceva, we shall need to add extra points and lines to the Menelaus diagram, apply Ceva's theorem *six* times, and then apply the *converse* of Ceva's theorem (once).

So suppose we are given $\triangle ABC$ with collinear points D, E, F, as before. Then (Figure 5) let BE, CF meet at X, let CF, AD meet at Y, and let AD, BE meet at Z. (We shall assume that these pairs of lines *do* meet. The case where one—or more?—pair is parallel is left as an exercise for the reader.)

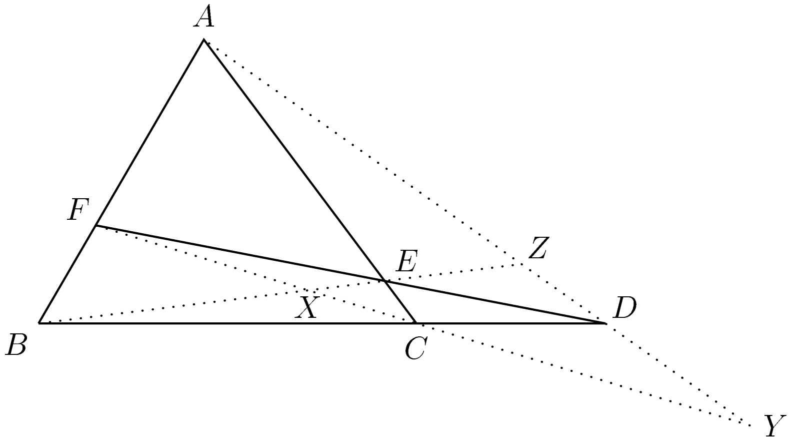

Figure 5: Menelaus from Ceva

With one eye on (2), we apply Ceva's theorem

(i) to $\triangle BCE$ and the concurrent lines BA, CX, ED;

(ii) to $\triangle CAF$ and the concurrent lines CB, AY, FE;

(iii) to $\triangle ABD$ and the concurrent lines AC, BZ, DF.

We obtain, respectively:

$$\frac{BD}{DC}\cdot\frac{CA}{AE}\cdot\frac{EX}{XB} = 1, \tag{6}$$

$$\frac{CE}{EA}\cdot\frac{AB}{BF}\cdot\frac{FY}{YC} = 1, \tag{7}$$

$$\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DZ}{ZA} = 1. \tag{8}$$

If we multiply these together, we shall get the left hand side of (2) and more, but we particularly want to be able to get rid of the terms involving X, Y, and Z. So, looking at Figure 5 again, we apply Ceva's theorem

(iv) to $\triangle BEF$ and the concurrent lines BD, EA, FX;

(v) to $\triangle CFD$ and the concurrent lines CE, FB, DY;

(vi) to $\triangle ADE$ and the concurrent lines AF, DC, EZ.

We obtain, respectively:

$$\frac{BX}{XE}\cdot\frac{ED}{DF}\cdot\frac{FA}{AB} = 1, \tag{9}$$

$$\frac{CY}{YF}\cdot\frac{FE}{ED}\cdot\frac{DB}{BC} = 1, \tag{10}$$

$$\frac{AZ}{ZD}\cdot\frac{DF}{FE}\cdot\frac{EC}{CA} = 1. \tag{11}$$

If we now multiply the left hand sides of (6)–(11) together, noting that

$$\frac{EX}{XB}\cdot\frac{BX}{XE} = \frac{FY}{YC}\cdot\frac{CY}{YF} = \frac{DZ}{ZA}\cdot\frac{AZ}{ZD} = 1,$$

and also that the terms AB, BC, CA, FE, ED, DF all cancel, we obtain

$$\left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)\left(\frac{DB}{CD}\cdot\frac{EC}{AE}\cdot\frac{FA}{BF}\right) = 1^6 = 1,$$

that is,

$$\left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)^2 = 1. \tag{12}$$

Since the lines AD, BE, CF are neither concurrent nor all parallel—OK?—we deduce from the *converse* of Ceva's theorem that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} \neq 1,$$

and so from (12) we finally have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1,$$

which is Menelaus' theorem.

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