The r-subsequences of the Fibonacci sequence

The Fibonacci sequence 0, 1, 1, 2, 3, ... is defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1. \tag{1}$$

In [1] some relations between the odd terms F_1, F_3, F_5, ... were obtained; here we show that similar relations hold between the terms of any r-subsequence, by which we mean a subsequence of the type F_n , F_{n+r} , F_{n+2r} , ...

Recall that the Fibonacci sequence can be extended to the left by rewriting the recurrence relation as $F_{n-1} = F_{n+1} - F_n$. Now,

$$\begin{aligned} F_n &= 2F_n - F_n, && (2)\\ \text{and} \quad F_{n+1} &= F_n + F_{n-1}. && (3) \end{aligned}$$

Adding (2) and (3), we obtain

$$F_{n+2} = 3F_n - F_{n-2}, \tag{4}$$

which is the lemma from [1]. If we now add (3) and (4), we get

$$F_{n+3} = 4F_n + F_{n-3}, \tag{5}$$

and so on. The coefficients 2, 1, 3, 4 of F_n in (2)–(5) are recognised as coming from the *Lucas* sequence, defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \text{ for } n \geq 1. \tag{6}$$

So if we continue the above process, we obtain a recurrence relation for any Fibonacci r-subsequence:

$$F_{n+r} = L_r F_n + (-1)^{r+1} F_{n-r}. \tag{7}$$

Indeed, since the Lucas numbers L_n satisfy the same recurrence relation as the F_n, a similar argument yields

$$L_{n+r} = L_r L_n + (-1)^{r+1} L_{n-r}. \tag{8}$$

Note, for future use, that on putting $n = r$ in this, we obtain

$$L_{2r} = L_r^2 + 2(-1)^{r+1}. \tag{9}$$

Likewise (7) gives

$$F_{2r} = L_r F_r. \tag{10}$$

(An alternative way of proceeding is to note that, if the quadratic polynomial $x^2 - x - 1$ associated to (1) and (6) has zeros σ, τ, then $\sigma + \tau = 1$ and $\sigma\tau = -1$. So $\sigma^2 + \tau^2 = (\sigma + \tau)^2 - 2\sigma\tau = 3$, and $\sigma^2\tau^2 = 1$. Thus the polynomial $x^2 - 3x + 1$ has zeros σ^2, τ^2, whence (4). More generally, solving (6) yields $L_r = \sigma^r + \tau^r$, and $\sigma^r\tau^r = (-1)^r$, so the polynomial $x^2 - L_r x + (-1)^r$ has zeros σ^r, τ^r, which gives (7) and (8).)

Next I shall recall some results from [2]. If we put

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix},$$

then an easy induction shows that

$$\mathbf{A}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Taking determinants, and noting that $\det \mathbf{A} = -1$, we have the well-known result

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \tag{11}$$

This should be compared with Corollary 1 of [1], which can be rewritten

$$F_{n-2}F_{n+2} - F_n^2 = 1 \quad (n \text{ odd}).$$

Surely this must come from another determinant? Indeed it does, and more generally if we put

$$\mathbf{B}_{n,\,r} = \begin{pmatrix} F_{n-r} & F_n \\ F_n & F_{n+r} \end{pmatrix}$$

we shall find the value of $\det \mathbf{B}_{n,\,r}$, that is, of $F_{n-r}F_{n+r} - F_n^2$, in part (iii) of the proposition below. Note that $\mathbf{B}_{n,\,1} = \mathbf{A}^n$; also, if we put

$$\mathbf{C}_r = \begin{pmatrix} 0 & 1 \\ (-1)^{r+1} & L_r \end{pmatrix},$$

then it is immediate from (7) that $\mathbf{C}_r\mathbf{B}_{n,\,r} = \mathbf{B}_{n+r,\,r}$. However, to avoid having r separate induction arguments for each value of r, we need a way of obtaining $\mathbf{B}_{n+1,\,r}$ (rather than $\mathbf{B}_{n+r,\,r}$) from $\mathbf{B}_{n,\,r}$, which we shall do in part (ii) of the proposition below.

First a couple more results from [2]. The equation $\mathbf{A}^m\mathbf{A}^n = \mathbf{A}^{m+n}$ yields

$$F_{m-1}F_n + F_mF_{n+1} = F_{m+n}. \tag{12}$$

Next, the equation $\mathbf{A}^{-n} = (\mathbf{A}^n)^{-1}$ gives

$$\begin{pmatrix} F_{-n-1} & F_{-n} \\ F_{-n} & F_{-n+1} \end{pmatrix} = (-1)^n \begin{pmatrix} F_{n+1} & -F_n \\ -F_n & F_{n-1} \end{pmatrix}$$

from which

$$F_{-n} = (-1)^{n+1}F_n. \tag{13}$$

Now put

$$\mathbf{D}_r = \begin{pmatrix} -F_{r-1} & 1 \\ (-1)^{r+1} & F_{r+1} \end{pmatrix}.$$

Proposition. (i) $\det \mathbf{D}_r = -F_r^2$ (ii) $\mathbf{D}_r\mathbf{B}_{n,\,r} = F_r\mathbf{B}_{n+1,\,r}$

(iii) $\det \mathbf{B}_{n,\,r} = (-1)^{n+r+1}F_r^2$ (iv) $\mathbf{D}_r^r = F_r^r\mathbf{C}_r$

Proof. (i) $\det \mathbf{D}_r = -F_{r-1}F_{r+1} + (-1)^r = -F_r^2$, by (11).

(ii)
$$\begin{aligned}\mathbf{D}_r\mathbf{B}_{n,\,r} &= \begin{pmatrix} -F_{r-1} & 1 \\ (-1)^{r+1} & F_{r+1}\end{pmatrix}\begin{pmatrix} F_{n-r} & F_n \\ F_n & F_{n+r}\end{pmatrix} \\ &= \begin{pmatrix} F_n - F_{r-1}F_{n-r} & F_{n+r} - F_{r-1}F_n \\ (-1)^{r+1}F_{n-r} + F_{r+1}F_n & (-1)^{r+1}F_n + F_{r+1}F_{n+r}\end{pmatrix}.\end{aligned}$$

In the first column, we have, by (12), $F_n - F_{r-1}F_{n-r} = F_rF_{n+1-r}$, as required; and then

$$\begin{aligned}(-1)^{r+1}F_{n-r} + F_{r+1}F_n &= (-1)^{r+1}(F_{n-r} - F_{-r-1}F_n), \quad \text{by (13)}, \\ &= (-1)^{r+1}F_{-r}F_{n+1}, \quad \text{by (12)}, \\ &= F_rF_{n+1}, \quad \text{by (13)},\end{aligned}$$

as required. Similarly for the second column, replacing n by $n+r$. This completes (ii).

(iii) Since $F_0 = 0$, $\det \mathbf{B}_{0,\,r} = F_{-r}F_r = (-1)^{r+1}F_r^2$, by (13). So the result is true for $n = 0$. The general result now follows by induction on n, using (i) and (ii).

(iv) This is an exercise for the reader. (It is not used elsewhere.) ∎

Writing out (iii) in full, we have

$$F_{n-r}F_{n+r} - F_n^2 = (-1)^{n+r+1}F_r^2, \tag{14}$$

which generalises both (11) and also Corollary 1 of [1]. Next, we generalise the theorem from [1]. If we substitute into (14) from (7), we get the generalisation

$$L_rF_nF_{n-r} = F_n^2 + (-1)^rF_{n-r}^2 + (-1)^{n+r+1}F_r^2. \tag{15}$$

When $r = 2$ this becomes $3F_nF_{n-2} = F_n^2 + F_{n-2}^2 + (-1)^{n+1}$, which in the case where n is odd is the theorem from [1].

Finally we shall obtain, in (16) below, a generalisation of Corollary 2 from [1]. First, replace r by $2r$ and n by $n+r$ in (15):

$$L_{2r}F_{n+r}F_{n-r} = F_{n+r}^2 + F_{n-r}^2 + (-1)^{n+r+1}F_{2r}^2.$$

Substituting for $F_{n+r}F_{n-r}$ from (14),

$$\begin{aligned}F_{n+r}^2 - L_{2r}F_n^2 + F_{n-r}^2 &= (-1)^{n+r+1}(L_{2r}F_r^2 - F_{2r}^2) \\ &= (-1)^{n+r+1}F_r^2(L_{2r} - L_r^2), \quad \text{by (10)}.\end{aligned}$$

By (9), we deduce

$$F_{n+r}^2 - L_{2r}F_n^2 + F_{n-r}^2 = 2(-1)^nF_r^2. \tag{16}$$

When $r = 2$ this becomes $F_{n+2}^2 - 7F_n^2 + F_{n-2}^2 = 2(-1)^n$, which, in the case where n is odd, is Corollary 2 from [1].

References

1. V. Rajesh and G. Leversha, Some properties of odd terms of the Fibonacci sequence, Math. Gaz. 88 (2004), 85–6.
2. J. R. Silvester, Fibonacci properties by matrix methods, Math. Gaz. 63 (1979), 188–91.

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