Extensions of a theorem of Van Aubel

1 Introduction and preliminaries

A theorem of Van Aubel, which first appeared in [2], concerns the figure obtained by erecting squares on the sides of a quadrilateral, and considering various line-segments such as the joins of the centres of opposite squares. In [1], parts of the theorem are extended to the case of (i) four similar rhombi, and (ii) four similar rectangles erected on the sides of a quadrilateral. Further, these two results are described as in some sense "dual" to each other. Here we show that they are in fact both special cases of a further extension of the original theorem (see Proposition 6, and the remarks that follow it), and we also find several extra squares in the original Van Aubel figure (Theorem 11).

We shall adopt the general convention that points A_1, B_2, ... are represented as complex numbers by the corresponding lower case letters a_1, b_2, ...

In a number of places we shall make use of the following well-known technical device:

Equal Fractions Lemma. If several fractions are equal, then each of them is equal to any linear combination of their numerators divided by the corresponding linear combination of their denominators (provided this is not zero).

Proof. If $u = \dfrac{x_j}{y_j}$ for $j = 1, 2, \ldots$, then for any $\lambda_1, \lambda_2, \ldots$ (with $\sum_j \lambda_j y_j \neq 0$),

$$\frac{\sum_j \lambda_j x_j}{\sum_j \lambda_j y_j} = \frac{\sum \lambda_j (u y_j)}{\sum_j \lambda_j y_j} = \frac{u \sum_j \lambda_j y_j}{\sum_j \lambda_j y_j} = u. \qquad \blacksquare$$

Is this well-known? I've mentioned it once or twice in company recently, and got blank looks all round. In my youth, it featured regularly in school examinations, and I had the impression that most people were familiar with it. Thirty years ago, I remember using it to solve a newspaper brain-twister, and saying to a very eminent colleague (who shall remain nameless) that any reasonably intelligent non-mathematician ought to be able to follow the argument. "If they were reasonably intelligent," he remarked mildly, "they wouldn't be non-mathematicians!"—but I digress.

2 The diagonal ratio of a quadrilateral

Let $A_1A_2A_3A_4$ be a quadrilateral, which might possibly be non-convex, or might even have two opposite sides crossing internally. The *diagonals* of $A_1A_2A_3A_4$ are the line segments A_1A_3 and A_2A_4, and we define the *diagonal ratio* $(A_1A_2A_3A_4)$ to be the complex number given by the formula

$$(A_1A_2A_3A_4) = \frac{a_1 - a_3}{a_2 - a_4}.$$

We shall say that the quadrilateral $A_1A_2A_3A_4$ is *equidiagonal* if the lengths A_1A_3, A_2A_4 are equal, that is, if $|(A_1A_2A_3A_4)| = 1$, and *orthodiagonal* if $A_1A_3 \perp A_2A_4$, that is, if $(A_1A_2A_3A_4)$ is pure imaginary. So, for example, every rectangle is equidiagonal and every rhombus is orthodiagonal.

If the quadrilateral Γ has diagonal ratio z, then a cyclic permutation, or a reversal (or both) of the vertex order of Γ will produce a quadrilateral with diagonal ratio z, $-z$, z^{-1} or $-z^{-1}$, and it would make some sense to regard these complex numbers as equivalent, and then work with equivalence classes. However, we choose not to do this, and instead are careful at various steps to name vertices in the order that produces the "correct" (i.e., desired) value of the diagonal ratio. Notice, however, that if z has modulus 1 (respectively, if z is pure imaginary), then the same can be said of $-z$, z^{-1}, and $-z^{-1}$.

Now let $A_1A_2A_3A_4$ be a quadrilateral, and let $B_1B_2B_3B_4$ be the mid-point quadrilateral: specifically, let B_1, B_2, B_3, B_4 be the mid-points of A_2A_1, A_1A_4, A_4A_3, A_3A_2 respectively (note the order). It is a well-known theorem of Varignon that $B_1B_2B_3B_4$ is always a parallelogram; but a parallelogram can have any diagonal ratio. However, it is easy to show that the diagonal ratios $z = (A_1A_2A_3A_4)$ and $w = (B_1B_2B_3B_4)$ are related:

PROPOSITION 1. With z, w as above, we have $w = \dfrac{z+1}{z-1}$.

PROOF.

$$w = \frac{b_1 - b_3}{b_2 - b_4} = \frac{\frac{1}{2}(a_2 + a_1) - \frac{1}{2}(a_4 + a_3)}{\frac{1}{2}(a_1 + a_4) - \frac{1}{2}(a_3 + a_2)} = \frac{(a_1 - a_3) + (a_2 - a_4)}{(a_1 - a_3) - (a_2 - a_4)} = \frac{z+1}{z-1}. \qquad \blacksquare$$

COROLLARY 2. Let $A_1A_2A_3A_4$ have mid-point parallelogram $B_1B_2B_3B_4$. Then

(i) $A_1A_2A_3A_4$ is equidiagonal iff $B_1B_2B_3B_4$ is orthodiagonal, and

(ii) $A_1A_2A_3A_4$ is orthodiagonal iff $B_1B_2B_3B_4$ is equidiagonal.

PROOF. The Möbius transformation $f : z \mapsto \dfrac{z+1}{z-1}$ is an involution, that is, $f^2 = 1$, and it interchanges 0 with -1, i with $-i$, and 1 with ∞. Consequently, it interchanges the unit circle $|z| = 1$ with the imaginary axis, and the result follows. $\blacksquare$

Of course, this corollary has a much easier geometric proof (not using complex numbers), which we leave to the reader.

PROPOSITION 3. Let $f : \mathbb{C} \to \mathbb{C}$ be a transformation, and for each point A write $f(A) = A'$. Then, for any quadrilateral $A_1A_2A_3A_4$,

(i) if f is a direct (that is, orientation-preserving) similarity, then $(A_1A_2A_3A_4) = (A'_1A'_2A'_3A'_4)$;

(ii) if f is a translation, then $(A_1A_2A_3A_4) = (A'_1A_2A'_3A_4) = (A_1A'_2A_3A'_4)$.

PROOF. There exist $u, v \in \mathbb{C}$, with $u \neq 0$, such that, for all $z \in \mathbb{C}$, (i) $f(z) = uz + v$; (ii) $f(z) = z + v$. The result is immediate. $\blacksquare$

The next corollary is about what we might call the *side ratio* $\dfrac{a_4 - a_1}{a_2 - a_1}$ of a parallelogram $A_1A_2A_3A_4$, though we shall simply regard this as the diagonal ratio of the degenerate quadrilateral $A_4A_2A_1A_1$:

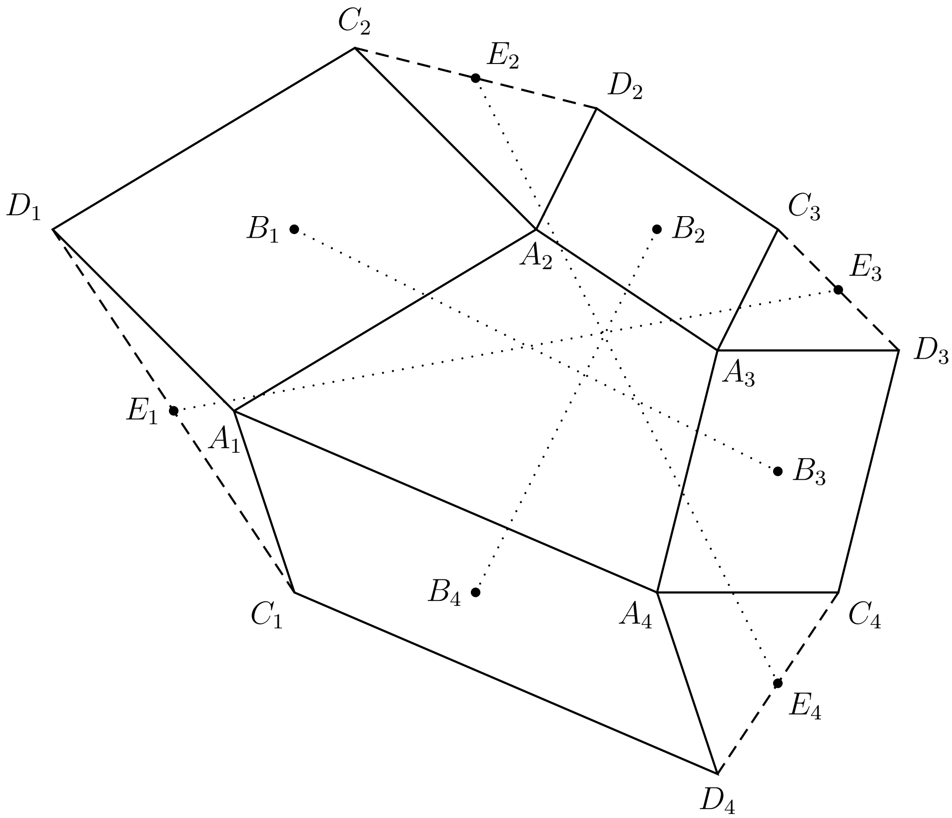

Figure 1: Proposition 5

Corollary 4. Let $A_1A_2A_3A_4$ be a parallelogram, and let B_1, B_2, B_3, B_4 be the mid-points of A_1A_2, A_2A_3, A_3A_4, A_4A_1 respectively. Then $(A_4A_2A_1A_1) = (B_3B_2B_1B_4)$.

Proof. Immediate from Proposition 3(ii) (or by direct calculation). ∎

3 A theorem about four parallelograms

Proposition 5. Given a quadrilateral $A_1A_2A_3A_4$, erect parallelograms $A_1A_2C_2D_1$, $A_2A_3C_3D_2$, $A_3A_4C_4D_3$, $A_4A_1C_1D_4$ on the sides, and let the parallelograms have centres B_1, B_2, B_3, B_4 respectively. Let the mid-point of C_kD_k be E_k, $k = 1, 2, 3, 4$. Then

(i) $B_1B_2B_3B_4$ is equidiagonal iff $E_1E_2E_3E_4$ is orthodiagonal, and

(ii) $B_1B_2B_3B_4$ is orthodiagonal iff $E_1E_2E_3E_4$ is equidiagonal.

(See Figure 1, which shows the second case, where $B_1B_2B_3B_4$ is orthodiagonal. For clarity, a case where all four of the parallelograms are *external* to $A_1A_2A_3A_4$ is shown, though in fact each one (separately) can be erected either externally or internally.)

Proof. B_1 is the mid-point of A_1C_2, so $b_1 = \frac{1}{2}(a_1 + c_2)$, whence $c_2 = 2b_1 - a_1$; similarly $d_2 = 2b_2 - a_3$, and so $e_2 = \frac{1}{2}(c_2 + d_2) = b_1 + b_2 - \frac{1}{2}(a_1 + a_3)$. Likewise $e_3 = b_2 + b_3 - \frac{1}{2}(a_2 + a_4)$,

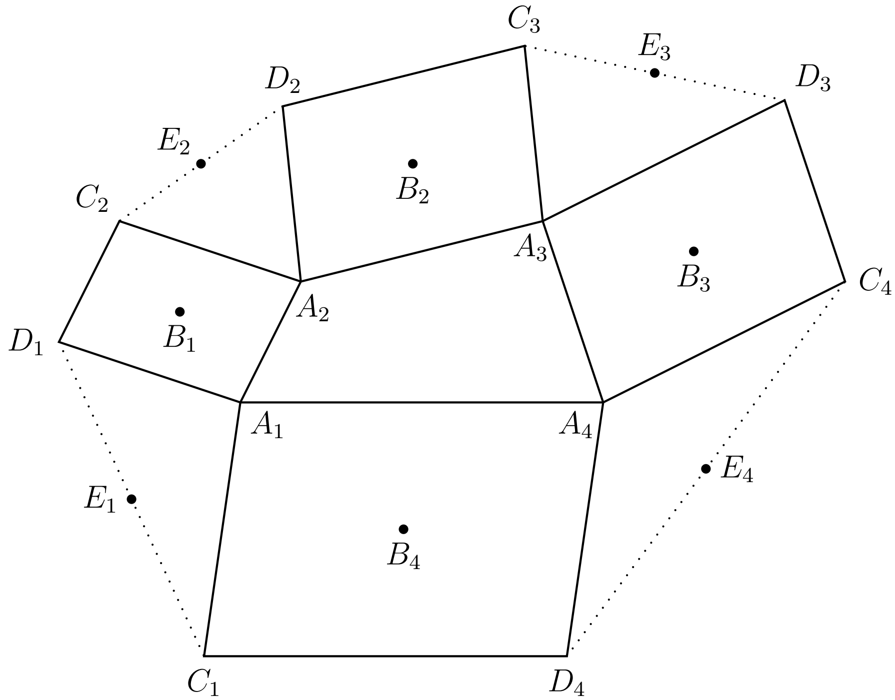

Figure 2: Four similar parallelograms

$e_4 = b_3 + b_4 - \frac{1}{2}(a_1 + a_3)$, and $e_1 = b_4 + b_1 - \frac{1}{2}(a_2 + a_4)$. By two applications of Proposition 3(ii) and one application of Proposition 3(i), it follows that $E_1E_2E_3E_4$ has the same diagonal ratio as the mid-point parallelogram of $B_1B_2B_3B_4$, and the result follows from Corollary 2. ∎

4 Theorems about four similar parallelograms

For the rest of this paper, we shall be concerned with a special case of Figure 1, where the four parallelograms are directly similar; but the vertices do not correspond in perhaps the most obvious order. We write $P_1P_2P_3P_4 \sim Q_1Q_2Q_3Q_4$ to mean that $P_1P_2P_3P_4$ is directly similar to $Q_1Q_2Q_3Q_4$ with P_k corresponding to Q_k, for each k. We shall assume that

$$A_1A_2C_2D_1 \sim A_3C_3D_2A_2 \sim A_3A_4C_4D_3 \sim A_1C_1D_4A_4. \tag{1}$$

The points E_k are defined as before, throughout. See Figure 2.

Proposition 6. Given (1), we have $(B_1B_2B_3B_4) = (D_1A_2A_1A_1)$.

Proof. By Proposition 3(i), we have

$$(D_1A_2A_1A_1) = (A_2C_3A_3A_3) = (D_3A_4A_3A_3) = (A_4C_1A_1A_1) = u \text{ (say)},$$

that is,

$$u = \frac{2b_1 - a_2 - a_1}{a_2 - a_1} = \frac{a_2 - a_3}{2b_2 - a_2 - a_3} = \frac{2b_3 - a_4 - a_3}{a_4 - a_3} = \frac{a_4 - a_1}{2b_4 - a_4 - a_1}. \tag{2}$$

Using the equal fractions lemma,

$$u = \frac{(2b_1 - a_2 - a_1) + (a_2 - a_3) - (2b_3 - a_4 - a_3) - (a_4 - a_1)}{(a_2 - a_1) + (2b_2 - a_2 - a_3) - (a_4 - a_3) - (2b_4 - a_4 - a_1)}$$

which is just $(B_1B_2B_3B_4)$. ∎

In particular, $\dfrac{B_1B_3}{B_2B_4} = \dfrac{A_1D_1}{A_1A_2}$, and also, if the lines B_1B_3 and B_2B_4 meet at O, then $\angle B_1OB_2 = \angle D_1A_1A_2$. From Proposition 5, with the E_k defined as before, $\dfrac{E_1E_3}{E_2E_4} = \dfrac{A_2D_1}{A_1C_2}$ and also, if the lines E_1E_3 and E_2E_4 meet at Q, then $\angle E_1QE_2 = \angle A_2B_1A_1$.

As special cases, if $A_1A_2C_2D_1$ is a rectangle, then $B_1B_2B_3B_4$ is orthodiagonal and $E_1E_2E_3E_4$ is equidiagonal; and if $A_1A_2C_2D_1$ is a rhombus, then $B_1B_2B_3B_4$ is equidiagonal and $E_1E_2E_3E_4$ is orthodiagonal. This shows that [1, Theorems 5 & 6], rather than being in some sense "dual" to each other, are in fact two special cases of the same theorem. Van Aubel's theorem itself is the case where $A_1A_2C_2D_1$ is a square, so that $B_1B_2B_3B_4$ is both equidiagonal and orthodiagonal, likewise $E_1E_2E_3E_4$.

We now turn to the question of when the four lines B_1B_3, B_2B_4, E_1E_3, E_2E_4 are concurrent, that is, when the points O and Q coincide. First, an easy special case:

PROPOSITION 7. Given (1), if $A_1A_2A_3A_4$ is a parallelogram, then the six lines A_1A_3, A_2A_4, B_1B_3, B_2B_4, E_1E_3, E_2E_4 are concurrent.

PROOF. One does not need complex numbers for this: if A_1A_3 and A_2A_4 meet at P, then the entire diagram is symmetrical by a half-turn about P; see Figure 3. (Nonetheless, the reader might like to check that, if $a_1 + a_3 = a_2 + a_4 = 2p$, then also $b_1 + b_3 = b_2 + b_4 = e_1 + e_3 = e_2 + e_4 = 2p$.) ∎

In the general case, let us take O as the origin; so now B_1, B_2 and O are collinear, and thus $\dfrac{b_1}{b_3} \in \mathbb{R}$; likewise $\dfrac{b_2}{b_4} \in \mathbb{R}$. Now $u = \dfrac{b_1 - b_3}{b_2 - b_4}$, by Proposition 6, and it follows that u is a real multiple of $\dfrac{b_1}{b_2}$, likewise of $\dfrac{b_3}{b_4}$. So there exist λ, $\mu \in \mathbb{R}$ with $\dfrac{b_1}{b_2} = \lambda u$ and $\dfrac{b_3}{b_4} = \mu u$.

Next,

$$\frac{e_2}{e_4} = \frac{2b_1 + 2b_2 - a_1 - a_3}{2b_3 + 2b_4 - a_1 - a_3}.$$

From (2),

$$u = \frac{2b_1 - a_2 - a_1}{a_2 - a_1} = \frac{a_2 - a_3}{2b_2 - a_2 - a_3} = \frac{(2b_1 - a_2 - a_1) + (a_2 - a_3)}{(a_2 - a_1) + (2b_2 - a_2 - a_3)} = \frac{2b_1 - a_1 - a_3}{2b_2 - a_1 - a_3},$$

whence $a_1 + a_3 = \dfrac{2(b_1 - ub_2)}{1 - u}$, and similarly $a_1 + a_3 = \dfrac{2(b_3 - ub_4)}{1 - u}$. So, substituting back,

$$\frac{e_2}{e_4} = \frac{(1-u)(b_1 + b_2) - (b_1 - ub_2)}{(1-u)(b_3 + b_4) - (b_3 - ub_4)} = \frac{b_2 - ub_1}{b_4 - ub_3} = \left(\frac{b_2}{b_4}\right)\left(\frac{1 - \lambda u^2}{1 - \mu u^2}\right).$$

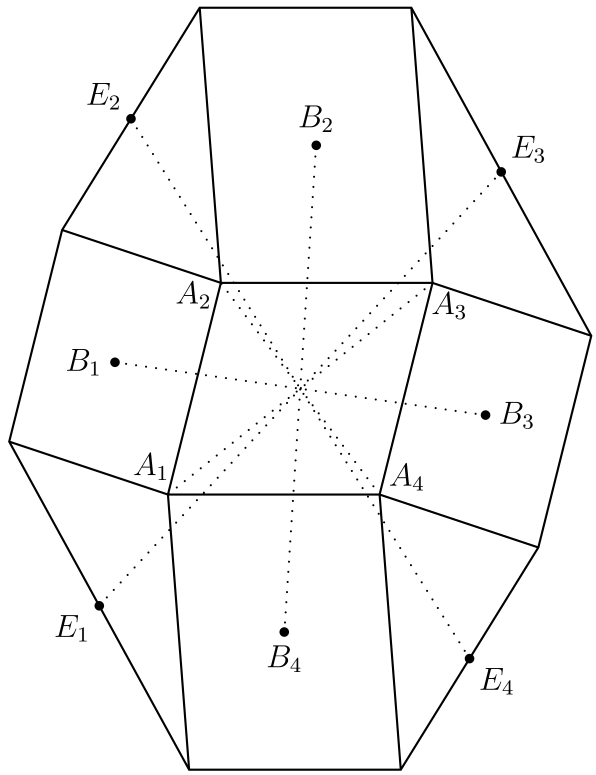

Figure 3: Proposition 7, and Proposition 8, case (ii).

It follows that

$$\begin{aligned}\frac{e_2}{e_4} \in \mathbb{R} &\iff (1-\lambda u^2)(1-\mu\overline{u}^2) \in \mathbb{R} \\ &\iff 1+\lambda\mu(u\overline{u})^2 - \lambda u^2 - \mu\overline{u}^2 \in \mathbb{R} \\ &\iff (\mu-\lambda)u^2 + \Big(1+\lambda\mu(u\overline{u})^2 - \mu(u^2+\overline{u}^2)\Big) \in \mathbb{R} \\ &\iff (\mu-\lambda)u^2 \in \mathbb{R}.\end{aligned}$$

But $\mu - \lambda \in \mathbb{R}$, and so $\frac{e_2}{e_4} \in \mathbb{R}$ iff either $\lambda = \mu$ or $u^2 \in \mathbb{R}$. This leads to:

Proposition 8. Given (1), the four lines B_1B_3, B_2B_4, E_1E_3, E_2E_4 are concurrent iff

(i) $A_1A_2C_2D_1$ is a rectangle, or

(ii) $A_1A_2A_3A_4$ is a parallelogram.

(See Figures 3 and 4. The sufficiency of (i) is part of [1, Theorem 5].)

Proof. If (i) holds, then $(A_1A_2C_2D_1) = u$ is pure imaginary, so that $u^2 \in \mathbb{R}$, and by the above calculation it follows that $\frac{e_2}{e_4} \in \mathbb{R}$ and thus E_2E_4 passes through O; similarly, $\frac{e_1}{e_3} \in \mathbb{R}$, and E_1E_3 passes through O also. If (ii) holds, use Proposition 7.

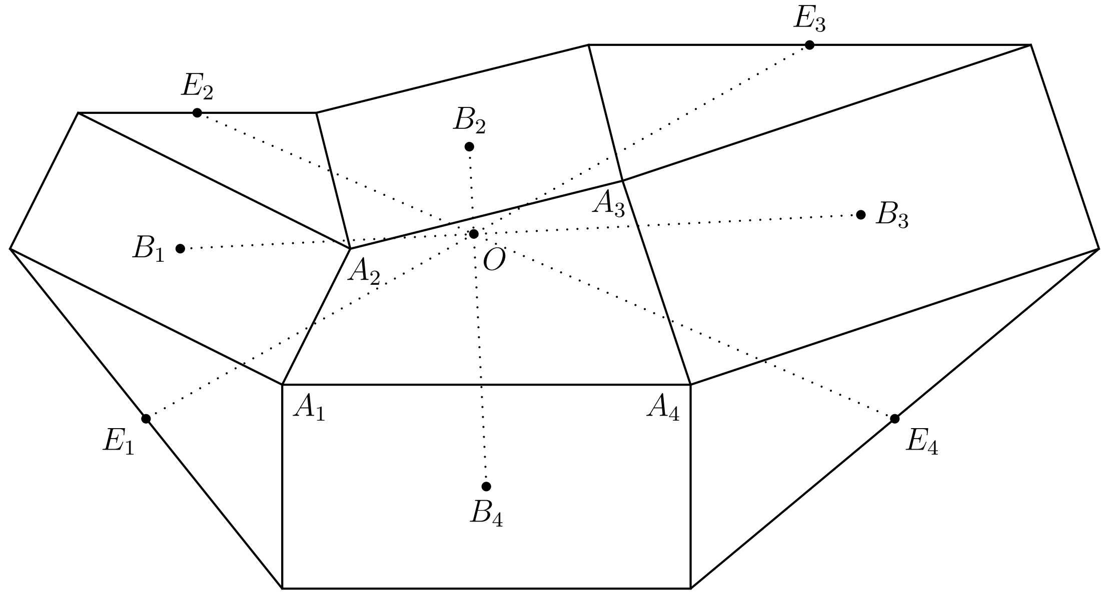

Figure 4: Proposition 8, case (i).

So now suppose we are not in case (i). We shall assume that u is not real, for if it is, then we are in a degenerate case in which the parallelograms collapse, and C_2, D_1 lie on the line A_1A_2. The reader might like to show that when this happens the four lines B_1B_3, B_2B_4, E_1E_3, E_2E_4 are all parallel, so that they are concurrent at a point at infinity.

If we are not in case (i) and u is not real, then u^2 is not real either: so, by the above,

$$\begin{aligned}
E_2E_4 \text{ passes through } O &\implies \lambda = \mu \\
&\implies \frac{b_1}{b_2} = \frac{b_3}{b_4} \\
&\implies \frac{b_1}{b_2} = \frac{b_3}{b_4} = \frac{b_1 - b_3}{b_2 - b_4} = u \\
&\implies b_1 - ub_2 = 0 = b_3 - ub_4 \\
&\implies a_1 + a_3 = 0.
\end{aligned}$$

Similarly, if E_1E_3 passes through O, then $a_2 + a_4 = 0$. So if the four lines B_1B_3, B_2B_4, E_1E_3, E_2E_4 are concurrent and we are not in case (i), then $a_1 + a_3 = 0 = a_2 + a_4$, and we are in case (ii). ∎

PROPOSITION 9. In the rectangular case (Proposition 8, case (i)), B_1B_3 and B_2B_4 are the bisectors of the angles between E_1E_3 and E_2E_4 at O. (This also appears in [1].)

PROOF. We need to show that $\angle E_2OB_2 = \angle B_2OE_3$, and for this we just need $\dfrac{e_2e_3}{b_2^2} \in \mathbb{R}$.

As before, we have

$$e_2 = b_1 + b_2 - \frac{1}{2}(a_1 + a_3) = b_1 + b_2 - \frac{b_1 - ub_2}{1 - u} = \frac{b_2 - ub_1}{1 - u}$$

and, by a similar calculation,

$$e_3 = b_2 + b_3 - \frac{1}{2}(a_2 + a_4) = b_2 + b_3 - \frac{ub_2 + b_3}{1+u} = \frac{b_2 + ub_3}{1+u},$$

so that

$$\frac{e_2 e_3}{b_2^2} = \frac{1}{1-u^2}\left(1 - u\frac{b_1}{b_2}\right)\left(1 + u\frac{b_3}{b_2}\right).$$

Since each of $\frac{b_1}{b_2}$ and $\frac{b_3}{b_2}$ is a real multiple of u, and $u^2 \in \mathbb{R}$, we see that $\frac{e_2 e_3}{b_2^2} \in \mathbb{R}$, and the result follows. ∎

We now return to the general case (1), where the four similar parallelograms are not necessarily rectangles. We are going to find two more parallelograms similar to these four. To this end, let F_1, F_2 be the mid-points of B_1B_3, B_2B_4 respectively; let G_1, G_2 be the mid-points of E_1E_3, E_2E_4 respectively; and let H_1, H_2 be the mid-points of A_1A_3, A_2A_4 respectively. Of course, if $A_1A_2A_3A_4$ is a parallelogram, then the six points we have just defined will all coincide; so let us assume that $A_1A_2A_3A_4$ is *not* a parallelogram. We have

$$\begin{aligned}
2f_1 &= b_1 + b_3, \\
2f_2 &= b_2 + b_4, \\
2g_1 &= e_1 + e_3 = b_1 + b_2 + b_3 + b_4 - (a_2 + a_4), \\
2g_2 &= e_2 + e_4 = b_1 + b_2 + b_3 + b_4 - (a_1 + a_3), \\
2h_1 &= a_1 + a_3, \\
2h_2 &= a_2 + a_4.
\end{aligned}$$

It is immediate that $f_1 + f_2 = g_1 + h_2 = g_2 + h_1$, so that the mid-points of F_1F_2, G_1H_2 and G_2H_1 coincide. Thus the quadrilaterals $F_1G_1F_2H_2$, $G_2F_1H_1F_2$ and $G_1G_2H_2H_1$ are parallelograms.

Proposition 10. Given (1), and assuming $A_1A_2A_3A_4$ is not a parallelogram, then $A_1A_2C_2D_1 \sim F_1G_1F_2H_2 \sim G_2F_1H_1F_2$.

Proof. We have

$$\begin{aligned}
(H_2G_1F_1F_1) &= \frac{h_2 - f_1}{g_1 - f_1} = \frac{a_2 + a_4 - b_1 - b_3}{b_2 + b_4 - a_2 - a_4}, \quad \text{and} \\
(F_2F_1G_2G_2) &= \frac{f_2 - g_2}{f_1 - g_2} = \frac{a_1 + a_3 - b_1 - b_3}{a_1 + a_3 - b_2 - b_4}.
\end{aligned}$$

Let $u = (D_1A_2A_1A_1)$, as in (2). From (2), and the lemma on equal fractions,

$$u = \frac{-(2b_1 - a_2 - a_1) \pm (a_2 - a_3) - (2b_3 - a_4 - a_3) \pm (a_4 - a_1)}{-(a_2 - a_1) \pm (2b_2 - a_2 - a_3) - (a_4 - a_3) \pm (2b_4 - a_4 - a_1)}.$$

Taking the four upper signs, we have $u = (H_2G_1F_1F_1)$, and taking the four lower signs, we have $u = (F_2F_1G_2G_2)$. ∎

5 Van Aubel revisited

THEOREM 11. Suppose the four parallelograms of condition (1) are squares. Then (using the above notation)

(i) $B_1B_2B_3B_4$ is equidiagonal and orthodiagonal;

(ii) $E_1E_2E_3E_4$ is equidiagonal and orthodiagonal;

(iii) the four lines B_1B_3, B_2B_4, E_1E_3, E_2E_4 are concurrent and equally inclined;

(iv) the mid-point parallelogram of $B_1B_2B_3B_4$ is a square;

(v) the mid-point parallelogram of $E_1E_2E_3E_4$ is a square;

(vi) $G_1 = H_1$, $G_2 = H_2$, and (provided $A_1A_2A_3A_4$ is not a parallelogram) $F_1G_1F_2G_2$ is a square; and finally

(vii) the square in (iv) is the mid-point parallelogram of the square in (v).

See Figure 5, where for clarity we have taken $A_1A_2A_3A_4$ convex and the squares external; we invite the reader to draw the corresponding diagram for some other cases, e.g., a case where the four original squares are erected *internally* on the sides of $A_1A_2A_3A_4$, or a case where two opposite sides of $A_1A_2A_3A_4$ cross internally.

PROOF. Most of this is done already: (i) follows from Proposition 6, as previously remarked, and then (ii) follows from (i) and Proposition 5. Then (iii) is Proposition 8, case (i), together with Proposition 9. This far, the theorem is Van Aubel's theorem, as stated in [1].

Next, (iv) and (v) follow from (i) and (ii) together with Corollary 2.

Finally, we prove (vi) and (vii). From Proposition 10, $F_1G_1F_2H_2$ and $G_2F_1H_1F_2$ are directly similar, and are squares. Since they share the opposite vertices F_1 and F_2, they coincide, so $G_1 = H_1$ and $G_2 = H_2$, which completes (vi). (The reader is invited to check that $G_1 = H_1$ and $G_2 = H_2$ even if $A_1A_2A_3A_4$ is a parallelogram.) From $g_1 = h_1$ we deduce that

$$b_1 + b_2 + b_3 + b_4 = a_1 + a_2 + a_3 + a_4. \tag{3}$$

There is an amusing alternative way to derive (3): we have

$$\frac{b_1 - a_1}{b_1 - a_2} = \frac{b_2 - a_2}{b_2 - a_3} = \frac{b_3 - a_3}{b_3 - a_4} = \frac{b_4 - a_4}{b_4 - a_1} = i \text{ (or } -i).$$

But here the numerators and denominators both add up to $b_1+b_2+b_3+b_4-a_1-a_2-a_3-a_4$, so if this is not zero, the lemma on equal fractions gives us $1 = i$ (or $-i$), a contradiction, and so (3) follows, again.

We now have

$$\begin{aligned} e_1 + 2e_2 + e_3 &= \left(b_4 + b_1 - \frac{a_2 + a_4}{2}\right) + 2\left(b_1 + b_2 - \frac{a_1 + a_3}{2}\right) + \left(b_2 + b_3 - \frac{a_2 + a_4}{2}\right) \\ &= 3b_1 + 3b_2 + b_3 + b_4 - (a_1 + a_2 + a_3 + a_4) \\ &= 2(b_1 + b_2), \end{aligned}$$

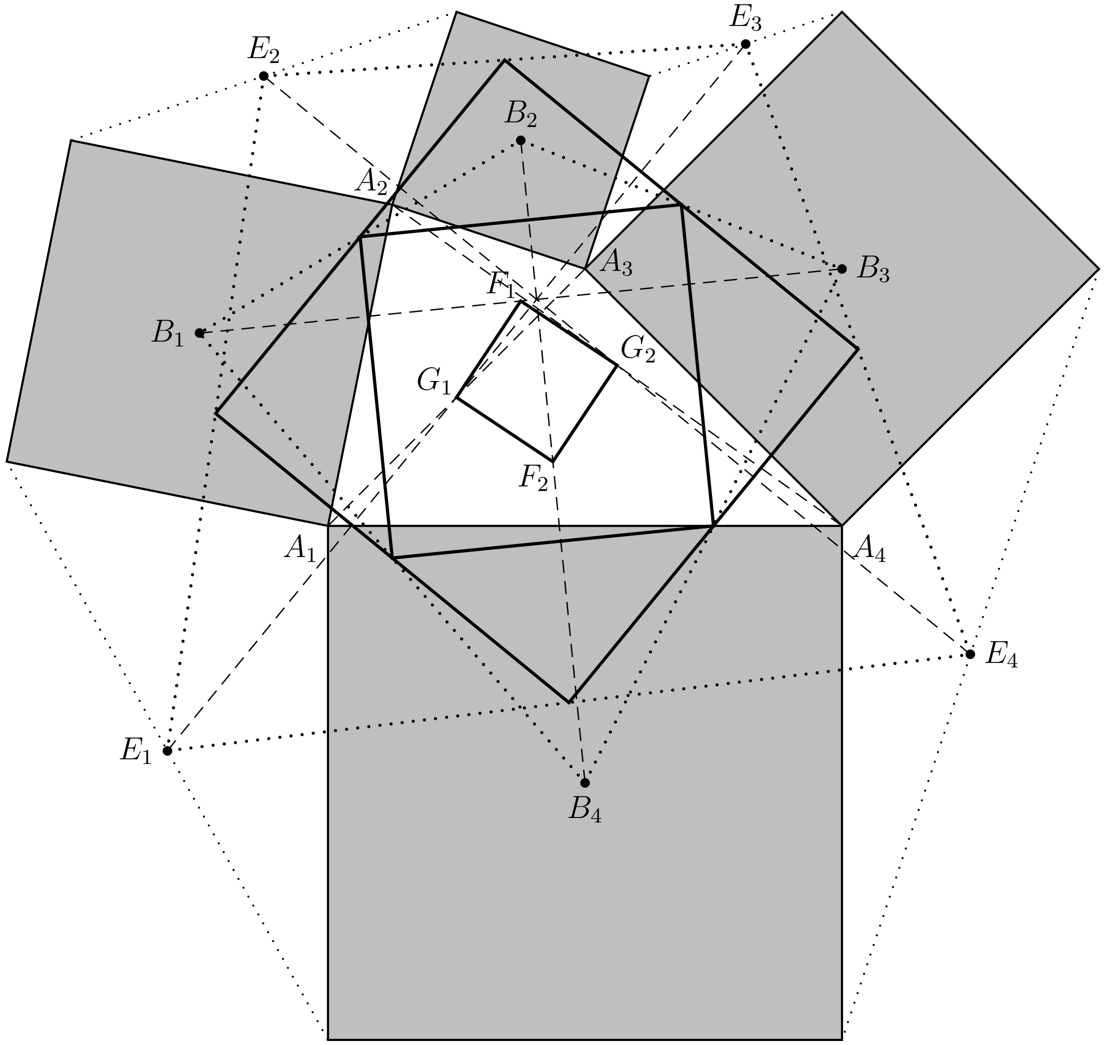

Figure 5: Theorem 11

or

$$\frac{1}{2}\left(\frac{e_1+e_2}{2}+\frac{e_2+e_3}{2}\right)=\frac{b_1+b_2}{2},$$

which says that the mid-point of the line joining the mid-points of E_1E_2 and E_2E_3 is the same as the mid-point of B_1B_2. A similar calculation for the other three vertices finishes (vii). ∎

As a parting shot, we invite the reader to draw the diagram in the special case where $A_3 = A_4$, so that in fact $A_3 = A_4 = B_3 = C_4 = D_3$. We then obtain a theorem about three squares erected on the sides of $\triangle A_1A_2A_3$, considered as a degenerate quadrilateral $A_1A_2A_3A_3$, so we can add to this the corresponding theorems about the degenerate quadrilaterals $A_1A_2A_2A_3$ and $A_1A_1A_2A_3$. Part of what results is a familiar theorem about the concurrency of the lines joining A_1, A_2, A_3 to the centres of the opposite squares (at the orthocentre of the triangle formed by these centres), but Theorem 11 also gives us nine other squares to be drawn within the figure. The author's recommended method for drawing this is to use a dynamic geometry program to produce figure 5, and then move A_4 to coincide with A_3. A macro can be used to reproduce this figure so as to superimpose the three degenerate quadrilaterals, and judicious use of colouring helps make sense of the resulting collection of lines and squares.

Reference

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