# GEOMETRY
# ANCIENT & MODERN

*from the book of the same name*
*(Oxford, 2001)*

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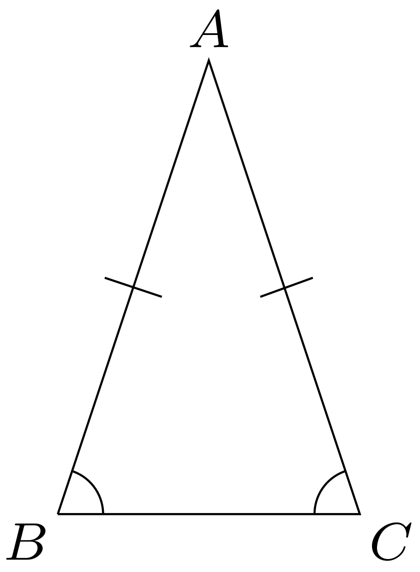
Geometry probably began in Egypt, where the River Nile regularly washed away field boundaries, and they had to have methods to recalculate the areas, for tax purposes.

In about 600BC, Thales travelled from Greece to Egypt, and brought back the methods, and it was the Greeks who first set out to support these methods with proofs.

“No one before them had thought of proving such a thing as that the two base angles of an isosceles triangle are equal: the idea was an inspiration unique in the history of the world, and the fruit of it was the creation of mathematics as a science.” [Heath]



# Pons asinorum

(The asses’ bridge)

If in $\triangle ABC$ we have $AB = AC$, then also $\angle ABC = \angle ACB$.

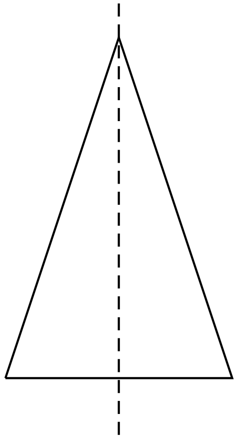
→S1

Pupil:

“Master, what shall I gain
by learning these things?”

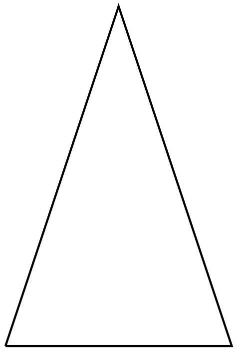
Euclid *(to his assistant)*:

“Give him threepence, since he must
*gain* by what he learns.”



Today's proof:
It's symmetrical, innit.

or, more likely:

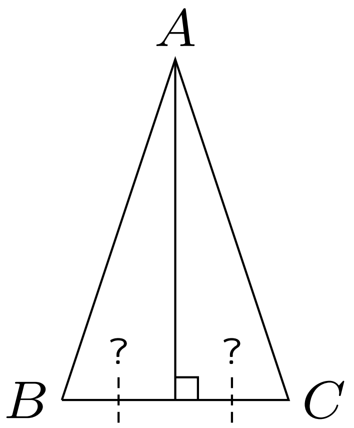


*Prove* it? **It's obvious!**

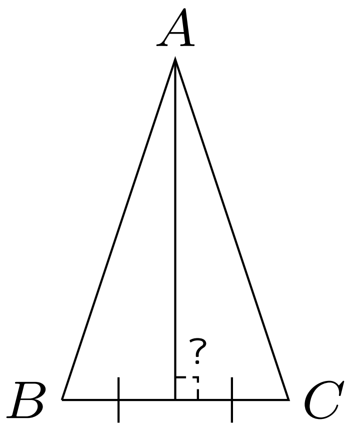
(This is called *progress*.)

COROLLARY: We are all Egyptians, now.

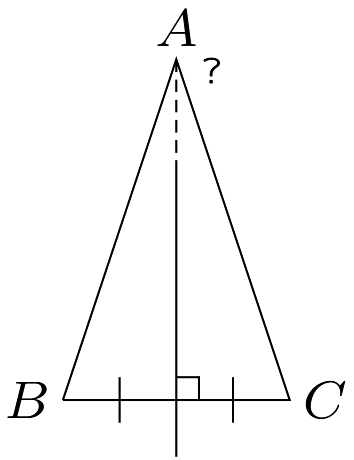
# Where, *precisely*, is the mirror?



If it is the perpendicular from $A$ to $BC$, why does it meet $BC$ at its mid-point?

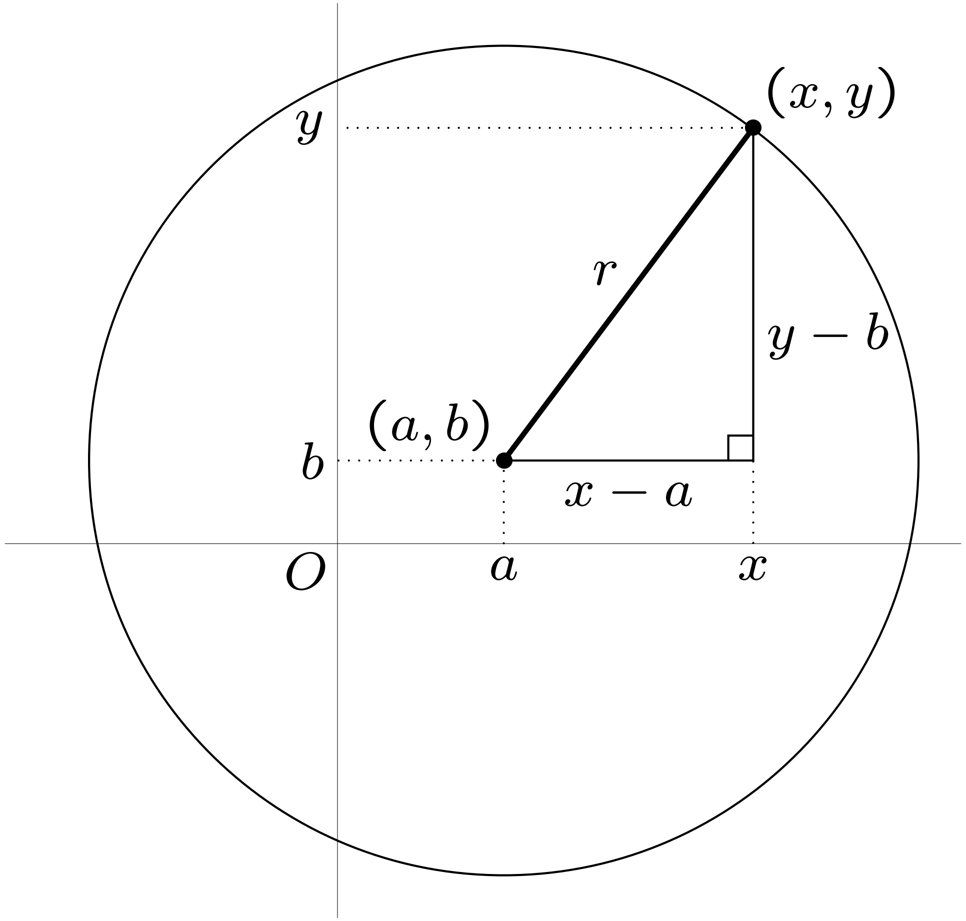


If it is the join of $A$ to the mid-point of $BC$, why is it perpendicular to $BC$?



If it is the perpendicular bisector of $BC$, why does it go through $A$?

→S3



$(x, y)$ moves on a circle, centre $(a, b)$, radius $r$.

By Pythagoras,

$$(x-a)^2 + (y-b)^2 = r^2.$$

That is,

$$x^2 + y^2 + \text{linear stuff} = 0.$$

Suppose the circles $S_1 = 0$ and $S_2 = 0$ meet at $A$ and $B$.

$$\begin{aligned} \text{Here } S_1 &= x^2 + y^2 + \text{some linear stuff,} \\ \text{and } S_2 &= x^2 + y^2 + \text{more linear stuff.} \\ \text{Subtracting, } S_1 - S_2 &= \text{yet more linear stuff,} \end{aligned}$$

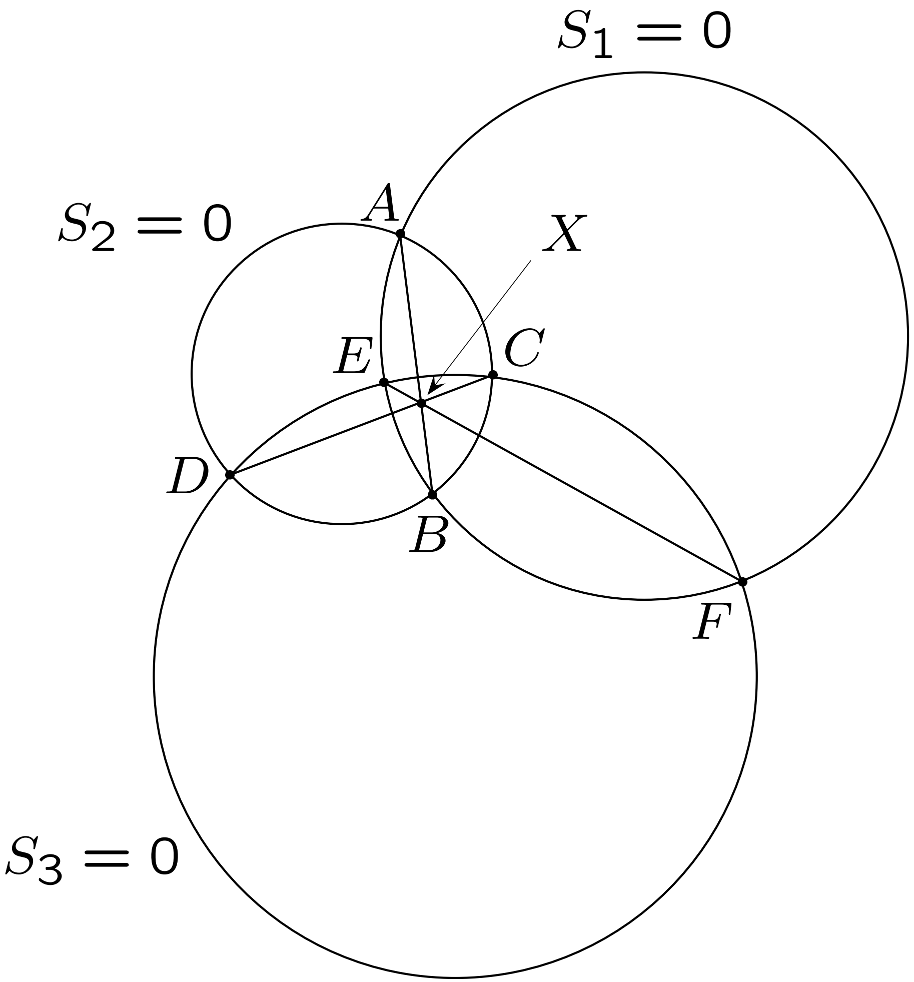
since $x^2$ and $y^2$ cancel. So $S_1 - S_2 = 0$ is the equation of a *line.*

Now $S_1 = 0$ at $A$, since $A$ is on the first circle. (This means that if we substitute the coordinates of $A$ for $x, y$ in $S_1$, we get 0.)

Similarly, at $A$, we have $S_2 = 0$. Therefore, at $A$, we have $S_1 - S_2 = 0 - 0 = 0$, that is, $A$ lies on the line $S_1 - S_2 = 0$.

Likewise, $B$ lies on the line $S_1 - S_2 = 0$. But there is only *one* line through $A$ and $B$, namely the line $AB$ !

**So the equation of the line $AB$ must be $S_1 - S_2 = 0$.**



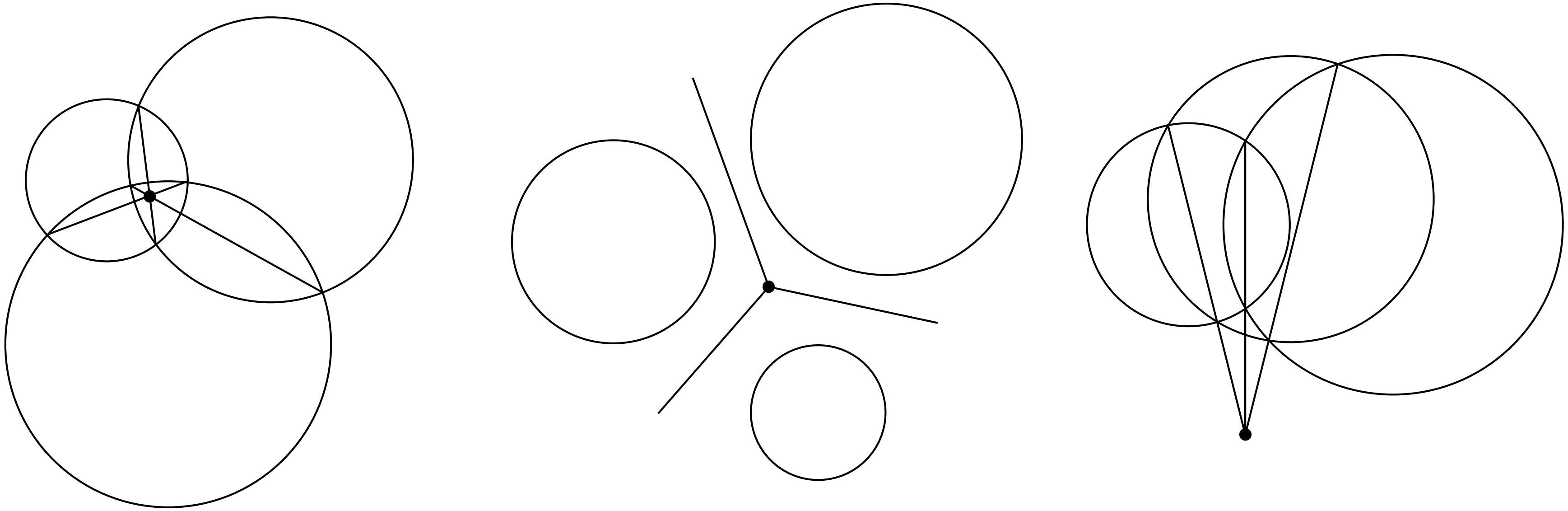
$AB$ has equation $S_1 - S_2 = 0$;
$CD$ has equation $S_2 - S_3 = 0$;
$EF$ has equation $S_1 - S_3 = 0$.

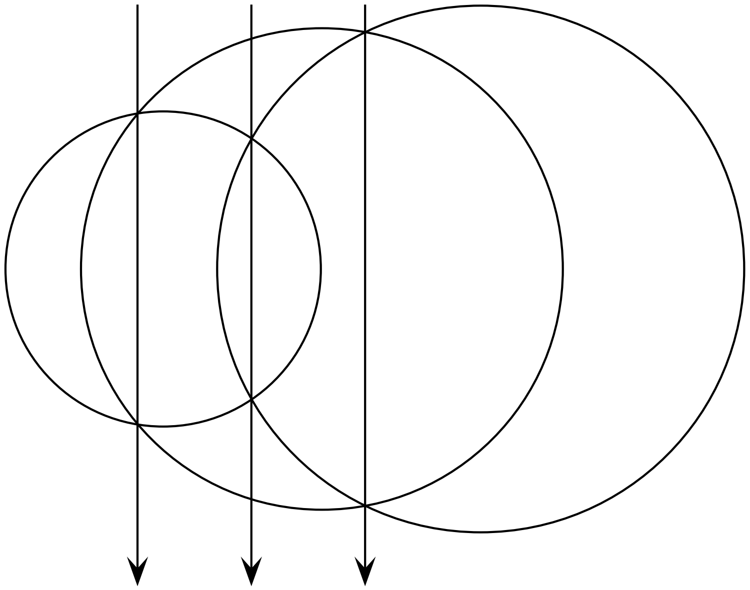
If $AB$, $CD$ meet at $X$, then at this point both $S_1 - S_2 = 0$ and also $S_2 - S_3 = 0$. So, at $X$,

$$\begin{aligned} (S_1 - S_2) \;&+\; (S_2 - S_3) \\ &= \; 0 + 0 \; = \; 0, \\ \text{i.e., } S_1 - S_3 \;&= \; 0. \end{aligned}$$

But this says $X$ lies on $EF$ !

**Theorem.** The radical axes of three circles, taken in pairs, meet in a point. [This point is called the *radical centre* of the circles.]





Actually, I lied: in *this* case the radical axes
don't meet, but are parallel.

(We wriggle out of this by saying they meet in a point at $\infty$ !)

Problem: find where two given lines meet.

Example 1: $x - 2y = 0$ and $x - y = 1$. Method: solve the simultaneous equations, giving $(2, 1)$, that is, $x = 2$ and $y = 1$.

Example 2: $x - 2y = 0$ and $x - 2y = 1$. We can't solve, because $0 \neq 1$. The lines are *parallel*.

Let's make a substitution: replace $x$ by $x/z$ and $y$ by $y/z$. The equations become

$$\frac{x}{z} - \frac{2y}{z} = 0 \quad \text{and} \quad \frac{x}{z} - \frac{2y}{z} = 1,$$

that is (multiplying through by $z$),

$$x - 2y = 0 \quad \text{and} \quad x - 2y = z.$$

These have the solution $(2, 1, 0)$, that is, $x = 2$, $y = 1$, $z = 0$.

We can represent any point in the plane by *three* coordinates $(x, y, z)$. The idea is that $(x, y, z)$ stands for the point with cartesian coordinates $(x/z, y/z)$, provided $z \neq 0$. Note:

1. For any $k \neq 0$, the triple $(kx, ky, kz)$ represents the same point as $(x, y, z)$, because $kx/kz = x/z$ and $ky/kz = y/z$.
2. The point with cartesian coordinates $(x, y)$ corresponds to the triple $(x, y, 1)$, or indeed $(kx, ky, k)$, where $k \neq 0$. So, for example, the origin $(0, 0)$ becomes $(0, 0, 1)$.
3. The line $ax + by + c = 0$ becomes $ax + by + cz = 0$, which is *homogeneous*. The triples $(x, y, z)$ are called *homogeneous coordinates*.
4. Triples $(x, y, 0)$ represent *points at infinity*, and the equation $z = 0$ represents the *line at infinity*.
5. The triple $(0, 0, 0)$ is *banned*.

Example: The circle $x^2 + y^2 + ax + by + c = 0$ becomes

$$(x/z)^2 + (y/z)^2 + ax/z + by/z + c = 0,$$

$$\text{or } x^2 + y^2 + axz + byz + cz^2 = 0.$$

This meets the line at infinity, $z = 0$, where $x^2 + y^2 = 0$, or $y^2 = -x^2$, or $(y/x)^2 = -1$, so that $y/x = \pm i$.

We get two points, $I = (1, i, 0)$ and $J = (1, -i, 0)$.

These (imaginary) points $I, J$ are the **same for every circle!**

They are called the *circular points at infinity*.

**A line can be found through 2 given points.** This is because its equation, $ax + by + c = 0$, has 3 coefficients, so we have to determine $a : b : c$, that is, 2 ratios.
(Note $kax + kby + kc = 0$ is the same line.)

A conic has a quadratic equation, so can involve terms in $x^2$, $xy$ and $y^2$ as well as $x$, $y$ and a constant: 6 coefficients, so 5 ratios. Thus **a conic can be found through 5 given points.**

(A circle is a conic through $I$ and $J$, so a circle can be found to go through 3 (other) points—as long as they are not collinear!)

A cubic can involve $x^3$, $x^2y$, $xy^2$ and $y^3$, as well as the 6 terms of lower degree: 10 coefficients, so 9 ratios. Thus **a cubic can be found through 9 given points.**

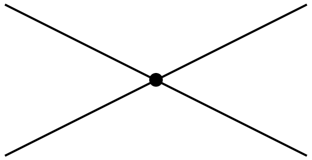
There is in general

- a unique line through 2 points,
- a unique conic through 5 points,
- a unique cubic through 9 points.

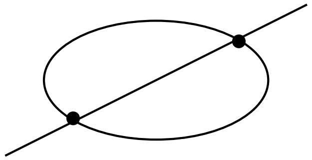
On the other hand, there are

- infinitely many lines through a single point,
- infinitely many conics through 4 points,
- infinitely many cubics through 8 points.

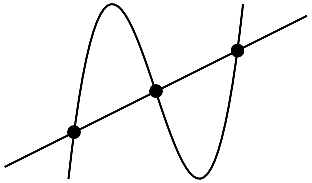
# Easy cases of Bezout's theorem



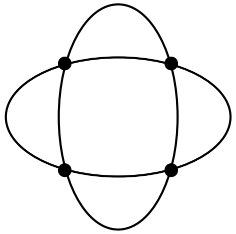
$1 \times 1 = 1$



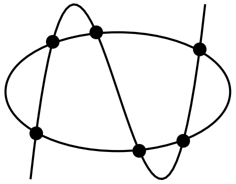
$1 \times 2 = 2$



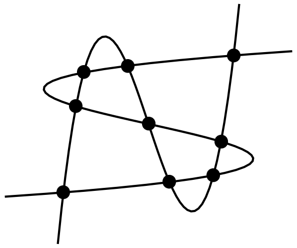
$1 \times 3 = 3$



$2 \times 2 = 4$



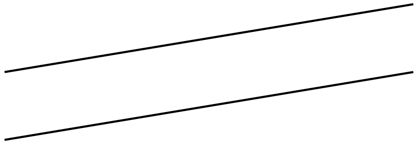
$2 \times 3 = 6$



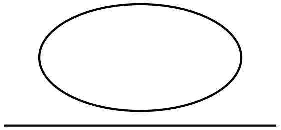
$3 \times 3 = 9$

**Bezout's theorem:** A curve of order $m$ and a curve of order $n$ meet in $m \times n$ points, properly counted.

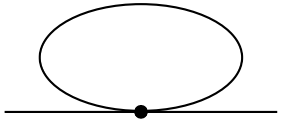
# Trickier cases of Bezout's theorem



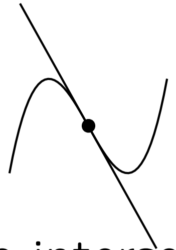
Intersection at $\infty$



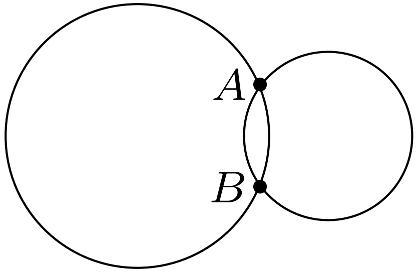
Imaginary intersections



Double intersection



Triple intersection



Two real intersections, $A$ and $B$, and two more ($I$ and $J$), both of which are imaginary *and* at $\infty$.

In general, then, two cubics meet in 9 points.

But in general, there is a *unique* cubic through 9 points!

"*In general*" is a let-out clause, meaning (roughly) *provided nothing peculiar happens*.

Here something peculiar *has* happened, and if two cubics meet in 9 points, then these points are rather special. They are called *nine associated points*.

**Theorem of nine associated points.** If two cubics meet in precisely nine points, then every cubic through eight of these points also contains the ninth point.

(For a proof, see JRS: Geometry A&M.)

A **conic** is any curve given by a 2nd degree equation. If the equation factorises, like (for example)

$$(2x + y - 3)(x - 3y + 7) = 0,$$

then the conic consists of two lines; in this example they are

$$2x + y - 3 = 0 \quad \text{and} \quad x - 3y + 7 = 0.$$

This type of conic is **reducible**.

If the equation does *not* factorise, the conic is **irreducible**, and then it will be an ellipse (possibly a circle), a parabola, or a hyperbola.

Likewise a **cubic** is any curve given by a 3rd degree equation, and if its equation factorises the cubic is **reducible**. It will then consist of a line and a conic, such as

$$(2x + y - 3)(x^2 + y^2 - 1) = 0,$$

or of three lines, such as

$$(2x + y - 3)(x - 3y + 7)(x + y + 1) = 0.$$

If the cubic does *not* factorise, it is **irreducible**.

→S11