8 The Spectral analysis of selfadjoint operators.

In this section A will always denote a selfadjoint operator with spectrum $\sigma = \sigma(A)$ where $\sigma(A) \subseteq [m, M]$ as defined in Section 6.

Theorem 8.1 (Spectral Mapping Theorem) *Let $T \in \mathcal{B}(\mathcal{H})$. For any polynomial p,*

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\} .$$

Proof. Let p be any polynomial. By the elementary scalar remainder theorem, $x - \lambda$ is a factor of $p(x) - p(\lambda)$, that is, $p(x) - p(\lambda) = (x - \lambda)q(x)$ for some polynomial q. Therefore,

$$p(T) - p(\lambda)I = (T - \lambda I)q(T) .$$

Now if $p(\lambda) \notin \sigma(p(T))$ then $p(T) - p(\lambda)I$ has an inverse X and so

$$I = X.[p(T) - p(\lambda)I] = X.q(T).(T - \lambda I) = [p(T) - p(\lambda)I].X = (T - \lambda I).q(T).X .$$

Therefore $(T - \lambda I)$ has both a left inverse (namely $X.q(T)$) and a right inverse $(q(T).X)$. An easy algebraic argument shows that these are equal and $(T - \lambda I)$ has an inverse, that is, $\lambda \notin \sigma(T)$. That is, if $\lambda \in \sigma(T)$ then $p(\lambda) \in \sigma(p(T))$.

Conversely, if $k \in \sigma(p(T))$ then the polynomial $p(x) - k$ factors over the complex field into linear factors: $p(x) - k = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_n)$. Where $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are the roots of $p(x) = k$. Then

$$p(T) - kI = (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_n I) ,$$

and these factors clearly commute. If each $T - \lambda_i I$ has an inverse then the product of all these would be an inverse of $p(T) - kI$. Therefore $T - \lambda_i I$ fails to have an inverse for at least one root λ_i of $p(x) = k$. That is $k = p(\lambda_i)$ for some $\lambda_i \in \sigma(T)$. ■

Corollary 8.2 *If A is a selfadjoint operator then, for any polynomial p,*

$$\|p(A)\| = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\}$$

Proof. If p is real, $p(A)$ is selfadjoint. We know from Theorem 5.5 that for selfadjoint operators, the norm equals the spectral radius. Therefore, using the result of the theorem,

$$\|p(A)\| = \sup\{|k| : k \in \sigma(p(A))\} = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} .$$

For the general case, since $(\bar{p}.p)(A) = \bar{p}(A).p(A) = p(A)^*.p(A)$ we have, using Corollary 5.4

$$\begin{aligned} \|p(A)\|^2 = \|p(A)^*.p(A)\| &= \sup\{|\bar{p}p(\lambda)| : \lambda \in \sigma(A)\} \\ &= \sup\{|p(\lambda)|^2 : \lambda \in \sigma(A)\} \\ &= (\sup\{|p(\lambda)| : \lambda \in \sigma(A)\})^2 . \ ■ \end{aligned}$$

Definition. The functional calculus. Let $f \in C[m, M]$. From Weierstrass' approximation theorem there is a sequence (p_n) of polynomials converging to f uniformly on $[m, M]$. [If $f = g + ih$ is complex, (p_n) is found by combining sequences approximating g and h.] The operator $f(A)$ is defined by

$$f(A) = \lim_{n\to\infty} p_n(A)\,.$$

Theorem 8.3 *The operator $f(A) \in \mathcal{B}(\mathcal{H})$ and is well defined. The map $f \mapsto f(A)$ is a *-algebra homomorphism of $C[m, M]$ into $\mathcal{B}(\mathcal{H})$ and*

$$\|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\}\,.$$

Proof. We first show that the above definition determines an operator $f(A)$. Let p_n be a sequence of polynomials converging to f uniformly on $[m, M]$; that is, in the norm of $C[m, M]$. Then Corollary 1.2 shows that

$$\begin{aligned}\|p_n(A) - p_m(A)\| &= \sup\{|p_n(\lambda) - p_m(\lambda)| : \lambda \in \sigma(A)\}\\ &\le \sup\{|p_n(\lambda) - p_m(\lambda)| : \lambda \in [m, M]\} = \|p_n - p_m\|\,.\end{aligned}$$

Since (p_n) is a Cauchy sequence in $C[m, M]$, it is clear that $(p_n(A))$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ and so is convergent. To show that this defines a unique operator we must show that it is independent of the choice of the sequence. Suppose that (p_n) and (q_n) are two sequence of polynomials converging to f uniformly on $[m, M]$. Write $\lim p_n(A) = X$ and $\lim q_n(A) = Y$. Then

$$\begin{aligned}\|X - Y\| &= \|X - p_n(A) + p_n(A) - q_n(A) + q_n(A) - Y\|\\ &\le \|X - p_n(A)\| + \|p_n(A) - q_n(A)\| + \|q_n(A) - Y\|\\ &\le \|X - p_n(A)\| + \|q_n(A) - Y\| + \sup_{m\le t\le M} |p_n(t) - q_n(t)|\end{aligned}$$

and, since $\|X - Y\|$ is independent of n and the right hand side $\to 0$ as $n \to \infty$, this implies that $X = Y = f(A)$ and that the limit depends only on the function f.

To demonstrate the *-algebra homomorphism statement, we need to show that,

$$\begin{aligned}(\alpha f + \beta g)(A) &= \alpha f(A) + \beta g(A)\\ (f.g)(A) &= f(A).g(A)\\ \bar{f}(A) &= f(A)^*\,.\end{aligned}$$

These follow easily from the definitions since, if (p_n) and (q_n) are sequences of polynomials converging uniformly on $[m, M]$ to f and g respectively, then

$$(\alpha f + \beta g)(A) = \lim_{n\to\infty}(\alpha p_n + \beta q_n)(A) = \alpha \lim_{n\to\infty} p_n(A) + \beta \lim_{n\to\infty} q_n(A) = \alpha f(A) + \beta g(A)$$

and the other statements are easily proved in the same way.

Finally,

$$\|f(A)\| = \|\lim_{n\to\infty} p_n(A)\| = \lim_{n\to\infty} \|p_n(A)\| = \lim_{n\to\infty} \sup_{\lambda\in\sigma} |p_n(\lambda)| = \sup_{\lambda\in\sigma} |f(\lambda)|\,.$$

■

The extension of the functional calculus. We wish to extend the functional calculus to limits of pointwise monotonically convergent sequences of continuous functions. (The immediate goal is to attach a meaning to $\chi_{[-\infty,t]}(A)$. Note that $\chi_{[-\infty,t]}$ is a real function which is equal to its square. Our hope is that $E_t = \chi_{[-\infty,t]}(A)$ will satisfy $E_t = E_t^2 = E_t^*$; that is, that E_t is a projection.)

We first need a definition some technical results.

Definition. An operator A is said to be positive (in symbols $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A \geq B$ to mean $A - B \geq 0$.

Note that, from the polarization identity, Theorem 2.3 and one of the exercises, any positive operator is selfadjoint. Also, it follows from Theorem 5.5 that $A \geq 0$ if and only if $\sigma(A) \subseteq \mathbb{R}^+$.

Lemma 8.4 *Let A be a positive operator. Then*
(i) $|\langle Ax, y\rangle| \leq \langle Ax, x\rangle.\langle Ay, y\rangle$
(ii) $\|Ax\| \leq \|A\|\langle Ax, x\rangle$.

Proof. (i) This is proved in exactly the same way as Theorem 1.1 from the fact that $\langle A(\lambda x + y), (\lambda x + y)\rangle \geq 0$ for all λ.

(ii) The result is obvious if $Ax = 0$. If $Ax \neq 0$, using (i) with $y = Ax$ we have

$$\begin{aligned}\|Ax\|^4 = |\langle Ax, Ax\rangle|^2 &\leq \langle Ax, x\rangle.\langle A^2x, Ax\rangle \\ &\leq \langle Ax, x\rangle.\|A^2x\|.\|Ax\| \\ &\leq \langle Ax, x\rangle.\|A\|.\|Ax\|^2\end{aligned}$$

and the result follows on dividing by $\|Ax\|^2$. ■

Theorem 8.5 *Let (A_n) be a decreasing sequence of positive operators. Then (A_n) converges pointwise (strongly) to an operator A such that $0 \leq A \leq A_n$ for all n.*

Proof. Note that for each n, using Lemma normsa, $\|A_n\| = sup_{\|x\|=1}\langle A_nx, x\rangle \leq sup_{\|x\|=1}\langle A_1x, x\rangle = \|A_1\|$. The hypothesis shows that, for each $x \in \mathcal{H}$, the sequence $(\langle A_nx, x\rangle)$ is a decreasing sequence of positive real numbers and hence is convergent. If $m > n$ then $A_m - A_n \geq 0$ and from Lemma 1.4 (ii),

$$\begin{aligned}\|(A_m - A_n)x\|^2 &\leq \|A_m - A_n\|.\langle (A_m - A_n)x, x\rangle \\ &\leq 2.\|A_1\|.[\langle A_mx, x\rangle - \langle A_nx, x\rangle]\end{aligned}$$

and this shows that (A_n) is a Cauchy sequence, and so convergent. Call the limit Ax. It is routine to show that A is linear, bounded, selfadjoint and $0 \leq A \leq A_n$ for all n. [E.g. to show that A is selfadjoint, we write

$$\langle Ax, y\rangle = \lim_{n\to\infty}\langle A_nx, y\rangle = \lim_{n\to\infty}\langle x, A_ny\rangle = \langle x, Ay\rangle .$$

■

Lemma 8.6 *If the sequences $(A_n), (B_n)$ converge pointwise (strongly) to an operators A, B respectively (and if $\|A_n\|$ is bounded) then $(A_n.B_n) \to AB$ pointwise.*

[Note that pointwise convergence is convergence in a topology on $\mathcal{B}(\mathcal{H})$ called the "strong operator topology"; hence the alternative terminology. Note also that by a theorem of Banach space theory (the Uniform Boundedness Theorem) the condition in brackets is implied by the convergence of (A_n).]

Proof. For any $x \in \mathcal{H}$ we have

$$\|(A_nB_n - AB)x\| = \|(A_nB_n - A_nB + A_nB - AB)x\| \leq \|A_n\|.\|(B_n - B)x\| + \|(A_n - A)Bx\|$$

and the right hand side $\to 0$ from the pointwise convergence of (A_n) and (B_n) [at the points Bx and x respectively]. ■

Definition. The extended functional calculus. Let ϕ be a positive function on $[m, M]$ that is the pointwise limit of a decreasing sequence (f_n) of positive functions $\in C[m, M]$. The operator $\phi(A)$ is defined as the pointwise limit of the sequence $(f_n(A))$.

It is a fact that $\phi(A)$ is well defined (that is, it depends only on ϕ and not on the approximating sequence).

Lemma 8.7 *Let ϕ, ψ be a positive functions on $[m, M]$ that are the pointwise limits of a decreasing sequences of positive functions $\in C[m, M]$. Then*
(i) $\phi(A) + \psi(A) = (\phi + \psi)(A)$ *.*
(ii) $\phi(A).\psi(A) = (\phi.\psi)(A)$ *.*
(iii) *If X commutes with A then X commutes with $\phi(A)$.*

Proof. (i) Let (f_n) and (g_n) be decreasing sequences of functions that are continuous on $[m, M]$ and converge pointwise $[m, M]$ to ϕ and ψ respectively. Then $(f_n + g_n)$ is a decreasing sequence of continuous functions converging pointwise to $\phi + \psi$ and

$$(\phi + \psi)(A) = \lim_{n\to\infty} (f_n + g_n)(A) = \lim_{n\to\infty} f_n(A) + \lim_{n\to\infty} g_n(A) = \phi(A) + \psi(A)$$

where the limits indicate pointwise convergence in $\mathcal{H}$.

(ii) As in (i) $(f_n.g_n)$ is a decreasing sequence of continuous functions converging pointwise to $\phi.\psi$ and

$$(\phi.\psi)(A) = \lim_{n\to\infty} (f_n.g_n)(A) = \lim_{n\to\infty} f_n(A). \lim_{n\to\infty} g_n(A) = \phi(A).\psi(A),$$

using Lemma 1.6, where the limits indicate pointwise convergence in $\mathcal{H}$.

(iii) If X commutes with A then X commutes with $f(A)$ for every $f \in C[m, M]$ since if (p_n) is a sequence of polynomials converging uniformly to f on $[m, M]$,

$$X.f(A) = \lim_{n\to\infty} X.p_n(A) = \lim_{n\to\infty} p_n(A).X = f(A).X\ .$$

Now if (f_n) is a decreasing sequences of functions in $C[m, M]$ that converge pointwise $[m, M]$ to ϕ then, using Lemma 1.6,

$$X.\phi(A) = \lim_{n\to\infty} X.f_n(A) = \lim_{n\to\infty} f_n(A).X = \phi(A).X\ . \quad ■$$

For every real λ we define the operator E_λ as follows: let

$$f_{n,\lambda} = \begin{cases} 1 & t < \lambda \\ 1 - n(t-\lambda) & \lambda \leq t \leq \lambda + \frac{1}{n} \\ 0 & t > \lambda + \frac{1}{n} \end{cases}$$

Then $(f_{n,\lambda})$ is is a decreasing sequence of continuous functions converging pointwise to $\chi_{[-\infty,\lambda]}$. Now let $E_\lambda = \chi_{[-\infty,\lambda]}(A)$. If $\lambda < m$ then for all $t \in [m, M]$ we have $f_{n,\lambda}(t) = 0$ for all sufficiently large n and so $E_\lambda = 0$. Similarly, $E_\lambda = I$ for $\lambda \geq M$.

Note that

$$E_\lambda . E_\mu = E_\nu \text{ where } \nu = \min[\lambda, \mu] \, .$$

It follows that $E_\lambda = E_\lambda^2$. Also $E_\lambda \geq 0$ and so $E_\lambda = E_\lambda^*$.

A family of projections with these properties is called a **bounded resolution of the identity**.

We call the family $\{E_\lambda : -\infty < \lambda < \infty\}$ as obtained above the bounded resolution of the identity for the operator A. It is fact that it is (essentially) uniquely determined by A.

We say that the integral

$$\int_m^M f(\lambda) \, dE_\lambda$$

of a function with respect to a bounded resolution of the identity exists and is equal to T if, given any $\epsilon > 0$ there exists a partition

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \lambda_{n-1} < \lambda_n$$

with $\lambda_0 < m$ and $\lambda_n > M$ such that, for any $\xi \in (\lambda_{i-1}, \lambda_i]$,

$$\left\| T - \sum_{i=1}^n f(\xi_i)(E_{\lambda_i} - E_{\lambda_{i-1}}) \right\| < \epsilon \, .$$

For the proof of the Spectral Theorem, we need the following lemmas.

Lemma 8.8 *Let $\{\Delta_i : 1 \leq i \leq n\}$ be orthogonal projections such that $I = \sum_{i=0}^n \Delta_i$ and $\Delta_i . \Delta_j = 0$ when $i \neq j$. If $X \in \mathcal{B}(\mathcal{H})$ commutes with each Δ_i then*

$$\|X\| = \max_{1 \leq i \leq n} \|\Delta_i X \Delta_i\| \, .$$

Proof. Let $h \in \mathcal{H}$. Then

$$\|h\|^2 = \| \sum_{i=0}^n \Delta_i h \|^2 = \left\langle \sum_{i=0}^n \Delta_i h, \sum_{j=0}^n \Delta_j h \right\rangle = \sum_{i=0}^n \|\Delta_i h\|^2$$

since the cross terms are zero. Therefore, since $X\Delta_i = X\Delta_i^2 = \Delta_i X \Delta_i = \Delta_i X$

$$\|Xh\|^2 \;=\; \sum_{i=0}^n \|\Delta_i X h\|^2 = \sum_{i=0}^n \|\Delta_i X \Delta_i . \Delta_i h\|^2$$

$$
\begin{aligned}
&\le \sum_{i=0}^{n} \|\Delta_i X \Delta_i\|^2 . \|\Delta_i h\|^2 \\
&\le \max_{1\le i\le n} \|\Delta_i X \Delta_i\|^2 . \sum_{i=0}^{n} \|\Delta_i h\|^2 \\
&= \max_{1\le i\le n} \|\Delta_i X \Delta_i\|^2 . \|h\|^2 .
\end{aligned}
$$

Thus $\|X\| \le \max_{1\le i\le n} \|\Delta_i X \Delta_i\|$. But the opposite inequality is clear, since for each i we have $\|\Delta_i X \Delta_i\| \le \|\Delta_i\|.\|X\|.\|\Delta_i\| = \|X\|$. ■

Corollary 8.9

$$
\left\| \int_m^M f(\lambda)\, dE_\lambda \right\| \le \sup_{m\le t\le M} |f(t)| .
$$

Proof. From the lemma,

$$
\left\| \sum_{i=1}^{n} f(\xi_i)(E_{\lambda_i} - E_{\lambda_{i-1}}) \right\| = \max_{1\le i\le n} \|f(\xi_i)(E_{\lambda_i} - E_{\lambda_{i-1}})\| = \max_{1\le i\le n} |f(\xi_i)| \le \sup_{m\le t\le M} |f(t)| .
$$

Since the integral is approximated arbitrarily closely in norm by these sums, the result follows. ■

Lemma 8.10 *For $\lambda > \mu$,*

$$
\mu(E_\lambda - E_\mu) \le (E_\lambda - E_\mu)A(E_\lambda - E_\mu) \le \lambda(E_\lambda - E_\mu) .
$$

Proof. First note the general fact that if $S \le T$ then for any operator X we have $X^*SX \le X^*TX$. This is clear since for any $h \in \mathcal{H}$,

$$
\langle X^*SXh, h\rangle = \langle SXh, Xh\rangle \le \langle TXh, Xh\rangle = \langle X^*TXh, h\rangle .
$$

Next we claim that $AE_\lambda \le \lambda E_\lambda$. Let $f_{\lambda,n}$ be as in the definition of E_λ and define $g_n(t) = (\lambda - t) f_{\lambda,n} + \frac{1}{n}$. It is easy to see that $g_n \ge 0$. [This is obvious when $t \le \lambda$, and also when $t \ge \lambda + \frac{1}{n}$, for then $f_{\lambda,n}(t) = 0$; for $t \in (\lambda, \lambda + \frac{1}{n})$ we have $\lambda - t > -\frac{1}{n}$ so, since $0 \le f_{\lambda,n}(t) \le 1$, it follows that $g_n(t) \ge 0$.] Also (g_n) is decreasing (this is an elementary but tedious exercise) and so the pointwise (strong) limit of $(g_n(A))$ exists and is positive. But $(g_n(t))$ is pointwise convergent to $\lambda\chi(-\infty, \lambda](t) - t\chi(-\infty, \lambda](t)$. Therefore

$$
\lambda E_\lambda - AE_\lambda \ge 0
$$

proving the claim.

Note that E_λ commutes with A and $AE_\lambda^2 = AE_\lambda$. Therefore we have that

$$
E_\lambda A E_\lambda = AE_\lambda^2 \le \lambda E_\lambda .
$$

We now use that $(I - E_\mu) = (I - E_\mu)^*$ and the general fact from the start of the proof to conclude that

$$
(E_\lambda - E_\mu)A(E_\lambda - E_\mu) = (I - E_\mu)E_\lambda A E_\lambda (I - E_\mu) \le (I - E_\mu)\lambda E_\lambda (I - E_\mu) = \lambda(E_\lambda - E_\mu) .
$$

The fact that $\mu(E_\lambda - E_\mu) \le (E_\lambda - E_\mu)A(E_\lambda - E_\mu)$ is proved in an exactly similar way. ■

Theorem 8.11 (The Spectral Theorem for bounded selfadjoint operators.) *For any bounded selfadjoint operator A there exists a bounded resolution of the identity E_λ such that*

$$A = \int_m^M \lambda \, dE_\lambda \, .$$

Proof. Let E_λ be the resolution of the identity as found above. Let $\epsilon > 0$ be given. Choose $\lambda_0 < \lambda_1 < \lambda_2 < \cdots \lambda_n$ with $\lambda_0 < m, \ \lambda_n > M$ such that $0 \leq \lambda_i - \lambda_{i-1} < \epsilon$. Write $\Delta_i = E_{\lambda_i} - E_{\lambda_{i-1}}$. From Lemma 1.10 we have that

$$\lambda_{i-1}\Delta_i \leq \Delta_i A \Delta_i \leq \lambda_i \Delta_i \, .$$

Hence, for any $\xi_i \in [\lambda_{i-1}, \lambda_i]$,

$$(\lambda_{i-1} - \xi_i)\Delta_i \leq \Delta_i A \Delta_i - \xi_i \Delta_i \leq (\lambda_i - \xi_i)\Delta_i \, .$$

Note that when $\|h\| = 1$, since Δ_i is an orthogonal projection, $0 \leq \langle \Delta_i h, h \rangle = \|\Delta_i h\|^2 \leq 1$. Therefore, using Theorem 5.3

$$\|\Delta_i A \Delta_i - \xi_i \Delta_i\| \leq \max[|\lambda_i - \xi_i|, |\lambda_{i-1} - \xi_i|] < \epsilon \, .$$

Observe that $\{\Delta_i : 1 \leq i \leq n\}$ satisfies the hypotheses of Lemma 1.8 and that $X = (A - \sum_{i=1}^n \xi_i \Delta_i)$ commutes with each Δ_i. Therefore, applying Lemma 1.8 we have that

$$\|(A - \sum_{i=1}^n \xi_i \Delta_i)\| = \max_{1 \leq i \leq n} \|\Delta_i A \Delta_i - \xi_i \Delta_i\| < \epsilon$$

and this is exactly what is required. ■

$\Delta_i . \Delta_j = 0$ when $i \neq j$.

Corollary 8.12 *If f is continuous on $[m, M]$ then*

$$f(A) = \int_m^M f(\lambda) \, dE_\lambda \, .$$

Proof. For any integer k, choose $n, \xi_1, \xi_2 \cdots \xi_n, \lambda_0, \lambda_1, \ldots, \lambda_n$ as in the Theorem so that

$$\left\| (A - \sum_{i=1}^n \xi_i \Delta_i) \right\| < \frac{1}{k}$$

and write $\mathcal{I}_k = \sum_{i=1}^n \xi_i \Delta_i$. Then $\lim_{k \to \infty} \mathcal{I}_k = A$. Therefore, for any integer r,

$$A^r = \lim_{k \to \infty} \mathcal{I}_k^r \, .$$

But

$$\mathcal{I}_k^r = \left(\sum_{i=1}^n \xi_i \Delta_i \right)^r = \sum_{i=1}^n \xi_i^r \Delta_i \, ,$$

and the right hand side is the approximating sum to $\int_m^M \lambda^r \, dE_\lambda$. Therefore

$$A^r = \int_m^M \lambda^r \, dE_\lambda \, .$$

and, by taking linear combinations,

$$p(A) = \int_m^M p(\lambda)\, dE_\lambda$$

for all polynomials p.

For $f \in C[m, M]$, given any $\epsilon > 0$, choose a polynomial p_ϵ such that

$$\sup_{m \le \lambda \le M} |f(\lambda) - p(\lambda)| < \epsilon\,.$$

Then, from Theorem 1.3 $\|f(A) - p(A)\| < \epsilon$ and

$$\begin{aligned}
\left\| f(A) - \int_m^M f(\lambda)\, dE_\lambda \right\| &\le \left\| f(A) - \int_m^M p(\lambda)\, dE_\lambda \right\| + \left\| \int_m^M p(\lambda)\, dE_\lambda - \int_m^M f(\lambda)\, dE_\lambda \right\| \\
&= \|f(A) - p(A)\| + \left\| \int_m^M p(\lambda) - f(\lambda)\, dE_\lambda \right\| \\
&< 2\epsilon\,,
\end{aligned}$$

the last step using Corollary 1.9. Since ϵ is arbitrary, it follows that

$$f(A) = \int_m^M f(\lambda)\, dE_\lambda\,.$$

■

1. (More spectral mapping results.) If $X \in \mathcal{B}(\mathcal{H})$ show that

(i) $\sigma(X^*) = \{\lambda : \bar{\lambda} \in \sigma(X)\}$,

(ii) if X is invertible then $\sigma(X^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(X)\}$.

Deduce that every member of the spectrum of a unitary operator has modulus 1.

2. For any selfadjoint operator A, prove that $\ker A = (\operatorname{ran} A)^{\perp}$.

3. Let A and B be selfadjoint operators. Show that if there exists an invertible operator T such that $T^{-1}AT = B$ then there exists a unitary operator U such that $U^*AU = B$ (that is, if A and B are similar then they are unitarily equivalent).

[Hint: use the polar decomposition of T.]

4. Suppose $X \in \mathcal{B}(\mathcal{H})$ is selfadjoint and $\| X \| \leq 1$. Observe that $X + i\sqrt{I - X^2}$ can be defined and prove that it is unitary. Deduce that any operator T can be written as a linear combination of at most 4 unitary operators. [First write $T = X + iY$ with X, Y selfadjoint.]

5. (i) Use results on the spectrum show that $A \geq kI$ (where $k \in \mathbb{R}$) if and only if for all $\lambda \in \sigma(A)$, $\lambda \geq k$. [Note that A is selfadjoint, since $\langle Ax, x \rangle$ is real – make sure you know how this follows!] Deduce that if $A \geq I$ then $A^n \geq I$ for every positive integer n. [Alternatively factorise $A^n - I$.]

(ii) If B and C commute and $B \geq C \geq 0$ then prove that $B^n \geq C^n$ for every positive integer n. [Factorise $B^n - C^n$.]

6. Let U be both selfadjoint and unitary. Prove that $\sigma(U) = \{-1, 1\}$ (unless $U = \pm I$). [Use Question 1.] Use the spectral theorem to find an orthogonal projection E such that $U = 2E - I$; (alternatively, if you are given the result it is trivial to verify that $E = \frac{1}{2}(U + I)$ is a suitable E). Note that a self adjoint isometry must be unitary [use Question 2]. Deduce that the only positive isometry is I.

[Definition: V is an isometry if $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$.]