6 The Spectral analysis of compact operators.

In this section K will always denote a compact operator.

Theorem 6.1 *If $\lambda \neq 0$ then either λ is an eigenvalue of K or $\lambda \in \rho(K)$.*

Proof. Suppose that $\lambda \neq 0$ is not an eigenvalue of K. We show that $\lambda \in \rho(K)$. The proof of this is in several stages.

(a) *For some $c > 0$, we have that that $\|(\lambda I - K)x\| \geq c\|x\|$ for all $x \in \mathcal{H}$.*
Suppose this is false. Then the inequality fails for $c = \frac{1}{k}$ for $k = 1, 2, \cdots$. Therefore there is a sequence of unit vectors such that

$$\|(\lambda I - K)x_k\| \leq \frac{1}{k},$$

that is, $((\lambda I - K)x_k) \to 0$. Applying the condition that K is compact, there is a subsequence (x_{k_i}) such that (Kx_{k_i}) is convergent. Call its limit y. Then

$$x_{k_i} = \frac{1}{\lambda}\left((\lambda I - K)x_{k_i} + Kx_{k_i}\right)$$

and so $(x_{k_i}) \to \frac{y}{\lambda}$. Since (x_{k_i}) is a sequence of unit vectors, $y \neq 0$. But then,

$$(\lambda I - K)y = \lim_{i\to\infty}(\lambda I - K)x_{k_i} = \lambda\frac{y}{\lambda} - y = 0\,.$$

This contradicts the fact that λ is not an eigenvalue, so (a) is established.

(b) $\operatorname{ran}(\lambda I - K) = \mathcal{H}$.
Let $H_n = \operatorname{ran}(\lambda I - K)^n$ and write $H_0 = \mathcal{H}$. It follows from (a) using Lemma 5.1 that (H_n) is a sequence of closed subspaces. Also

$$(\lambda I - K)H_n = H_{n+1}$$

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots .$$

Note that, if $y \in H_n$ then $Ky = ((K - \lambda I)y + \lambda y) \in H_n$ so that $K(H_n) \subseteq H_n$.

We now use the compactness of K to show that the inclusion $H_n \subseteq H_{n+1}$ is not always proper. Suppose, on the contrary that

$$H_0 \supset H_1 \supset H_2 \supset H_3 \supset \cdots .$$

Using Lemma 1.7, for each n we can find a unit vector x_n such that $x_n \in H_n$ and $x_n \perp H_{n+1}$. We show that (Kx_n) cannot have a Cauchy subsequence. Indeed, if $m > n$

$$\begin{aligned} Kx_n - Kx_m &= (K - \lambda I)x_n + \lambda x_n - Kx_m \\ &= \lambda x_n + [(K - \lambda I)x_n - Kx_m] \\ &= \lambda x_n + z \end{aligned}$$

where $z \in H_{n+1}$ [$Kx_m \in H_m \subseteq H_{n+1}$ follows from $m > n$]. Thus

$$\|Kx_n - Kx_m\|^2 = |\lambda|^2 + \|z\|^2 \geq |\lambda|^2$$

and so (Kx_n) has no convergent Cauchy subsequence. Therefore, the inclusion is not always proper. Let k be the smallest integer such that $H_k = H_{k+1}$. If $k \neq 0$ then choose $x \in H_{k-1} \setminus H_k$. Then $(\lambda I - K)x \in H_k = H_{k+1}$ and so, for some y,

$$(\lambda I - K)x = (\lambda I - K)^{k+1}y = (\lambda I - K)z$$

where $z = (\lambda I - K)^k y \in H_k$. Now $x \notin H_k$ so $x - z \neq 0$ and

$$(\lambda I - K)(x - z) = 0$$

contradicting the fact that λ is not an eigenvalue. Therefore $k = 0$, that is $\operatorname{ran}(\lambda I - K) = H_1 = H_0 = \mathcal{H}$.

(c) *Completing the proof.* This is done exactly as in Theorem 5.5 (i). For any $y \in \mathcal{H}$, there is a unique $x \in \mathcal{H}$ such that $y = (\lambda I - K)x$. Define $(\lambda I - K)^{-1}y = x$. Then $\|y\| \geq c\|x\|$ so

$$\|(\lambda I - K)^{-1}y\| = \|x\| \leq \tfrac{1}{c}\|y\|$$

showing that $(\lambda I - K)^{-1} \in \mathcal{B}(\mathcal{H})$ (i.e. it is continuous). Thus $\lambda \notin \sigma(K)$. ■

Lemma 6.2 *If $\{x_n\}$ are eigenvectors of K corresponding to different eigenvalues $\{x_n\}$, then $\{x_n\}$ is a linearly independent set.*

Proof. This is exactly as in an elementary linear algebra course. Suppose the statement is false and k is the first integer such that $x_1, x_2, \cdots, x_k$ is linearly dependent. Then $\sum_{i=1}^{k} \alpha_i x_i = 0$ and $\alpha_k \neq 0$. Also, by hypothesis $Kx_i = \lambda_i x_i$ with the λ_i's all different. Now $x_k = \sum_{i=1}^{k-1} \beta_i x_i$ (where $\beta_i = -\alpha_i/\alpha_k$) and so

$$0 = (\lambda_k I - K)x_k = \sum_{i=1}^{k-1} (\lambda_k - \lambda_i)\beta_i x_i$$

showing that $x_1, x_2, \cdots, x_{k-1}$ is linearly dependent, contradicting the definition of k. ■

Theorem 6.3 *$\sigma(K)\setminus\{0\}$ consists of eigenvalues with finite-dimensional eigenspaces. The only possible point of accumulation of $\sigma(K)$ is 0.*

Proof. Let λ be any non-zero eigenvalue and let $N = \{x : Kx = \lambda x\}$ be the eigenspace of λ. If N is not finite-dimensional, we can find an orthonormal sequence (x_n) of elements of N [apply the Gram-Schmidt process (Theorem 3.4) to any linearly independent sequence]. Then

$$\|Kx_n - Kx_m\|^2 = \|\lambda x_n - \lambda x_m\|^2 = 2|\lambda|^2$$

which is impossible, since K is compact.

To show that $\sigma(K)$ has no points of accumulation other than (possibly) 0, we show that $\{\lambda \in \mathbb{C} : |\lambda| > \delta\} \cap \sigma(K)$ is finite for any $\delta > 0$. Suppose this is false and there is a sequence of distinct eigenvalues (λ_i) with $|\lambda_i| > \delta$ for all i. Then we have vectors x_i with $Kx_i = \lambda_i x_i$.

Let $H_n = \text{span}\{x_1, x_2, \cdots, x_n\}$. Then, since $\{x_n\}$ is a linearly independent set, we have the proper inclusions

$$H_1 \subset H_2 \subset H_3 \subset H_4 \subset \cdots .$$

It is easy to see that $K(H_n) \subseteq H_n$ and $(\lambda_n I - K)H_n \subseteq H_{n-1}$. Choose, as in Theorem 1.1 a sequence of unit vectors (y_n) with $y_n \in H_n$ and $y_n \perp H_{n-1}$. Then, for $n > m$,

$$Ky_n - Ky_m = \lambda_n y_n - [(\lambda_n I - K)y_n - Ky_m] .$$

Since $(\lambda_n I - K)y_n \in H_{n-1}$ and $Ky_m \in H_m \subseteq H_{n-1}$, the vector in square brackets is in H_{n-1}. Therefore, since $y_n \perp H_{n-1}$,

$$\|Ky_n - Ky_m\| > |\lambda_n| > \delta$$

showing that (Ky_n) has no convergent subsequence. ■

Corollary 6.4 *The eigenvalues of K are countable and whenever they are put into a sequence (λ_i) we have that $\lim_{\to\infty} \lambda_i = 0$.*

Proof. [The set of all eigenvalues is (possibly) 0 together with the countable union of the finite sets of eigenvalues $> \frac{1}{n}, (n = 1, 2, \cdots)$.

If $\epsilon > 0$ is given then, since $\lambda : \lambda$ an eigenvalue of $K, |\lambda| \geq \epsilon$ is finite, we have that $|\lambda_i| < \epsilon$ for all but a finite number of values of i. Hence $(\lambda_i) \to 0$.] ■

Corollary 6.5 *If A is a compact selfadjoint operator then $\|A\|$ equals its eigenvalue of largest modulus.*

Proof. This is immediate from Theorem 5.5 (ii). ■

The Fredholm alternative. For any scalar μ, either

$$(I - \mu K)^{-1} \text{ exists}$$

or the equation

$$(I - \mu K)x = 0$$

has a finite number of linearly independent solutions.

(Fredholm formulated this result for the specific operator $(Kf)(x) = \int_a^b k(x,t)f(t)\,dt$. In fact, he said : EITHER the integral equation

$$f(x) - \mu \int_a^b k(x,t)f(t)\,dt \; = g(x)$$

has a unique solution, OR the associated homogeneous equation

$$f(x) - \mu \int_a^b k(x,t) f(t)\, dt \ = 0$$

has a finite number of linearly independent solutions.)

We now turn to compact selfadjoint operators. For the rest of this section A will denote a compact selfadjoint operator.

Note that every eigenvalue of of A is real. This is immediate from Theorem 5.5, but can be proved much more simply since if $Ax = \lambda x$, where x is a unit eigenvector,

$$\bar{\lambda} = \overline{\langle Ax, x\rangle} = \langle x, Ax\rangle = \langle A^* x, x\rangle = \langle Ax, x\rangle = \lambda\,.$$

Lemma 6.6 *Distinct eigenspaces of A are mutually orthogonal.*

Proof. Let x and y be eigenvectors corresponding to distinct eigenvalues λ and μ. Then,

$$\lambda\langle x, y\rangle = \langle Ax, y\rangle = \langle x, A^* y\rangle = \langle x, Ay\rangle = \bar{\mu}\langle x, y\rangle = \mu\langle x, y\rangle$$

(since μ is real) and so $\langle x, y\rangle = 0$. ■

Theorem 6.7 *If A is a compact selfadjoint operator on a Hilbert space $\mathcal{H}$ then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of A.*

Proof. Let $(\lambda_i)_{i=1,2,\cdots}$ be the sequence of all the non-zero eigenvalues of A and let N_i be the eigenspace of λ_i. Take an orthonormal basis of each N_i and an orthonormal basis of $N_0 = \ker A$. Let (x_n) be the union of all these, put into a sequence. It follows from Lemma 1.6 that this sequence is orthonormal.

Let $M = \{z : z \perp x_n \text{ for all } n\}$. Then, if $y \in M$ we have that $\langle x_n, Ay\rangle = \langle Ax_n, y\rangle = \lambda_n\langle x_n, y\rangle = 0$ and so $A(M) \subseteq M$. Therefore A with its domain restricted to M is a compact selfadjoint operator on the Hilbert space M. Clearly this operator is selfadjoint $[\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$ so certainly for all $x, y \in M]$. Also it cannot be have a no-zero eigenvector [for then $M \cap N_k \neq (0)$ for some $k > 0$]. Therefore, by Corollary 1.5, it is zero. But then $M \subseteq N_0$. But also $M \perp N_0$ and so $M = (0)$. Therefore (x_n) is a basis. ■

Corollary 6.8 *Then there is an orthonormal basis $\{x_n\}$ of $\mathcal{H}$ such that, for all h,*

$$Ah = \sum_{n=1}^{\infty} \lambda_n \langle h, x_n\rangle x_n\,.$$

Proof. Let (x_n) be the basis found in the Theorem and let $\lambda_n = \langle Ax_n, x_n\rangle$ (this is merely re-labeling the eigenvalues. The from Theorem 3.3 (iii), for any $h \in \mathcal{H}$,

$$h = \sum_{n=1}^{\infty} \langle h, x_n\rangle x_n\,.$$

Acting on this by A, since A is continuous and $Ax_n = \lambda_n x_n$ we have that

$$Ah = \sum_{n=1}^{\infty} \lambda_n \langle h, x_n \rangle x_n \,.$$

■

Theorem 6.9 *If A is a compact selfadjoint operator on a Hilbert space $\mathcal{H}$ then there is an orthonormal basis $\{x_n\}$ of $\mathcal{H}$ such that*

$$A = \sum_{n=1}^{\infty} \lambda_n (x_n \otimes x_n)$$

where the series is convergent in norm.

Proof. Let $\{x_n\}$ be the basis found as above so that $Ax_n = \lambda_n x_n$ and

$$Ah = \sum_{n=1}^{\infty} \lambda_n \langle h, x_n \rangle x_n \,.$$

Note that $(\lambda_n) \to 0$. Let

$$A_k = \sum_{n=1}^{k} \lambda_n (x_n \otimes x_n) \,.$$

We need to show that $\|A - A_k\| \to 0$ as $k \to \infty$.

Now

$$(A - A_k)h = \sum_{n=k+1}^{\infty} \lambda_n \langle h, x_n \rangle x_n \,.$$

and, using Theorem 3.3 (v)

$$\begin{aligned}
\|(A - A_k)h\|^2 &= \sum_{n=k+1}^{\infty} |\lambda_n \langle h, x_n \rangle|^2 \\
&\leq \sup_{n \geq k+1} |\lambda_n|^2 \sum_{n=k+1}^{\infty} |\langle h, x_n \rangle|^2 \\
&\leq \sup_{n \geq k+1} |\lambda_n|^2 \sum_{n=1}^{\infty} |\langle h, x_n \rangle|^2 \\
&= \sup_{n \geq k+1} |\lambda_n|^2 \, \|h\|^2 \,.
\end{aligned}$$

Thus $\|(A - A_k)\| \leq \sup_{n \geq k+1} |\lambda_n|$, and so since $(\lambda_n) \to 0$, we have that $\|A - A_k\| \to 0$ as $k \to \infty$. ■

Alternatively, Theorem 4.4 may be used to prove the above result. Let $\{x_n\}$ and A_k and A be as above and let

$$P_k = \sum_{n=1}^{k} (x_n \otimes x_n) \,.$$

Then, since $\{x_n\}$ is a basis, Theorem 3.3 (iii) shows that (P_k) converges pointwise to the identity operator I. Since $A_k = AP_k$, Theorem 4.4 shows that (A_k) converges to A in norm.

Exercises 6

1. Let K be a compact operator on a Hilbert space $\mathcal{H}$ and let $\lambda \neq 0$ be an eigenvalue of K. Show that $\lambda I - K$ has closed range. [Hint : let $N = \ker(\lambda I - K)$ and let $M = N^{\perp}$. If $y \in \operatorname{ran}(\lambda I - K)$, show that $y = \lim_{n\to\infty}(\lambda I - K)z_n$ with $z_n \in M$. Now imitate the proof for the case when λ is not an eigenvalue.]

2. Find the norm of the compact operator V defined on $L^2[0,1]$ by

$$(Vf)(x) = \int_0^x f(t)\,dt$$

.

Hints: Use Corollary 5.4 and the fact that the norm of the compact selfadjoint operator V^*V is given by its largest eigenvalue. Now use the result of Exercises 2 Question 6 to show that if f satisfies $V^*Vf = \lambda f$ then it satisfies

$$\begin{cases} \lambda f'' + f = 0 \\ f(1) = 0, \qquad f'(0) = 0. \end{cases}$$

[You may assume that any vector in the range of V^*V (being in the range of two integrations) is twice differentiable (almost everywhere).]

Note that a direct approach to evaluating $\| V \|$ seems to be very difficult (try it !).

3. Let $\{x_n\}$ be an orthonormal basis of $\mathcal{H}$ and suppose that $T \in \mathcal{B}(\mathcal{H})$ is such that the series $\sum_{n=1}^{\infty} \|Tx_n\|^2$ converges. Prove that

(i) T is compact,

(ii) $\sum_{n=1}^{\infty} \|Ty_n\|^2$ converges for every orthonormal basis $\{y_n\}$ of $\mathcal{H}$ and for the sum is the same for every orthonormal basis.

Note : an operator satisfying the above is called a *Hilbert-Schmidt* operator.

Hints: (i) write $h \in \mathcal{H}$ as a Fourier series, $h = \sum_{i=1}^{\infty} \alpha_i x_i$ where $\alpha_i = \langle h, x_i\rangle$. Define $T_n h = \sum_{i=1}^{n} \alpha_i T x_i$ and show that

$$\|(T - T_n)h\|^2 \leq \left(\sum_{n+1}^{\infty} |\alpha_i|.\|Tx_i\|\right) \leq \|h\|^2.\left(\sum_{n+1}^{\infty} \|Tx_i\|^2\right).$$

(ii) Take an orthonormal basis ϕ_k of $\mathcal{H}$ consisting of eigenvectors of the compact operator T^*T. Observe that if $T^*T\phi_k = \mu_k\phi_k$ then $\mu_k = \langle T^*T\phi_k, \phi_k\rangle = \|T\phi_k\|^2 \geq 0$. Now use the spectral theorem for T^*T to prove that if for any orthonormal basis $\{x_n\}$, $\sum_{n=1}^{\infty} \|Tx_n\|^2$ converges then

$$\sum_{n=1}^{\infty} \|Tx_n\|^2 = \sum_{n=1}^{\infty} \langle T^*Tx_n, x_n\rangle = \sum_{k=1}^{\infty} \mu_k\,.$$

Note that for a double infinite series with all terms positive, the order of summation may be interchanged.