5 The Spectrum.

Definition. The *spectrum* of an operator T is the set of all complex numbers λ such that $\lambda I - T$ has no inverse in $\mathcal{B}(\mathcal{H})$.

The *spectrum* of T is denoted by $\sigma(T)$.

The complement (in $\mathbb{C}$) of $\sigma(T)$, that is, the set of all complex numbers λ such that $\lambda I - T$ has an inverse in $\mathcal{B}(\mathcal{H})$, is called the *resolvent set* of T and is denoted by $\rho(T)$.

For any element T of $\mathcal{B}(\mathcal{H})$, it is a fact that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$. We shall not need this general fact in this course. For the two classes of operators that we shall be concerned with (compact operators and selfadjoint operators) the required facts about the spectrum will be established by simple methods.

Note that every eigenvalue of an operator T is in the spectrum of T.

Also, if the K is a compact operator on an infinite-dimensional Hilbert space then $0 \in \sigma(K)$ (this merely repeats the fact that K does not have an inverse).

Lemma 5.1 *Let T be an operator such that for all $x \in \mathcal{H}$, $\| Tx \| \geq c \| x \|$, where c is a positive constant. Then the range of T is closed.*

Proof. Let (y_n) be a convergent sequence of elements of $\operatorname{ran}(T)$ converging to y. Then $y_n = Tx_n$ for some sequence (x_n) and we need to show that $y = Tx$ for some x.

Since (y_n) is convergent it is a Cauchy sequence. Now,

$$\|x_n - x_m\| \leq \tfrac{1}{c}\|T(x_n - x_m)\| = \tfrac{1}{c}\|y_n - y_m)\|$$

so it follows easily that (x_n) is a Cauchy sequence and so convergent to some element x. Then, since T is continuous, $y = \lim y_n = \lim Tx_n = \lim Tx$, as required. ■

Corollary 5.2 *If T is an in the lemma, the range of T^n is closed for each positive integer n.*

Proof. $\|T^n x\| \geq c^n \|x\|$ for all $x \in \mathcal{H}$. ■

We now derive some simple properties of the spectrum of a selfadjoint operator. For the rest of this section, A will denote a selfadjoint operator. Recall that $\langle Ax, x\rangle$ is real for all x since $\overline{\langle Ax, x\rangle} = \langle x, Ax\rangle = \langle A^* x, x\rangle = \langle Ax, x\rangle$.

Lemma 5.3

$$\|A\| = \sup_{\|x\|\leq 1} |\langle Ax, x\rangle| \,.$$

Proof. Let $k = \sup_{\|z\|\leq 1} |\langle Az, z\rangle|$. Then $|\langle Ax, x\rangle| \leq k\|x\|^2$ for all x and, from the Cauchy-Schwatrz in equality, $k \leq \|A\|$. Since

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |\langle Ax, y\rangle| \,,$$

to show that $\|A\| \le k$, we need to show that $|\langle Ax, y\rangle| \le k$ whenever $\|x\| \le 1$ and $\|y\| \le 1$. It is sufficient to prove this when $\langle Ax, y\rangle$ is real, since if $|\langle Ax, y\rangle| = e^{i\theta}\langle Ax, y\rangle$ then applying the result for the real case for $\langle Ax', y\rangle$ where $x' = e^{i\theta}x$, proves the general result.

Now, using the polarization identity (Lemma 2.3) and the paralellogram law (Lemma 1.4),

$$\begin{aligned} 4\langle Ax, y\rangle &= \langle A(x+y), (x+y)\rangle - \langle A(x-y), (x-y)\rangle \\ &\quad +i[\langle A(x+iy), (x+iy)\rangle - \langle A(x-iy), (x-iy)\rangle] \\ &\le k\{\|x+y\|^2 + \|x-y\|^2\} \\ &= k(2\|x\|^2 + 2\|y\|^2) \le 4k\,, \end{aligned}$$

(the expression in square brackets being zero since $\langle Ax, y\rangle$ is real). ■

Note that $\sup_{\|x\|\le 1} |\langle Ax, x\rangle| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. We write

$$m = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M = \sup_{\|x\|=1} \langle Ax, x\rangle\,.$$

Corollary 5.4 *For all $T \in \mathcal{B}(\mathcal{H})$*

$$\|T^*T\| = \|T\|^2\,.$$

Proof. Since T^*T is selfadjoint,

$$\|T^*T\| = \sup_{\|x\|\le 1} |\langle T^*Tx, x\rangle| = \sup_{\|x\|\le 1} \|Tx\|^2 = \|T\|^2\,.$$

■

The key to the next result is proving that $\| (\lambda I - A)x \| \ge c \| x \|$ whenever $\lambda \notin [m, M]$. This is done by a single calculation in the body of the proof. However, it can also be established by a sequence of simpler proofs as follows. Note that, if X is selfadjoint then

$$\begin{aligned} \|(iI - X)x\|^2 &= \langle (iI - X)x, (iI - X)x\rangle \\ &= \|x\|^2 + \|Xx\|^2 - i\langle x, Xx\rangle + i\langle Xx, x\rangle \\ &= \|Xx\|^2 + \|x\|^2 \ge \|x\|^2\,. \end{aligned}$$

Thus, if $\lambda = \xi + i\eta$ is not real (i.e. $\eta \ne 0$), then, using the above result for $X = \frac{1}{\eta}(A - \xi I)$, we have

$$\| (\lambda I - A)x \| = \| \eta(iI - X)x \| \ge |\eta|\|x\|\,.$$

If λ is real with $\lambda > M$, we have that for $\|x\| = 1$,

$$\|(\lambda I - A)x\| = \sup_{\|y\|\le 1} |\langle (\lambda I - A)x, y\rangle| \ge \langle (\lambda I - A)x, x\rangle \ge \lambda - M$$

so that (dividing by $\|x\|$) we have $\|(\lambda I - A)x\| \ge (\lambda - M)\|x\|$ for all x. A similar proof holds when $\lambda < m$.

Theorem 5.5
(i) $\sigma(A) \subseteq [m, M]$,
(ii) $m \in \sigma(A)$ *and* $M \in \sigma(A)$.

Proof. (i) Suppose $\lambda \notin [m, M]$ and let $d = \operatorname{dist}(\lambda, [m, M])$. Let $x \in \mathcal{H}$ be any unit vector and write $\alpha = \langle Ax, x\rangle$. Then $\langle(\alpha I - A)x, x\rangle = \langle x, (\alpha I - A)x\rangle = 0$ and

$$\begin{aligned}
\|(\lambda I - A)x\|^2 &= \|[\lambda I - \alpha I + (\alpha I - A)]x\|^2 \\
&= \langle[\lambda I - \alpha I + (\alpha I - A)]x, [\lambda I - \alpha I + (\alpha I - A)]x\rangle \\
&= |\lambda - \alpha|^2\|x\|^2 + (\bar{\alpha} - \bar{\lambda})\langle(\alpha I - A)x, x\rangle \\
&\qquad + (\alpha - \lambda)\langle x, (\alpha I - A)x\rangle + \|(\alpha I - A)x\|^2 \\
&\geq |\lambda - \alpha|^2 \geq d^2 .
\end{aligned}$$

It follows that $\|(\lambda I - A)x\| \geq d\|x\|$ [apply the above for $\frac{x}{\|x\|}$]. Hence $\lambda I - A$ is injective and, by Lemma 1.1, it has closed range. Further, if $0 \neq z \perp \operatorname{ran}(\lambda I - A)$ then $0 = \langle(\lambda I - A)x, z\rangle = \langle x, (\bar{\lambda} I - A)z\rangle$ for all x and so $(\bar{\lambda} I - A)z = 0$. But this is impossible, since, from above, noting that $d = \operatorname{dist}(\lambda, [m, M]) = \operatorname{dist}(\bar{\lambda}, [m, M])$, we have $\|\bar{\lambda} I - A)z\| \geq d\|z\|$. Therefore, $\operatorname{ran}(\lambda I - A) = \mathcal{H}$, (being both dense and closed).

Therefore, for any $y \in \mathcal{H}$, there is a unique $x \in \mathcal{H}$ such that $y = (\lambda I - A)x$. Define $(\lambda I - A)^{-1}y = x$. Then $\|y\| \geq d\|x\|$ so

$$\|(\lambda I - A)^{-1}y\| = \|x\| \leq \tfrac{1}{d}\|y\|$$

showing that $(\lambda I - A)^{-1} \in \mathcal{B}(\mathcal{H})$ (i.e. it is continuous). Thus $\lambda \notin \sigma(A)$, proving (i).

(ii) From Lemma 1.3, $\|A\|$ is either M or $-m$. If $\|A\| = M = \sup_{\|x\|=1}\langle Ax, x\rangle$; (if $\|A\| = -m$ the same proofs, applied to $-A$, hold) there exists a sequence (x_n) of unit vectors such that $(\langle Ax_n, x_n\rangle) \to M$. Then

$$\|(A - MI)x_n\|^2 = \|Ax_n\|^2 + M^2 - 2M\langle Ax_n, x_n\rangle \leq 2M^2 - 2M\langle Ax_n, x_n\rangle \to 0 .$$

Hence $A - MI$ has no inverse in $\mathcal{B}(\mathcal{H})$ [since if X were such an operator, $1 = \|x_n\| = \|X(A - MI)x_n\| \leq \|X\|.\|X(A - MI)x_n\| \to 0$] and so $M \in \sigma(A)$. For m, note that

$$\sup_{\|x\|=1} \langle(MI - A)x, x\rangle = M - m = \|MI - A\|$$

since $\inf_{\|x\|=1}\langle(MI - A)x, x\rangle = 0$. Applying the result just proved to the operator $MI - A$ shows that $M - m \in \sigma(MI - A)$, that is, $(M - m)I - (MI - A) = A - mI$ has no inverse. Hence $m \in \sigma(A)$. ■

The *spectral radius*, $\nu(T)$, of an operator T is defined to be

$$\nu(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} .$$

Thus we have shown that the spectrum of a selfadjoint operator is non-empty and real and its norm is equal to its spectral radius.

Exercises 5

1. Let $X, T \in \mathcal{B}(\mathcal{H})$ and suppose that X is invertible. Prove that $\sigma(T) = \sigma(X^{-1}TX)$.

2. Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator. Show that $U = (A - iI)(A + iI)^{-1}$ is a unitary operator.