4 Compact Operators.

Definition. An operator $K \in \mathcal{B}(\mathcal{H})$ is said to be *compact* if for every bounded set $\mathcal{S}$ of vectors of $\mathcal{H}$ the set $\overline{\{Ks : s \in \mathcal{S}\}}$ is compact.

Equivalently :

Definition. An operator $K \in \mathcal{B}(\mathcal{H})$ is said to be *compact* if for every bounded sequence (x_n) of vectors of $\mathcal{H}$ the sequence (Kx_n) has a convergent subsequence.

We shall denote the set of all compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$.

Definition. The *rank* of an operator is the dimension of its range.

Note that every operator of finite rank is compact. This is an immediate consequence of the Bolzano-Weierstrass theorem which states that every bounded sequence in $\mathbb{C}^n$ has a convergent subsequence. Note also that the identity operator on a Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

Theorem 4.1 *$\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$.*

Proof. We need to show that, if $A, B \in \mathcal{K}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ then $\alpha A, A + B, TA$ and AT are all in $\mathcal{K}(\mathcal{H})$. That is, for any a bounded sequence (x_n), we must show that $(\alpha Ax_n), ([A + B]x_n), (TAx_n)$ and (ATx_n) all have convergent subsequences.

Since A is compact, (Ax_n) has a convergent subsequence (Ax_{n_i}). Then clearly (αAx_{n_i}) is a convergent subsequence of (αAx_n) showing that αA is compact. Also, (x_{n_i}) is a bounded sequence and so, since B is compact, (Bx_{n_i}) has a convergent subsequence $(Bx_{n_{i_j}})$. Then $([A+B]x_{n_{i_j}})$ is a convergent subsequence of $([A+B]x_n)$, showing that $A + B$ is compact.

Again, since $T \in \mathcal{B}(\mathcal{H})$, T is continuous and so (TAx_{n_i}) is a convergent subsequence of (TAx_n) showing that TA is compact. The proof for AT is slightly different. Here, since (x_n) is bounded and $\|Tx_n\| \leq \|T\|.\|x_n\|$ we have that (Tx_n) is bounded and so, since A is compact, (ATx_n) has a convergent subsequence, showing that TA is compact. ■

A consequence of the above theorem is that, it $\mathcal{H}$ is infinite-dimensional then and $T \in \mathcal{B}(\mathcal{H})$ has an inverse $T^{-1} \in \mathcal{B}(\mathcal{H})$ then T is not compact.

Theorem 4.2 *$\mathcal{K}(\mathcal{H})$ is closed.*

Proof. Let (K_n) be a sequence of compact operators converging to K. To show that K is compact, we need to show that if (x_i) is a bounded sequence the (Kx_i) has a convergent subsequence.

Let (x_i^1) be a subsequence of (x_i) such that $(K_1x_i^1)$ is convergent,
let (x_i^2) be a subsequence of (x_i^1) such that $(K_2x_i^2)$ is convergent,
let (x_i^3) be a subsequence of (x_i^2) such that $(K_3x_i^3)$ is convergent,
and continue in this way.

[The notation above is slightly unusual and is adopted to avoid having to use subscripts on subscripts on $\cdots$.]

Let $z_i = x_i^i$. Then (z_i) is a subsequence of (x_i). Also, for each n, apart from the first n terms, (z_i) is a subsequence of (x_i^n) and so $(K_n z_i)$ is convergent.

We now show that (Kz_i) is convergent by showing that it is a Cauchy sequence. For all i, j, n we have

$$\begin{aligned} \|Kz_i - Kz_j\| &= \|(K-K_n)z_i + K_n z_i - K_n z_j - (K-K_n)z_j\| \\ &\leq \|K-K_n\|(\|z_i\| + \|z_j\|) + \|K_n(z_i - z_j)\| \,. \end{aligned}$$

Let $\epsilon > 0$ be given. Since $(K_n) \to K$ we can find n_0 such that $\|K - K_n\| < \frac{\epsilon}{4c}$ for $n > n_0$ where c satisfies $\|x_i\| \leq c$ for the bounded sequence (x_i). Choose one fixed such n. Now, since $(K_n z_i)$ converges, it is a Cauchy sequence and so there is an i_0 such that for $i > i_0, j > i_0$ we have $\|K_n z_i - K_n z_j\| < \frac{\epsilon}{2}$. Combining these with the displayed inequality shows that for $i > i_0, j > i_0$, $\|Kz_i - Kz_j\| < \epsilon$ so (Kz_i) is convergent as required. ■

Example. The operator K on $L^2[a,b]$ defined by

$$(Kf)(x) = \int_a^b k(x,t)f(t)\,dt\,,$$

where $\int_a^b \int_a^b |k(x,t)|^2\,dx\,dt = M^2 < \infty$, is compact.

We have already seen that operators of the above type are continuous with $\|K\| \leq M$ (Recall that $k(x,t)$ is called the *kernel* of the integral operator K). We shall show that K is the norm limit of a sequence of finite rank operators. Note that if $k(x,t)$ is of the form $u(x)v(t)$ then

$$(Rf)(x) = \int_a^b u(x)v(t)f(t)\,dt \;= \langle f, \overline{v}\rangle u = (\overline{v} \otimes u)f$$

is a rank one operator.

Let S be the square $[a,b] \times [a,b]$. We shall apply Hilbert space theory to $L^2(S)$ which is a Hilbert space of functions of 2 variables with the inner product

$$\langle \phi, \psi \rangle = \int_a^b \int_a^b \phi(x,t)\overline{\psi(x,t)}\,dx\,dt\,.$$

Let (u_i) be an orthonormal basis of $L^2[a,b]$. Then $(u_i(x)u_j(t))_{i,j=1}^{\infty}$ is an orthonormal basis of $L^2(S)$. Indeed,

$$\begin{aligned} \langle u_i(x)u_j(t), u_k(x)u_l(t)\rangle &= \int_a^b \int_a^b u_i(x)u_j(t)\overline{u_k(x)u_l(t)}\,dx\,dt \\ &= \int_a^b u_i(x)\overline{u_k(x)}\,dx \int_a^b u_j(t)\overline{u_l(t)}\,dt = 0 \end{aligned}$$

unless $i = k$ and $j = l$, in which case the integral is 1. Thus $(u_i(x)u_j(t))_{i,j=1}^{\infty}$ is an orthonormal sequence. To show that it is a basis, suppose $\phi(x,t) \perp u_i(x)u_j(t)$ for all i, j. Then

$$0 = \int_a^b \int_a^b \phi(x,t)\overline{u_i(x)u_j(t)}\,dx\,dt = \int_a^b \left(\int_a^b \phi(x,t)\overline{u_i(x)}\,dx\right)\overline{u_j(t)}\,dt\,.$$

This shows that, for each i, the function $\int_a^b \phi(x,t)\overline{u_i(x)}\,dx$ of t is orthogonal to $u_j(t)$ for each j. Therefore, since (u_j) is a basis of $L^2[a,b]$, it is (equivalent to) the zero function. Then, for fixed t the function $\phi(x,t)$ is orthogonal to $u_i(x)$ for each i and so it is zero.

Returning to the operator K, note that $k \in L^2(S)$. Therefore, by Theorem 3.3 (iii) it has a fourier expansion using the basis $(u_i u_j)$ of the type

$$k(x,t) = \sum_{i,j=1}^{\infty} \alpha_{ij} u_i(x) u_j(t) \,.$$

Thus, writing $k_n(x,t) = \sum_{i,j=1}^{n} \alpha_{ij} u_i(x) u_j(t)$ and

$$(K_n f)(x) = \int_a^b k_n(x,t) f(t)\,dt \,,$$

we have that K_n is a finite rank operator (of rank at most n^2). Note that $K - K_n$ is an integral operator (of the same type as K) with kernel $k(x,t) - k_n(x,t)$. Thus

$$\|K - K_n\|^2 \leq \int_a^b \int_a^b |k(x,t) - k_n(x,t)|^2 \,dx\,dt = \|k - k_n\|^2_{L^2(S)}$$

and the right hand side $\to 0$. Therefore Theorem 1.2 shows that K is compact.

Lemma 4.3 *Let K be a compact operator on $\mathcal{H}$ and suppose (T_n) is a bounded sequence in $\mathcal{B}(\mathcal{H})$ such that, for each $x \in \mathcal{H}$ the sequence $(T_n x)$ converges to Tx, where $T \in \mathcal{B}(\mathcal{H})$. Then $(T_n K)$ converges to TK in norm.*

Briefly, the above can be rephrased as :

If $K \in \mathcal{K}(\mathcal{H})$ and $\|T_n x - Tx\| \to 0$ for all $x \in \mathcal{H}$ then $\|T_n K - TK\| \to 0$.

In words : multiplying by a compact operator on the right converts a pointwise convergent sequence of operators into a norm convergent one.

Proof. Since (T_n) is a bounded sequence, $\|T_n\| \leq C$ for some constant C. Then for all $x \in \mathcal{H}$, $\|Tx\| = \lim_n \|T_n x\| \leq C\|x\|$ and so $\|T\| \leq C$.

Let K be compact and suppose that $\|TK - T_n K\| \not\to 0$. Then there exists some $\delta > 0$ and a subsequence $(T_{n_i} K)$ such that $\|TK - T_{n_i} K\| > \delta$. Choose unit vectors (x_{n_i}) of $\mathcal{H}$ such that $\|(TK - T_{n_i} K)x_{n_i}\| > \delta$. [That this can be done follows directly from the definition of the norm of an operator.] Using the fact that K is compact, we can find a subsequence (x_{n_j}) of (x_{n_i}) such that (Kx_{n_j}) is convergent. Let the limit of this sequence be y. Then for all j

$$\delta < \|(TK - T_{n_j} K)x_{n_j}\| \leq \|(T - T_{n_j})(Kx_{n_j} - y)\| + \|(T - T_{n_j})y\| \,.$$

Now, using the convergence of (Kx_{n_j}) to y, there exists n so that, for $n_j > n$, $\|Kx_{n_j} - y\| < \frac{\delta}{8C}$. Also, using the convergence of (T_{n_j}) to T, there exists m so that, for $n_j > m$, $\|(T - T_{n_j})y\| < \frac{\delta}{4}$. Then, for $j > \max[n,m]$ the right hand side of the displayed inequality is less than $\frac{\delta}{2}$, and this contradiction shows that the supposition that $\|TK - T_n K\| \not\to 0$ is false. ∎

The theorem below is true for all Hilbert spaces, but we shall only prove it for the case when the space is separable.

Theorem 4.4 *Every compact operator on $\mathcal{H}$ is a norm limit of a sequence of finite rank operators.*

Proof. Let x_i be an orthonormal basis of $\mathcal{H}$. Define P_n by

$$P_n h = \sum_{i=1}^{n} \langle h, x_i \rangle x_i \,.$$

[Note that P_n is the projection onto span $x_1, x_2, \cdots, x_n$. Also, P_n could be written as $P_n h = \sum_{i=1}^{n} x_i \otimes x_i \,.$] From Theorem 3.3(iii), for all $x \in \mathcal{H}$, $P_n x$ converges to x (that is, P_n converges pointwise to the identity operator I). Now, if K is any compact operator, $P_n K$ is of finite rank and, from Theorem 1.3 $(P_n K)$ converges to K in norm. ■

Exercises 4

1. Let T be the operator on $l^2 \oplus l^2$ defined by $T(x,y) = (0,x)$. Show that $T^2 = 0$ and that T is not compact.

2. Let (x_n) be an orthonormal sequence in a Hilbert space H and let (α_n) be a bounded sequence of complex numbers. Prove that the operator A defined by

$$Ax = \sum_{n=1}^{\infty} \alpha_n \langle x, x_n \rangle x_n$$

is bounded with

$$\| A \| \leq \sup_n |\alpha_n| \,.$$

Hence prove that if $\lim_{n\to\infty} (\alpha_n) = 0$ then A is compact.

Show that, when $m \neq n$,

$$\| Ax_m - Ax_n \|^2 = |\alpha_m|^2 + |\alpha_n|^2 \,.$$

Hence prove that, conversely if $\lim_{n\to\infty} (\alpha_n) \neq 0$ then A is not compact.

3. Given that K^*K is compact, prove that K is compact.

[Hint: if (K^*Kx_n) is convergent, prove that (Kx_n) is a Cauchy sequence.]

4. Let K be a compact operator. Using the hints below, prove that for any orthonormal sequence $\{x_n\}$, $(Kx_n) \to 0$ as $n \to \infty$

Hints: Observe that, for any vector z, $\langle x_n, z\rangle \to 0$. [A result of the course states that $\sum |\langle x_n, z\rangle|^2$ is convergent.] Apply this, with $z = K^*y$ for any y, and show that no subsequence of (Kx_n) can converge to a non-zero vector.

5. Let A_n be a bounded sequence in $\mathcal{B}(\mathcal{H})$ such that, for all $x, y \in \mathcal{H}, \lim_{n\to\infty}\langle A_n x, y\rangle = 0$. Prove that, for any compact operator K,

$$\lim_{n\to\infty} \|KA_nK\| = 0 \,.$$

[Use the ideas in the proof of Lemma 4.3.]