2 Linear Operators.

Some of the results in this section are stated for normed linear spaces but they will be used in the sequel only for Hilbert spaces.

Lemma 2.1 *Let X and Y be normed linear spaces and let $L : X \to Y$ be a linear map. Then the following are equivalent :*

1. *L is continuous;*
2. *L is continuous at 0;*
3. *there exists a constant K such that $\|Lx\| \leq K\|x\|$ for all $x \in X$.*

Proof. 1 implies 2 is obvious. If 2 holds, take any $\epsilon > 0$. Continuity at 0 shows that there is a corresponding $\delta > 0$ such that $\|LX\| < \epsilon$ whenever $\|x\| < \delta$. Take some c with $0 < c < \delta$. Then for any $x \neq 0$, $\left\|\frac{cx}{\|x\|}\right\| = c < \delta$ and so

$$\left\|L\left(\frac{cx}{\|x\|}\right)\right\| = c\frac{\|Lx\|}{\|x\|} < \epsilon.$$

This shows that $\|Lx\| < K\|x\|$ where $K = \frac{\epsilon}{c}$.

If 3 holds, to show continuity at any point x_0, note that

$$\|Lx - Lx_0\| = \|L(x - x_0)\| \leq K\|x - x_0\|.$$

Therefore, given any $\epsilon > 0$, let $\delta = \frac{\epsilon}{K}$. Then if $\|x - x_0\| < \delta$ we have $\|Lx - Lx_0\| < \epsilon$.
■

The set of all continuous (bounded) linear maps $X \to Y$ is denoted by $\mathcal{B}(X, Y)$. When $X = Y$ we write $\mathcal{B}(X)$.

For $L \in \mathcal{B}(X, Y)$, define $\|L\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|}$.

Exercise. $\|\cdot\|$ is a norm on $\mathcal{B}(X, Y)$ and

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Lx\| = \sup_{\|x\| = 1} \|Lx\|.$$

If Y is complete then so is $\mathcal{B}(X, Y)$

When $Y = \mathbb{C}$ then $\mathcal{B}(X, \mathbb{C})$ is called the *dual* of X and denoted by X' (sometimes by X^*). The elements of the dual are called (continuous) *linear functionals.*

We shall be concerned with Hilbert spaces; $\mathcal{H}$ will always denote a Hilbert space.

Theorem 2.2 (Riesz representation theorem) *Every linear functional f on $\mathcal{H}$ is of the form*

$$f(x) = \langle x, h \rangle$$

for some $h \in \mathcal{H}$, where $\|f\| = \|h\|$.

Proof. If $f = 0$, take $h = 0$. For $f \neq 0$ then $N = f^{-1}(0) = \{x : f(x) = 0\} \neq \mathcal{H}$. Also, since f is continuous, N is closed. Thus $N^{\perp} \neq (0)$ so take $y \perp N$. Then $f(y) \neq 0$. Write $z = \frac{y}{f(y)}$ so that $f(z) = 1$ [using $f(\alpha x) = \alpha f(x)$]. For any $x \in \mathcal{H}$

$$f\left(x - f(x)z\right) = f(x) - f(x).f(z) = 0 \text{ and so } x - f(x)z \in N\,.$$

Since $z \perp N$,

$$\langle x - f(x)z, z\rangle = \langle x, z\rangle - f(x)\|z\|^2 = 0.$$

Writing $h = \frac{z}{\|z\|^2}$ we obtain

$$f(x) = \langle x, h\rangle\,.$$

For the norm, note that $|f(x)| = |\langle x, h\rangle| \leq \|x\|.\|h\|$ so $\|f\| \leq \|h\|$. Also

$$\|f| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(h)|}{\|h\|} = \|h\|\,.$$

■

Note that the result $\|f\| = \|h\|$ shows that the correspondence between $\mathcal{H}$ and its dual is one to one.

Lemma 2.3 (Polarization identity for operators)

$$\begin{aligned}\langle Ax, y\rangle &= \frac{1}{4}[\langle A(x+y),(x+y)\rangle - \langle A(x-y),(x-y)\rangle \\ &\qquad +i\langle A(x+iy),(x+iy)\rangle - i\langle A(x-iy),(x-iy)\rangle]\,.\end{aligned}$$

Proof.

$$\begin{aligned}\langle A(x+y),(x+y)\rangle &= \langle Ax,x\rangle + \langle Ay,y\rangle + \langle Ax,y\rangle + \langle Ay,x\rangle \\ -\langle A(x-y),(x-y)\rangle &= -\langle Ax,x\rangle - \langle Ay,y\rangle + \langle Ax,y\rangle + \langle Ay,x\rangle \\ i\langle A(x+iy),(x+iy)\rangle &= i\langle Ax,x\rangle + i\langle Ay,y\rangle + \langle Ax,y\rangle - \langle Ay,x\rangle \\ -i\langle A(x-iy),(x-iy)\rangle &= -i\langle Ax,x\rangle - i\langle Ay,y\rangle + \langle Ax,y\rangle - \langle Ay,x\rangle\,.\end{aligned}$$

Adding the above gives the result. ■

Corollary 2.4 *If* $\langle Ax, x\rangle = 0$ *for all* $x \in \mathcal{H}$ *then* $A = 0$.

Proof. If $\langle Ax, x\rangle = 0$ for all $x \in \mathcal{H}$ the above shows that $\langle Ax, y\rangle = 0$ for all $x, y \in \mathcal{H}$ and so using $y = Ax$ it follows that $\|Ax\|^2 = 0$ for all $x \in \mathcal{H}$. Thus $A = 0$. ■

Definition Let $\mathcal{H}$ be a Hilbert space. A *bilinear form* (also called a *sesquilinear form*) ϕ on $\mathcal{H}$ is a map $\phi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

$$\begin{aligned}\phi(\alpha x + \beta x', y) &= \alpha\phi(x,y) + \beta\phi(x',y) \\ \phi(x, \alpha y + \beta y') &= \bar{\alpha}\phi(x,y) + \bar{\beta}\phi(x,y')\,.\end{aligned}$$

A bilinear form is said to be *bounded* if, for some constant K, $|\phi(x,y)| \leq K\|x\|.\|y\|$ for all $x, y \in \mathcal{H}$.

Theorem 2.5 (Riesz) *Every bounded bilinear form ϕ on $\mathcal{H}$ is of the form*

$$\phi(x,y) = \langle Ax, y\rangle$$

for some $A \in \mathcal{B}(\mathcal{H})$.

Proof. Consider x fixed for the moment. Then $\phi(x,y)$ is conjugate linear in y, so that $\overline{\phi(x,y)}$ is linear in y. Using Theorem 1.2 we have that there is a (unique) $h \in \mathcal{H}$,

$$\overline{\phi(x,y)} = \langle y, h\rangle\,, \text{ that is } \phi(x,y) = \langle h, y\rangle\,.$$

One can find such an h corresponding to each $x \in \mathcal{H}$. Define a function $\mathcal{H} \to \mathcal{H}$ by $Ax = h$. Then A is linear since , for all x, x', y,

$$\langle A(x+x'), y\rangle \;= \phi((x+x'), y) = \phi(x,y) + \phi(x',y) = \langle Ax, y\rangle \; + \langle Ax', y\rangle$$

so $A(x+x') = Ax + Ax'$ [since $A(x+x') - Ax - Ax' \in \mathcal{H}^\perp = (0)$]. Similarly $A(\alpha x) = \alpha Ax$. Also,

$$\|Ax\| = \sup_{y\neq 0} \frac{\langle Ax, y\rangle}{\|y\|} = \sup_{y\neq 0} \frac{|\phi(x,y)|}{\|y\|} \leq K\|x\|$$

so A is continuous. ■

Definition The adjoint. Let $A \in \mathcal{B}(\mathcal{H})$. Then $\psi(x,y) = \langle x, Ay\rangle$ is a bounded bilinear form on $\mathcal{H}$ so, by Theorem 1.5 there is an operator $A^* \in \mathcal{B}(\mathcal{H})$ such that

$$\langle A^*x, y\rangle = \psi(x,y) = \langle x, Ay\rangle\,.$$

A^* is called the *adjoint* of A.

Exercise.

(i) $(A^*)^* = A$,

(ii) $(\lambda A)^* = \bar{\lambda} A^*$,

(iii) $(A+B)^* = A^* + B^*$,

(iv) $(AB)^* = B^*A^*$,

(v) $\|A\| = \|A^*\|$.

Note. Bilinear forms could have been defined as maps ϕ from $\mathcal{H} \times \mathcal{K}$ to $\mathbb{C}$ where $\mathcal{H}$ and $\mathcal{K}$ are different Hilbert spaces. All the above can be done with essentially no change; (the adjoint of $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is then an operator in $\mathcal{B}(\mathcal{K}, \mathcal{H})$).

Definition.

If $A = A^*$ then A is said to be *selfadjoint.*
If $AA^* = A^*A$ then A is said to be *normal.*
If $UU^* = U^*U = I$ then U is said to be *unitary.*

Projections.

Let N be a closed subspace of $\mathcal{H}$. Then from Theorem 1.11,

$$\mathcal{H} = N \oplus N^{\perp}$$

that is, any $h \in \mathcal{H}$ has a unique decomposition as $h = x + y$ with $x \in N$ and $y \in N^{\perp}$.

The *orthogonal projection* P onto N is defined by $Ph = x$ (where $h = x + y$ is the decomposition above). Note that then $y = (I - P)h$ and $I - P$ is the projection onto $N^{\perp}$.

In this course we shall not consider projections that are not orthogonal and usually call these operators "projections".

Lemma 2.6 *Let N be a closed subspace of $\mathcal{H}$ and let P be the orthogonal projection onto N. Then*
(i) P *is linear,*
(ii) $\|P\| = 1$ *(unless* $N = 0$*),*
(iii) $P^2 = P$,
(iv) $P^* = P$.

Also, if $E \in \mathcal{B}(\mathcal{H})$ satisfies $E = E^2 = E^*$ then E is the (orthogonal) projection onto some (closed) subspace.

Proof. (i) Let $h, h' \in \mathcal{H}$ and suppose $h = x + y$ and $h' = x' + y'$ are the unique decompositions of h and h' with $x, x' \in N$ and $y, y' \in N^{\perp}$. Then $\alpha h + \beta h' = (\alpha x + \beta x') + (\alpha y + \beta y')$ is the decomposition of $\alpha h + \beta h'$ and

$$P(\alpha h + \beta h') = \alpha x + \beta x' = \alpha Ph + \beta Ph' .$$

(ii) If $h = x + y$ with $x \in N$ and $y \in N^{\perp}$,

$$\|Ph\|^2 = \|x\|^2 \le \|x\|^2 + \|y\|^2 = \|h\|^2$$

and so $\|P\| \le 1$. But if $0 \ne h \in N$ then $Ph = h$ and so $\|P\| = 1$.

(iii) If $h \in N$ [then $h = h + 0$ is the decomposition of h and] $Ph = h$. But for any $h \in \mathcal{H}$, $Ph \in N$ so $P(Ph) = Ph$, that is, $P^2 = P$.

(iv) If $h = x + y$ and $h' = x' + y'$ with $x, x' \in N$ and $y, y' \in N^{\perp}$.

$$\langle Ph, h' \rangle = \langle x, x' + y' \rangle = \langle x, x' \rangle$$

since $x \in N$ and $y' \in N^{\perp}$. Similarly $\langle h, Ph' \rangle = \langle x, x' \rangle$ and so $P = P^*$.

Finally, if $E \in \mathcal{B}(\mathcal{H})$ satisfies $E = E^2 = E^*$ let $N = \{x : Ex = x\}$. Then $N = \ker(I - E)$, so N is closed. For any $h \in \mathcal{H}$, write

$$h = Eh + (I - E)h .$$

Then $Eh \in N$ since $E(Eh) = E^2 h = Eh$ and $(I - E)h \perp N$ since if $x \in N$, $Ex = x$ and

$$\langle (I - E)h, x \rangle = \langle (I - E)h, Ex \rangle = \langle E^*(I - E)h, x \rangle = \langle (E^2 - E)h, x \rangle = 0.$$

This shows that E is the projection onto N. ■

Lemma 2.7 *If P is the orthogonal projection onto a subspace N then for all $h \in \mathcal{H}$,*

$$d(h, N) = \|(I - P)h\|.$$

Proof. For any $h \in \mathcal{H}$ we have $Ph \in N$ and $\langle (I - P)h, n \rangle = 0$ for all $n \in N$. Therefore from Lemma 1.7

$$d(h, N) = \|h - Ph\| = \|(I - P)h\| .$$

■

Lemma 2.8 *Let $A \in \mathcal{B}(\mathcal{H})$ and P be the orthogonal projection onto a subspace N.*

(i) N *is invariant under* $A \Longleftrightarrow AP = PAP$.
(ii) $N^{\perp}$ *is invariant under* $A \Longleftrightarrow PA = PAP$.

If $A = A^*$ *then* N *is invariant under* $A \Longleftrightarrow N^{\perp}$ *is invariant under* $A \Longleftrightarrow PA = AP$.

Proof. (i) $\Longrightarrow$ Suppose $An \in N$ for all $n \in N$. Then since $Ph \in N$ for all $h \in \mathcal{H}$, we have $APh \in N$. Therefore then $PAPh = APh$ [since $Pn = n$ for all $n \in N$].

$\Longleftarrow$ If $n \in N$ then $Pn = n$ and so $An = APn = PAPn \in N$ [since N is the range of P].

(ii) The projection onto $N^{\perp}$ is $I - P$. Trivial algebra shows that

$$A(I - P) = (I - P)A(I - P) \Longleftrightarrow PA = PAP$$

and so (ii) follows from (i).

Finally $AP = PAP \Longleftrightarrow (AP)^* = (PAP)^* \Longleftrightarrow PA = PAP$ since $A = A^*$. If these equalities hold then $PA = AP$. ■

Exercises 2

1. Let $X \in \mathcal{B}(\mathcal{H})$. Show that :

(i) X is selfadjoint $\iff \langle Xx, x\rangle$ is real for all $x \in \mathcal{H}$,

(ii) X is normal $\iff \| Xx \| = \| X^*x \|$ for all $x \in \mathcal{H}$,

(iii) X is unitary $\iff \| Xx \| = \| X^*x \| = \| x \|$ for all $x \in \mathcal{H}$.

2. Let S be the one-dimensional subspace of ℓ^2 spanned by the element $(1, -1, 0, 0, \ldots)$. Show explicitly that any element $x = (\xi_k) \in \ell^2$ can be written as $x = x_1 + x_2$ where $x_1 \in S$ and $x_2 \perp S$.

3. Let A be a selfadjoint operator such that for all $x \in H$, $\| Ax \| \geq c \| x \|$, where c is a positive constant. Show that A has a continuous inverse.

[Hints : Show (i) A is injective, (ii) the range of A is closed (iii) $(\operatorname{ran}(A))^\perp = (0)$.] Note that the selfadjointness condition **is** needed – consider the operator S on ℓ^2 defined by $S(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \xi_3, \ldots)$.

4. The operators D and W on ℓ^2 are defined by

$$D(\xi_1, \xi_2, \xi_3, \ldots) = (\alpha_1\xi_1, \alpha_2\xi_2, \alpha_3\xi_3, \ldots)\,,$$

$$W(\xi_1, \xi_2, \xi_3, \ldots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \alpha_3\xi_3, \ldots)\,,$$

where (α_n) is a bounded sequence of complex numbers. Show that W and D are bounded operators and find their adjoints.

5. Given that $X \in \mathcal{B}(\mathcal{H})$ is invertible (that is, there exists $X^{-1} \in \mathcal{B}(\mathcal{H})$ such that $XX^{-1} = X^{-1}X = I$) prove that X^* is invertible and $(X^*)^{-1} = (X^{-1})^*$.

6. Find the adjoint of the operator V defined on $L^2[0,1]$ by $(Vf)(x) = \int_0^x f(t)\,dt$.

7. Let $T : L^2[0,1] \to L^2[0,1]$ be defined by

$$(Tf)(x) = \sqrt{2x}\, f(x^2)\,.$$

Find the adjoint of T and deduce that T is unitary.

8. Let E, F be the orthogonal projections onto subspaces M and N respectively. Prove that,

(i) $EF = F \iff N \subseteq M \iff E - F$ is an orthogonal projection,

(ii) $EF = 0 \iff N \subseteq M^\perp \iff E + F$ is an orthogonal projection,

(iii) $EF = FE \iff E + F - FE$ is an orthogonal projection.

9. The operator $A \in \mathcal{B}(\mathcal{H})$ satisfies $Ax = x$ for some $x \in \mathcal{H}$. Prove that $A^*x = x + y$ where $y \perp x$. If, further, $\|A\| \leq 1$, show that $A^*x = x$.

Suppose that $E^2 = E$ and $\|E\| = 1$. Use the above to show that $\operatorname{ran}(E) = \operatorname{ran}(E^*)$ and $\ker(E) = \ker(E^*)$ and deduce that $E = E^*$ (so that E is the orthogonal projection onto some subspace of $\mathcal{H}$).

10. Let L_o and L_e be subspaces of $L^2[-1,1]$ defined by

$$\begin{aligned} L_o &= \{f : f(t) = -f(-t) \quad \text{(almost everywhere)}\} \\ L_e &= \{f : f(t) = f(-t) \qquad \text{(almost everywhere)}\}. \end{aligned}$$

Show that $L_o \oplus L_e = L^2[-1,1]$ and find the projections of $L^2[0,1]$ onto L_o and L_e. Find expressions for the distances of any element f to L_o and L_e. Calculate the values in the specific case where $f(t) = t^2 + t$.

11. Let M and N be vector subspaces of $\mathcal{H}$ such that $M \perp N$ and $M \dot{+} N = \mathcal{H}$. Prove that M and N are closed.

12. Show that the set of sequences $x = (\xi_n)$ such that $\sum_n n^2 |\xi_n|^2$ converges, forms a dense subset of l^2.

Define the operator D on l^2 by

$$D(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, \tfrac{1}{2}\xi_2, \tfrac{1}{3}\xi_3, \ldots)\,,$$

and let M and N be linear subspaces of $l^2 \oplus l^2$ defined by

$$M = \{(x,0) : x \in l^2\} \quad N = \{(x, Dx) : x \in l^2\}\,.$$

Observe that M is closed and use the continuity of D to show that N is also closed. Show that $M \cap N = (0)$ and that the algebraic direct sum of M and N is dense in $l^2 \oplus l^2$ but is not equal to $l^2 \oplus l^2$ (and so it is not closed).