Introduction

These are notes for a King's College course to fourth year undergraduates and MSc students. They cover the theoretical development of operators on Hilbert space up to the spectral theorem for bounded selfadjoint operators. I have tried to make the treatment as elementary as possible and to include only what is essential to the main development. The proofs are the simplest and most intuitive that I know. The exercises are culled from various sources; some of them are more or less original. They are designed to be worked as the course progresses and, in some cases, they anticipate results and techniques that come in the later theorems of the course.

It should be emphasized that these notes are not complete. Although the theoretical development is covered rather fully, examples, illustrations and applications which are essential for the understanding of the subject, are left to be covered in the lectures. There are good reasons for doing this. Experience has shown that audiences lose concentration if they are provided with comprehensive notes which coincide with the lectures. Also, in many cases examples and such are best treated in a less formal way which is more suited to oral presentation. In this way it is possible to cater for different sections of an audience with a mixed background. A formal proof may be indicated to some while others may have to take certain statements on trust. This is especially the case when integration spaces are involved.

I would like to thank the many students and colleagues who have pointed out errors and obscurities in earlier versions of these notes. A few proofs contain some sentences in square brackets. These indicate explanations that I consider rather obvious and should be superfluous to a formal proof but were added in response to some query.

For the benefit of a wider audience, here is a brief indication of what might be covered to supplement the notes and also a few comments.

Section 1. Examples of inner product spaces : $\ell^2_n(=\mathbb{C}^n)$ and ℓ^2. Continuous functions on $[a,b]$ with $\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$ and problems with extending to larger classes of functions (equivalence classes as elements of the space). Completeness of $\ell^2_n(=\mathbb{C}^n)$ and ℓ^2 (and continuous functions on $[a,b]$ not complete).

$L^2[a,b]$. Some brief discussion of the Lebesgue integral. The following statement to be known or accepted: there is a definition of the integral such that the (equivalence classes) of all functions f such that $\int_a^b |f(t)|^2\,dt$ exists and is finite forms a Hilbert space with the inner product $\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$ (that is, it is complete). Some more general L^2 spaces might be mentioned (e.g. $L^2(S)$ where $S = [a,b]\times[a,b]$ or some other subset of $\mathbb{R}^n$).

Examples of normed spaces which cannot be Hilbert spaces because they do not satisfy the parallelogram law ($C[a,b], \ell^p_n(n=2)$ for $p \neq 2$).

Examples of normed spaces with closed convex sets where the distance from a point is not attained uniquely (e.g. the unit ball in ℓ^1_2 with the point $(1,1)$) or not attained at all (e.g. the space $X = \{f \in C[0,1] : f(0) = 0\}$, the set $\{f \in X : \int_0^1 f(t)\,dt = 1\}$ and the zero function as the point).

Some indications of the applications of the minimum distance theorem, e.g. to approximation theory and optimal control theory.

Section 2. This section is supplemented by specific examples of operators on ℓ^2 and $L^2[a,b]$. These include diagonal operators, shifts (forward, backward and weighted) on ℓ^2, the bilateral shift on $\ell^2(\mathbb{Z})$ and the following operators on $L^2[a,b]$.

Multiplication operator : $(M_\phi f)(x) = \phi(x).f(x) \qquad \phi$ (essentially) bounded.

Fredholm integral operator : $(Kf)(x) = \int_a^b k(x,t)f(t)\,dt$ where $\int\int |k|^2 < \infty$,

with the Volterra operator, $(Vf)(x) = \int_a^x f(t)\,dt$ as a special case.

The boundedness of these operators should be established and the adjoints identified. Other examples of finding adjoints (similar to those in the exercises) might be done.

Section 3. The main additional topic for this section is the connection to classical Fourier series. The fact that the normalized trigonometric functions for an orthonormal basis of $L^2[-\pi,\pi]$ should be established or accepted. One route uses the Stone-Weierstrass Theorem and the density of $C[-\pi,\pi]$ in $L^2[-\pi,\pi]$. Inevitably, this requires background in metric spaces and Lebesgue theory. Note that this fact is also established, albeit in a roundabout way, by the work in Section 7.

The projection onto a subspace can be written down in terms of an orthonormal basis of the subspace : $P_N h = \sum \langle h, y_i \rangle y_i$ where $\{y_i\}$ is an orthonormal basis of N.

Applying the Gram-Schmidt process to the polynomials in the Hilbert space $L^2[-1,1]$ gives (apart from constant factors) the Legendre polynomials. Similarly the Hermite and the Laguerre polynomials arise from orthonormalizing $\{x^n e^{-x^2}\}$ and $\{x^n e^{-x}\}$ in the spaces $L^2[-\infty,\infty]$ and $L^2[0,\infty]$ respectively.

Section 4. Some of the operators introduced in Section 2 should be examined for compactness. In particular, the conditions for a diagonal operator on ℓ^2 to be compact should be established.

Theorem 4.4 and Lemma 4.3 on which its proof depends are the only results in these notes which are not strictly needed for what comes later. Note that this result is not valid in general Banach spaces.

Section 5. The spectra of some specific operators should be identified. In particular, the spectrum of M_ϕ where $\phi(x) = x$ on $L^2[0,1]$ should be identified as $[0,1]$ and the fact that M_ϕ has no eigenvalues should be noted.

Section 6. The fact that the Volterra operator has no eigenvalues should be established, hence showing that some compact operators may have spectrum equal to $\{0\}$.

It is useful to review the orthogonal diagonalization of real symmetric matrices and/or unitary diagonalization of Hermitian (i.e. selfadjoint) matrices. It is instructive to re-write both these results and Theorem 6.9 in terms of projections onto eigenspaces.

Section 7. This section is motivated by an informal discussion of of the Green's function as the response of a system to the input of a unit pulse. This is illustrated by the elementary example of finding the shape under gravity of a heavy string (of variable density) fixed at $(0,0)$ and $(\ell,0)$. This is found by calculating the (triangular) shape $k(x,t)$ of a weightless string with a unit weight at $x = t$ and then using an integration process. The differential equation is also found and shown to give the same answer. (Naturally, usual elementary applied mathematical assumptions - small displacements, constant tension - apply.)

Additionally, a brief, very informal discussion of delta functions and the Green's function as the solution of the system with the function f being a delta function is of interest.

It should be stressed, however, that the proof of Theorem 7.1 is purely elementary and quite independent of the discussions above.

Section 8. The most important part of this final section is Theorem 8.3, the continuous functional calculus. This is sufficient for the vast majority of applications of the spectral theorem for bounded self-adjoint operators. These include, for example, the polar decomposition and the properties of e^{At} and these are done in the course.

The approach here to the general spectral theorem is elementary and very pedestrian. It should be noted that, given the appropriate background, there are more elegant ways. These include using the identification of the dual of $C[m, M]$ (actually of $C(\sigma)$) as a space of measures. There is also a Banach algebra treatment.

In an elementary course such as this, the technicalities of the spectral theorem need not be strongly emphasized. However, a down to earth approach should clarify the meaning of theorem and remove the mystery often attached by students to these operator integrals.