Expected shortfall is ineffective against tail-risk seekers

Statistical arbitrage of coherent risk measures.

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Agenda

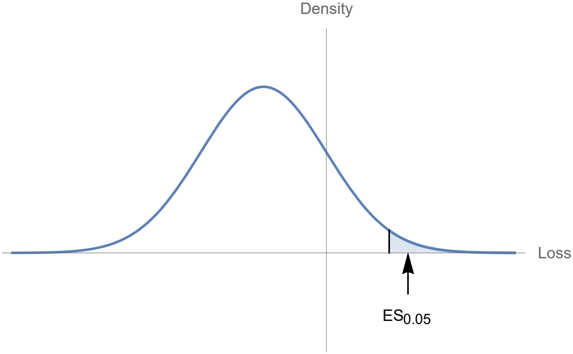


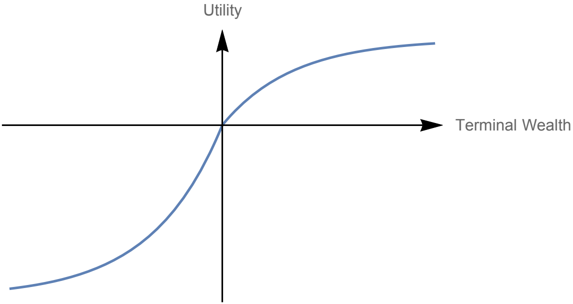
Figure 1:

S-shaped utility and the ineffectiveness expected shortfall

ρ-arbitrage and ineffective risk constraints

Does ρ arbitrage exist in practice?

Kahneman and Tversky proposed modelling observed human behaviour as maximization of expected S-shaped utility.



Definition

A utility function is risk-seeking in the left tail if $\exists N \leq 0$, $\eta \in (0,1)$ *and* $c > 0$ *such that*

$$u(x) > -c|x|^{\eta} \qquad \forall x \leq N.$$

Example: A trader with limited liability.

Theorem ([AB19a])

Let $(\Omega, \mathbb{P}, \mathbb{Q})$ be a one-period complete market satisfying

$$\operatorname{ess\,sup} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \infty$$

Let u be a utility function that is risk-seeking in the left tail. Let C be a cost constraint, and L be an expected shortfall risk limit at confidence level α. Then

$$\begin{aligned} &\sup \left\{ \mathbb{E}_{\mathbb{P}}(u(X)) \mid X \in L^1(\Omega),\, \mathbb{E}_{\mathbb{Q}}(X) \leq C \text{ and } \mathit{ES}_{\alpha}(X) \leq L \right\} \\ = &\sup \left\{ \mathbb{E}_{\mathbb{P}}(u(X)) \mid X \in L^1(\Omega),\, \mathbb{E}_{\mathbb{Q}}(X) \leq C \right\}. \end{aligned}$$

i.e. rogue traders are not materially affected by Expected Shortfall constraints.

In particular this applies to one period investments in Black–Scholes–Merton market since this market satisfies the $\operatorname{ess\,sup}$ condition.

Definition

Two complete markets are isomorphic if there is a measurable map which is a bijection up to null sets which preserves both $\mathbb{P}$ and $\mathbb{Q}$.

Definition

A standard probability space is isomorphic to the union of an interval with a number of atoms.

Definition

A casino is the complete market $\mathcal{C} := ([0,1], \mathbb{P}, \mathbb{Q})$ with $\mathbb{P} = \mathbb{Q}$ being the Lebesgue measure.

Theorem (Classification of complete markets [Arm18a])

Let $(\Omega, \mathbb{P}, \mathbb{Q})$ and $(\Omega', \mathbb{P}', \mathbb{Q}')$ be one period complete markets on a standard probability space, then

$$(\Omega, \mathbb{P}, \mathbb{Q}) \times \mathcal{C} \cong (\Omega', \mathbb{P}', \mathbb{Q}') \times \mathcal{C}$$

if and only if

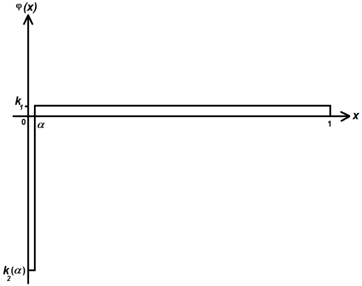
$$F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}} = F_{\frac{\mathrm{d}\mathbb{Q}'}{\mathrm{d}\mathbb{P}'}}.$$

(Proof idea to show that ES constraints are ineffective) WLOG we may write the optimal investment problem as finding the payoff function $\phi : [0,1]^2 \to \mathbb{R}$ to

$$\begin{aligned} \text{maximize} \quad & \int_{[0,1]^2} u(\phi(x,y))\,\mathrm{d}x\,\mathrm{d}y \\ \text{subject to} \quad & \int_{[0,1]^2} q(x)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y \le e^{-rt}C \\ \text{and} \quad & \mathrm{ES}_\alpha(f) \le L \end{aligned}$$

for positive decreasing q with $\lim_{x\to 0} q(x) = \infty$. q corresponds to the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.

Now consider payoffs functions $f(x,y) = f(x)$ of this form



Summary

Expected shortfall, and hence VaR, at any confidence level is ineffective as a risk-constraint in typical complete markets.

Questions for the remainder of the talk:

- Q: What is it about expected shortfall that makes it ineffective?
  A: It is a positively homogeneous (coherent) risk measure.
- What about incomplete markets?
  A: In incomplete markets there will be a minimum confidence level α such that ES_α is ineffective. In realistic examples this may be 0.01% or lower.

Definition ([Pen11], [Pen12])

A market consists of a probability space $(\Omega, \mathbb{P})$ *and a price function* $\mathcal{P}$ *mapping random variables (representing asset payoffs) to* $\mathbb{R} \cup \{+\infty\}$*. The domain of* $\mathcal{P}$ *defines the set of traded assets. In most classical markets* $\mathcal{P}$ *is linear on its domain, but we relax this assumption.*

The market is positive-homogeneous if $\mathcal{P}(\lambda X) = \lambda \mathcal{P}(X)$ *for* $\lambda \geq 0$*. We only require positive homogeneity to allow a bid ask spread.*

The market is coherent if: it is positive homogeneous; a portfolio costs no more than its components parts, i.e.

$$\mathcal{P}(X + Y) \leq \mathcal{P}(X + Y);$$

and there is a risk-free asset, possibly with a bid-ask spread, i.e.

$$\mathcal{P}(1) < \infty, \qquad \mathcal{P}(-1) < \infty.$$

Definition

A trading constraint, $\mathcal{A}$, is a subset of the set of random variables of finite price representing the assets a trader is allowed to purchase.

Definition

Let $\tilde{u}(x) = x^+$. This is a "worst case" version of an S-shaped utility. A trading constraint $\mathcal{A}$ is ineffective if for any cost $C \in \mathbb{R}$

$$\sup_{X \in \mathcal{A}, \mathcal{P}(X) \leq C} E(\tilde{u}(X)) = \infty.$$

Definition ([ADEH99])

A coherent risk measure $\rho : L^\infty(\Omega;\mathbb{R}) \to \mathbb{R}$ satisfies

(i) *Normalization:* $\rho(0) = 0$

(ii) *Montonicity:* $\rho(X) \geq \rho(Y)$ *if* $X \leq Y$ *almost surely.*

(iii) *Sub-additivity:* $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$*.*

(iv) *Translation invariance:* $\rho(X + a) = \rho(X) - a$ *for* $a \in \mathbb{R}$*.*

(v) *Positive homogeneity:* $\rho(\lambda X) = \lambda\rho(X)$ *for* $\lambda \in \mathbb{R}^+$*.*

Definition

If ρ is function on the space of random variables, then a random variable is called a ρ-arbitrage if $\mathcal{P}(X) \leq 0$, $\rho(X) \leq 0$ and X has a positive probability of taking a positive value.

If ρ^c assigns the value $c > 0$ to any random variable which takes negative values with positive probability, then a classical arbitrage is equivalent to a ρ^c-arbitrage. This justifies the name ρ-arbitrage.

Theorem ([AB19b])

Let ρ be a coherent risk-measure. If a coherent market contains a ρ-arbitrage X then for any random variable Y of finite expectation

$$\lim_{\lambda \to \infty} \mathbb{E}(\tilde{u}(Y + \lambda X)) = \infty \quad (1)$$

$$\mathcal{P}(Y + \lambda X) \leq \mathcal{P}(Y) \quad (2)$$

$$\rho(Y + \lambda X) \leq \rho(Y). \quad (3)$$

i.e. Given a financial positive X, you may add a multiple of Y to obtain a new position that meets your constraints with arbitrarily high $\mathbb{E}(\tilde{u})$.

If risk free assets in this market have a finite price, then the constraint

$$\mathcal{A}^{\rho,\alpha} := \{Y \mid \rho(Y) \leq \alpha\}$$

is ineffective for all α. Conversely if $\mathcal{A}^{\rho,\alpha}$ is ineffective then the market admits a ρ-arbitrage.

The previous theorem shows that ρ-arbitrage opportunities are good for the trader. The next theorem shows they are bad the risk-manager.

Theorem ([AB19b])

Let ρ be a coherent risk-measure. Let u_R be any concave increasing utility function satisfying

$$\lim_{\lambda \to \infty} \frac{u_R(-\lambda)}{\lambda} = -\infty, \tag{4}$$

If X is a ρ-arbitrage and not a true arbitrage, and if both $\mathbb{E}(|X|)$ and $\mathbb{E}(u_R(-\beta Y))$ are finite for some $\beta > 0$ then

$$\lim_{\lambda \to \infty} \mathbb{E}(u_R(Y + \lambda X)) = -\infty. \tag{5}$$

To detect whether a ρ-arbitrage, solve the convex optimization problem

$$\begin{array}{ll} \underset{X \in L^0(\Omega;\mathbb{R})}{\text{minimize}} & \operatorname{ess\,inf} -X \\ \text{subject to} & \mathcal{P}(X) \leq 0 \\ \text{and} & \rho(X) \leq 0. \end{array}$$

The convexity of the problem ensures that this is easy to solve in practice (unlike the S-shaped utility maximization problems we started with).

Theorem ([AB19b])

In complete markets an ES_α-arbitrage exists if and only if

$$\mathbb{P}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \geq \frac{1}{\alpha}\right) > 0.$$

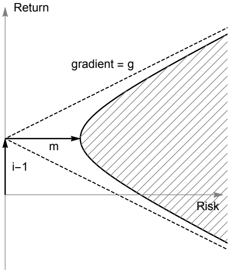
In complete markets it is easy to out-manouevre ES_α limits by financial engineering.

Definition

Two markets $(\Omega, \mathbb{P}, \mathcal{P})$ and $(\Omega, \mathbb{P}, \mathcal{P}')$ are isomorphic if there is a probability space isomorphism which preserves the price functions.

Theorem ([Arm18b])

Markowitz markets are classified up to isomorphism by their efficient frontiers.



Theorem ([AB19b])

A Markowitz market admits an ES_α-arbitrage for $\alpha < 0.5$ if and only if either $i < 0$ or $g > \Phi^{-1}(\alpha)$.

In realistic Markowitz markets there will not be any ES_α-arbitrage at the level $\alpha = 0.01$.

Exchange traded options

The market of exchange traded options on the S&P 500 is nearly complete.

On a given day we downloaded the end of day bid and ask prices at all available strikes.

We calibrated a GARCH-(1,1) model to the historic index data. We then used a Monte Carlo simulation of this model to obtain a plausible, discrete $\mathbb{P}$ measure model for that day.

Letting $\mathbf{p}$ be the vector of prices and $\mathbf{x}$ the vector of portfolio weights we then solved:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ES_\alpha(\mathbf{x}) \\ \text{subject to} \quad & \\ \text{cost constraint} \quad & \mathbf{p} \cdot \mathbf{x} \leq 0, \\ \text{quantity constraints} \quad & 0 \leq x_i \leq 1 \quad (1 \leq i \leq N_I). \end{aligned} \tag{6}$$

The minimizing portfolio will be an ES_α arbitrage portfolio if and only if an ES_α arbitrage portfolio exists.

Rockafellar and Uryasev [RU$^+$00] showed how expected shortfall optimization problems with a discrete probability measure can be solved by performing linear programming.

Date	GARCH$(1,1)$ run 1	GARCH$(1,1)$ run 2	Mixture
10 Feb	$< 0.01\%$	0.19%	$< 0.01\%$
11 Feb	0.29%	$< 0.01\%$	$< 0.01\%$
12 Feb	0.33%	0.39%	$< 0.01\%$
13 Feb	$< 0.01\%$	$< 0.01\%$	$< 0.01\%$
14 Feb	0.26%	$< 0.30\%$	$< 0.01\%$

The table above shows the minimum α values for which an ES$_\alpha$ arbitrage existed. Our conclusion is that ES$_\alpha$-arbitrage opportunities do not occur every day, but did occur on 13 Feb.

We also calibrated a mixture model to the data, in which case ES$_\alpha$-arbitrage existed for low α every day.

Conclusions

- The coherence of expected shortfall means that ρ-arbitrage opportunities can be exploited.
- Whether ρ-arbitrage exists depends upon
  - How complete the market is.
  - How large a discrepancy exists between the $\mathbb{P}$ and $\mathbb{Q}$ measures.
- The problem can be eliminated by using convex rather than coherent risk measures.

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