Optimizing S-shaped utility and risk management: ineffectiveness of VaR and ES constraints

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Agenda

Motivation

- Standard XVA methodology uses risk-neutral pricing, assumes all risk can be hedged. This is a questionable assumption, e.g. for KVA (see [7]).
- We wish to consider indifference pricing. Indifference price P^I of liability L solves:

$$\sup_{\text{strategies}} E(u(X)) = \sup_{\text{strategies}} E(u(X + L + P^I))$$

  where X is the payoff achieved by following a strategy.
- We will focus on the question of how to compute

$$\sup_{\text{strategies}} E(u(X))$$

- Challenge: shareholders have limited liability so their utility function will not be concave.
- Our results will have implications that go beyond XVA calculation.

S-shaped utility I

In [14], Kahneman & Tversky observed that individuals appear to have preferences governed by an S-shaped utility function u.

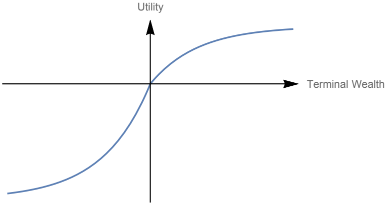
(i) u is increasing

(ii) strictly convex on the left

(iii) strictly concave on the right

(iv) non-differentiable at the origin

(v) asymmetrical: negative events are considered worse than positive events are considered good.



S-shaped utility II

An increasing function $u : \mathbb{R} \longrightarrow \mathbb{R}$ (to be thought of as a utility function) is said to be *"risk-seeking in the left tail"* if there exist constants $N \leq 0$, $\eta \in (0, 1)$ and $c > 0$ such that:

$$u(x) > -c|x|^\eta \qquad \forall x \leq N. \tag{1}$$

Similarly u is said to be *"risk-averse in the right tail"* if there exists $N \geq 0$, $\eta \in (0, 1)$ and $c > 0$ such that

$$u(x) < c|x|^\eta \qquad \forall x \geq N. \tag{2}$$

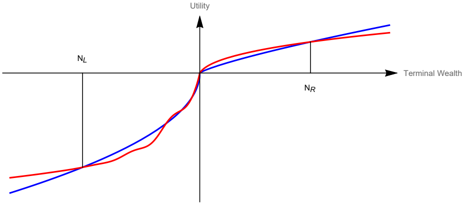
The standard pictures of "S-shaped" utility functions in the literature appear to have these properties. Furthermore the S-shaped utility functions that would arise due to a limited liability would be bounded below and so would certainly be risk-seeking in the left tail.

S-shaped utility III

We give a formal definition of S-shaped for the purposes of this work.

u is "*S-shaped*" if

1. u is increasing
2. $u(x) \leq 0$ for $x \leq 0$
3. $u(x) \geq 0$ for $x \geq 0$
4. $u(x)$ concave for $x \geq 0$.
5. u risk-seeking in the left tail.
6. u risk-averse in the right tail.

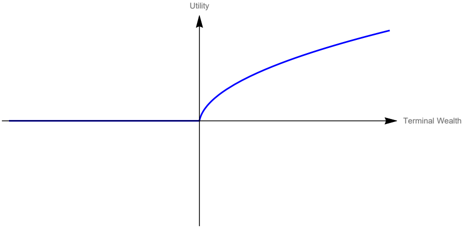


S-shaped utility IV

Example

Limited liability:

$$u^+(x) = \max\{u(x), 0\}$$



Solving optimization problems for expected concave utility typically yields convex optimization problems.
Our optimal utility problem will be non-convex, but we can still solve it.

Law invariant portfolio optimization

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ be a positive random variable with $\int_\Omega \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \, \mathrm{d}\mathbb{P}(\omega) = 1$.
We will use this model to represent a complete financial market as follows:

(i) We assume there is a fixed deterministic risk free interest rate r.

(ii) Given a random variable f, one can purchase a derivative security with payoff at maturity T (simple claim) given by $f(\omega)$ for the price

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT} f] := \int_\Omega e^{-rT} f(\omega) \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega) \, \mathrm{d}\mathbb{P} \tag{3}$$

assuming that this integral exists.

Law invariant portfolio optimization

- *Investor preferences are law-invariant.* i.e. The investor's preferences are encoded by some function

$$v : \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$$

  where $\mathcal{M}_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$, so that an investor will prefer a security with payoff f over a security with payoff g iff $v(F_f) > v(F_g)$ (F is the cdf).
- *Risk constraints are law-invarant.* i.e. whether a payoff f is acceptable depends only on F_f. Examples include VaR and CVaR constraints.
- *We have a cost constraint.*

The optimization problem

In summary, our investor wishes to solve the optimization problem:

$$
\begin{aligned}
&\sup_{f \in L^0(\Omega,\mathbb{P})} && v(F_f) \\
&\text{subject to a price constraint} && \int_\Omega e^{-rT} f(\omega) \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega)\,\mathrm{d}\mathbb{P}(\omega) \leq C \\
&\text{risk management constraints} && F_f \in \mathcal{A} \subseteq \mathcal{M}_1(\mathbb{R}).
\end{aligned}
\tag{4}
$$

Rearrangement Theorem

Theorem (Rearrangement)

Assume Ω is non-atomic. Then there exists a standard uniformly distributed random variable U such that:

(i) $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = (1 - F_{\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}})^{-1} \circ U$ *almost surely.*

(ii) *If f satisfies the price and risk management constraints of our problem then*

$$\varphi(U) = F_f^{-1} \circ U$$

also satisfies the constraints of our problem and is equal to f in distribution, and hence has the same objective value as f.

The implication of this is that we can simplify our market model so that we are simply betting on the final value of a single uniform variable U. We may assume that the payoff of our investment is an increasing function of U, while $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is a decreasing function of U.

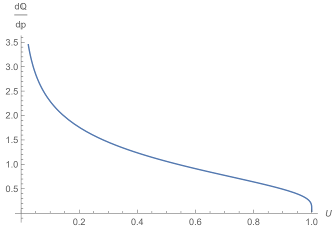
Example: The Black-Scholes case

Consider derivatives in a Black Scholes market with (deterministic) r, we can write the log-stock price as

$$s_T = s_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}N^{-1}(U)$$

The $\mathbb{P}$ and $\mathbb{Q}$ measure densities are

$$p\{q\}^{\mathsf{BS}}(s_T) = \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(-\frac{(s_T - (s_0 + (\mu\{r\} - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T}\right).$$



Equivalent problem

Let $q(U) = \frac{d\mathbb{Q}}{d\mathbb{P}}(U)$ then we can reduce our optimization problem to solving

$$\sup_{\varphi:[0,1]\to\mathbb{R},\varphi \text{ increasing}} \quad \mathcal{F}(\varphi) := \int_0^1 u(\varphi(x))\, \mathrm{d}x \tag{5}$$

$$\text{subject to the price constraint} \quad \int_0^1 \varphi(x) q(x)\, \mathrm{d}x \leq C \tag{6}$$

$$\text{and risk management constraints} \quad F_\varphi \in \mathcal{A} \subseteq \mathcal{M}_1(\mathbb{R}). \tag{7}$$

- The only feature of the market that is relevant is the decreasing function $q(U)$.
- Intuition: $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is a measure of how good value you think a bet is compared to the market. You should place your bets on events where $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is high. Nothing else matters.

Example: expected shortfall

Consider the optimization problem with expected shortfall constraints

$$\sup_{\varphi:[0,1]\to\mathbb{R},\varphi \text{ increasing}} \quad \mathcal{F}(\varphi) := \quad \int_0^1 u(\varphi(x))\,\mathrm{d}x \tag{8}$$

$$\text{subject to the price constraint} \quad \int_0^1 \varphi(x)q(x)\,\mathrm{d}x \leq C \tag{9}$$

$$\text{and the expected shortfall constraint} \quad \frac{1}{p}\int_0^p \varphi(x)\,\mathrm{d}x \geq L. \tag{10}$$

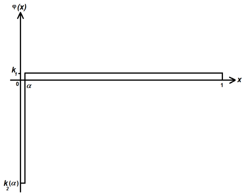
Theorem

If u is risk seeking in the tail and $q(x)$ is essentially unbounded, then the supremum of this problem is equal to $\sup u$. i.e. investors with S-shaped utility are untroubled by expected shortfall constraints.

Sketch proof

Consider the payoff

$$\phi(x) = \begin{cases} k_2 & \text{when } x < \alpha \\ k_1 & \text{otherwise} \end{cases}$$



Since $q(U) \to \infty$ as $U \to 0$, the market contains events of arbitrarily good value:

- The price constraint requires roughly $k_2 \lesssim -\frac{c_1}{\alpha q(\alpha)} k_1$.
- The ES constraint requires roughly $k_2 \gtrsim -\frac{c'}{\alpha} k_1$.

By taking α small enough we can find k_2 meeting our constraints whatever we choose for k_1. By the ES constraint

$$E(u) = \alpha u(k_1) + (1-\alpha)u(k_2) \gtrsim -\alpha \left(\frac{c'}{\alpha} k_1 \right)^{\eta} + (1-\alpha)u(k_1)$$

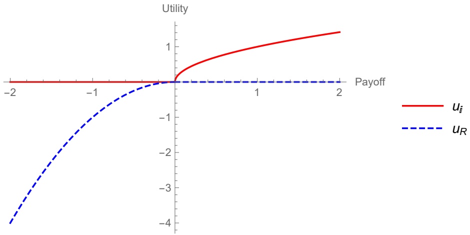
$$\to u(k_1) \quad \text{as } \alpha \to 0.$$

Implications

- In the Black–Scholes model with $\mu \neq r$, an investor with S-shaped utility subject to just Expected Shortfall constraints will be unconcerned by these constrains.
- In the Black–Scholes model $\mu \neq r$, an investor with S-shaped utility subject to just Value at Risk constraints will be unconcerned by these constrains.
- In fact, in any reasonable complete market model with non-zero market price of risk we expect that Expected Shortfall constraints and Value at Risk constraints will be ineffective.
- In general indifference prices cannot be defined for an investor with S-shaped utility subject only to price, expected shortfall constraints and value at risk constraints.

Optimization with limited liability & utility constraints

We now specialize to the case of an investor with limited liability u_I & We suppose a regulator is indifferent if portfolio payoff is positive, and imposes risk constraint with 2nd utility u_R on negative payoff part.



Risk constraint is

$$\mathbb{E}(u_R) \geq L$$

Solving the optimization problem

We know the optimal payoff function ϕ is increasing, so it must be negative for values less than some $p \in [0,1]$ and positive for values greater than p.

Given $p \in [0,1]$, define $C_1(p)$ and $V(p)$ as

$$C_1(p) = \inf_{f_1:[0,p]\to(-\infty,0),\ \text{with } f_1 \text{ increasing}} \int_0^p f_1(x)q(x)\mathrm{d}x \quad \text{subject to} \quad \int_0^p u_R(f_1(x))\,\mathrm{d}x \geq L. \tag{11}$$

$$V(p) = \sup_{f_2:[p,1]\to[0,\infty),\ \text{with } f_2 \text{ increasing}} \int_p^1 u_I(f_2(x))\mathrm{d}x \quad \text{subject to} \quad \int_p^1 f_2(x)q(x)\mathrm{d}x \leq e^{rT}C - C_1(p) \tag{12}$$

Theorem

The supremum of the optimization problem for an investor with limited liability subject to a concave utility constraint on the loss is $\sup_{p \in [0,1]} V(p)$*.*

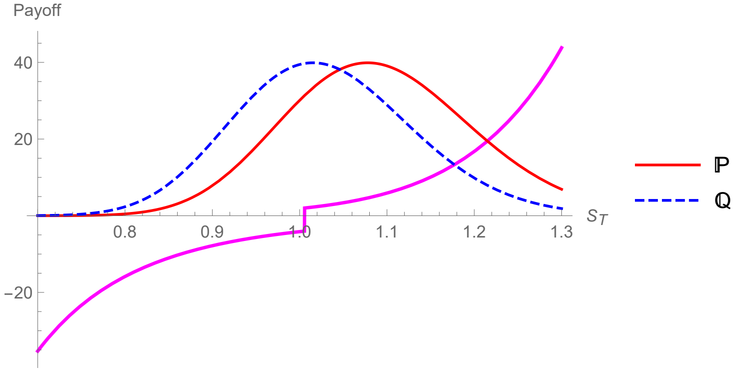
- The value of the theorem comes from the fact that the optimization problems to compute $C_1(p)$ and $V(p)$ are convex problems and so are easy to solve by standard techniques.
- We may then compute $\sup_{p \in [0,1]} V(p)$ by line search.
- Unlike the case of expected shortfall, these utility constraints are typically binding.

Implications

- Utility constraints are typically effective in constraining investors with S-shaped utility.
- In typical cases, for example in the Black-Scholes model when u_I is unbounded, an investor with S-shaped utility will choose investments with infinitely bad u_R utility if they are not subject to u_R constraints but only expected shortfall constraints.
- Indifference prices can be defined and calculated if we consider investors with limited liability under utility constraints.

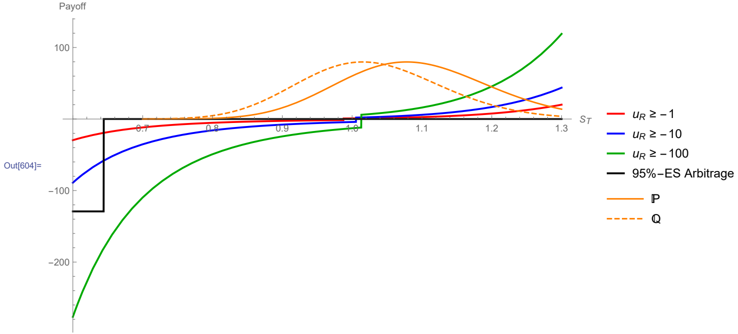
Example: the Black–Scholes case

The optimal payoff function is plotted against the stock price for the Black-Scholes model. We also show the $\mathbb{P}$ and $\mathbb{Q}$ measure density functions.



Example: the Black–Scholes case

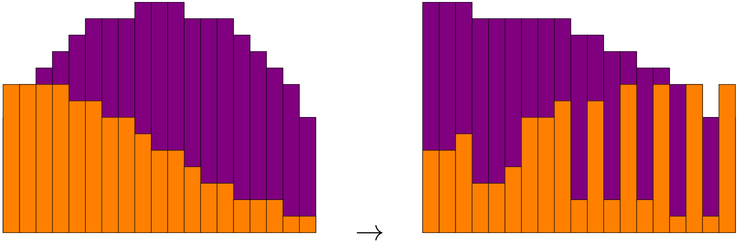
This figure shows how the strategies vary as the risk limit is changed.



In black we see a portfolio with positive payoff, negative expected shortfall and negative price. An investor with limited liability could purchase an arbitrary quantity of this asset to achieve any desired utility.

Proof of Rearrangement Theorem I

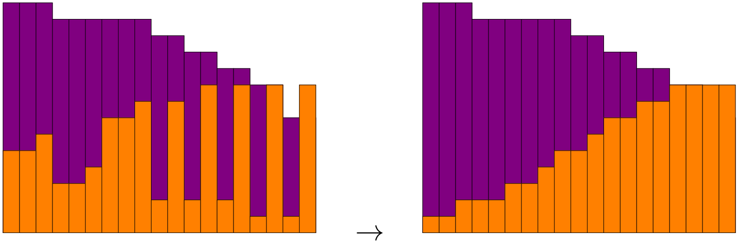
Consider a finite probability space where each atomic event has the same $\mathbb{P}$ measure. We show a graph of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in purple and the payoff f of some option in orange (LHS).



The x-axis corresponds to the different events. The choice of the x-axis completely arbitrary, so we might as well choose our plot so that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is decreasing (RHS).

Proof of Rearrangement Theorem II

Now swap the order of the payoff columns so that we get taller bars on the right. Swapping bars doesn't change the $\mathbb{P}$ distribution of the payoff, but clearly lowers the price.



We have a cheaper portfolio with identical $\mathbb{P}$ distribution. Our objective and risk-management constraints are law invariant, so the payoff on the right meets all our constraints.

Proof of Rearrangement Theorem III

To generalize this to the continuous case we need to define what we mean by rearrangement.

Definition 1

Given random variables $X, f \in L^0(\Omega, \mathbb{R})$ with X having a continuous distribution we define the *X-rearrangement of f*, denoted f^X by:

$$f^X(\omega) = F_f^{-1}(\mathbb{P}(X \leq X(\omega))) = F_f^{-1}(F_X(X(\omega))).$$

We know that $X = F_X^{-1}(U)$ for some uniformly distributed U

- f^X is equal to f in distribution
- f^X depends on U alone
- f^X is an increasing function of U

Proof of Rearrangement Theorem IV

To prove our theorem we will take an arbitrary payoff f and replace it with the rearranged payoff $f^{-\frac{d\mathbb{Q}}{d\mathbb{P}}}$.
(We assume for simplicity that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ has a continuous distribution. A minor technical lemma is needed to prove the general case.)
Our rearrangement theorem will then follow from:

Lemma 2

If $f, g \in L^0(\Omega; \mathbb{R})$ *and:*

(i) $\int f g \, d\mathbb{P} > -\infty$;

(ii) $g \geq 0$;

(iii) $\int_\Omega g \, d\mathbb{P}$ *exists;*

(iv) X *has a continuous distribution;*

then

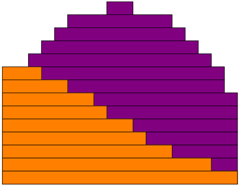
$$-\infty < \int_\Omega f g \, d\mathbb{P} \leq \int_\Omega f^X g^X \, d\mathbb{P} \leq \infty.$$

Proof of Rearrangement Theorem V

For simplicity assume that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and payoff are defined on $[0,1]$, with f and g also taking values in $[0,1]$. We consider the layer cake representations like

$$f(U) = \int_0^1 1_{f(U)\geq \ell} \mathrm{d}\ell, \qquad g(U) = \int_0^1 1_{g(U)\geq \ell} \mathrm{d}\ell$$

this is depicted below (LHS).



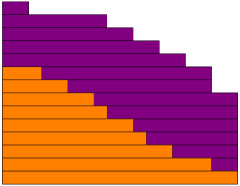
Proof of Rearrangement Theorem VI

$$\begin{aligned}\int f(U)g(U)\mathrm{d}U &= \int \int_0^1 \int_0^1 1_{f(U)\geq \ell} 1_{g(U)\geq k} \mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}U \\ &= \int \int_0^1 \int_0^1 1_{f(U)\geq \ell \text{ and } g(U)\geq k}\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}U\end{aligned}$$

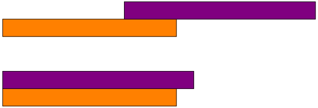
So the integral aggregates the intersections of the layers.

Proof of Rearrangement Theorem VII

The layer cake representations for f^U and g^U look as shown



This increases the integral of the product is because this rearrangement increases the amount any layers intersect as shown below



Hence

$$\int f^U(U)g^U(U)\mathrm{d}U \geq \int f(U)g(U)\mathrm{d}U.$$

Proof of Rearrangement Theorem VIII

This layer cake argument is due to Hardy and Littlewood who used it to prove their inequality on so-called “symmetric decreasing rearrangements”. We use exactly the same idea, but applied to a notion we call X-rearrangement.

Summary

- Indifference pricing provides a pricing methodology that can be used when not all risks can be perfectly hedged.
- The key to indifference pricing is computing optimal investment strategies.
- Market players may have limited liability and hence non-concave utility functions, nevertheless we can apply rearrangement to obtain tractable optimization problems.
- Expected shortfall and value at risk constraints typically **do not** constrain investors with S-shaped utility functions.
- Utility constraints typically **do** constrain investors with S-shaped utility functions.
- Future Research 1: Investigate S-shaped utility optimization in incomplete market models.
- Future Research 2: Apply these techniques to indifference pricing of XVA type liabilities. See [7] for initial results in this area.

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