Coordinate Free Stochastic Geometry with Jets

Drawing SDEs

John Armstrong (KCL), Damiano Brigo (Imperial)

Collegio Carlo Alberto
Torino

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Geometry of SDEs

Motivation:

- SDEs = Analysis + Geometry
- Itô: Brownian motion on a manifold
- How do you draw an SDE?

Applications:

- Visualisation tools
- Pedagogy
- Elegant reformulation of Itô's lemma
- Geometric interpretation of Fokker-Planck
- Asymptotic properties of SDEs
- Projection of SDEs

Analogy:

- Maxwell's equations easier in terms of differential forms
- Drawing differential forms is illuminating

Existing work

Not the first people to consider coordinate free stochastic differential geometry

- Coordinate free operator formalism for diffusions
- Coordinate free approach best on Stratonovich calculus
- Emery's approach based on the Schwarz-Morphism

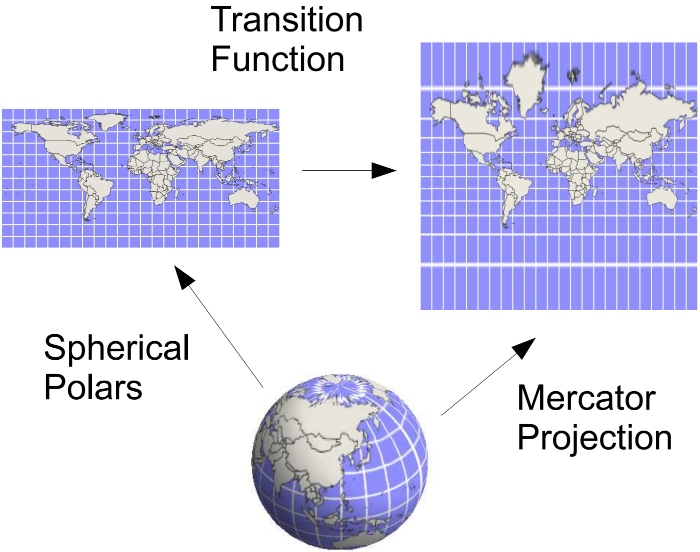
What's new?

- Very straightforward
- Based on Itô calculus so has good probabilistic properties
- Simple intrepretation in terms of numerical schemes
- Pictures!

Outline

- Differential geometry 101
  - Manifolds
  - Different perspectives on vectors
  - “Coordinate free” geometry
- Drawing SDEs (1)
- Itô’s Lemma
- Differential operators
- Drawing SDEs (2)
- Stratonovich calculus
- Drawing SDEs (3)

Manifolds


Transition
Function
Spherical
Polars
Mercator
Projection

Manifold Definition

Very informally a manifold is:

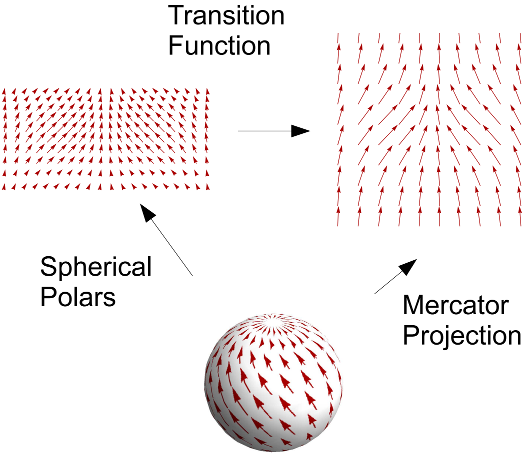
- A set of charts covering the manifold.
- Smooth coordinate change rules from one chart to another

Formally:

- A paracompact Hausdorff topological space M
- A family of *charts* $\phi_i : U_i \to \mathbb{R}^n$. Each chart is a homeomorphism defined on an open set U.
- The *transition functions* $\phi_i \circ \phi_j^{-1}$ are smooth on their domain of definition.
- $\cup U_i = M$.

Example: 2 charts needed for sphere
Example: London

Vector Fields



A vector field can be defined as an equivalence class of pairs (chart, vector field on $\mathbb{R}^n$)

Vector fields: coordinate definition

- Vector field is equivalence class (ϕ, X) where ϕ is a chart and X is the vector field on $\mathbb{R}^r$.
- We must choose the equivalence class so that the solutions of one ODE are mapped to the solutions of the other ODE by the transition functions.
- So by the chain rule, the correct definition is:

$$(\phi_1, X) \sim (\phi_2, Y)$$

if

$$\begin{aligned} X^i &= \sum_j \frac{\partial \tau^i}{\partial x^j} Y^j \\ &= (\partial_j \tau^i) Y^j \end{aligned}$$

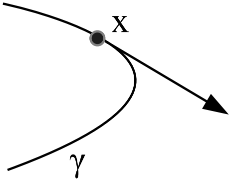
where we're using the Einstein summation convention.

Vector: 1-jet definition

- A k-jet of a smooth path is defined as an equivalence class of paths with the same Taylor series up to given order.
- Given two paths $\gamma_1, \gamma_2 : \mathbb{R} \to M$ satisfying $\gamma_i(0) = x$ we say

$$j_k(\gamma_1) = j_k(\gamma_2)$$

  if γ_1 and γ_2 have the same Taylor series expansion (in any chart) up to order k.
- A vector is a 1-jet of a path



Vector: Operator definition

Derivation:

- A function $D : C^\infty(x) \to \mathbb{R}$ satisfying:
  - $D(a\,f + b\,g) = a\,D(f) + b\,D(g)$ when $a,\, b \in \mathbb{R}$
  - $D(fg) = f, D(g) + g\,D(f)$ when $f,\, g \in C^\infty(x)$
- where $C^\infty(x)$ is set of *germs* of smooth functions
- Germ at x: $f \sim g$ if $f(y) = g(y)$ for all y in some neighbourhood $U \ni x$

Example

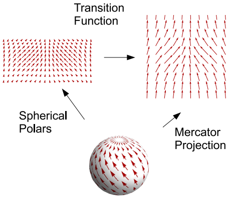
1. $\frac{\partial}{\partial x}$ is a derivation.
2. Given a vector $V \in \mathbb{R}^n$

$$V(f) := \lim_{h \to 0} \frac{f(x + hV) - f(x)}{h}$$

is a derivation on $\mathbb{R}^n$.

Vectors: Summary

1. First order ODEs on a manifold.
2. Vector fields defined as equivalence classes under change of coordinates
3. A smoothly varying choice of a 1-jet at each point of a manifold
4. Linear operators on germs satisfying the Leibniz rule (a.k.a. derivations)



- All of these view points are helpful.
- 3 is the most "visual". 3 + 4 are "coordinate free".

Euler Scheme

- All being well in the limit the Euler scheme

$$\delta X_t = a(X)\,\delta t + b(X)\,\delta W_t$$

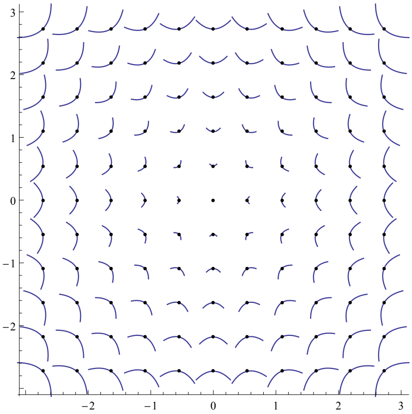
  converges to a solution of the SDE

$$\mathrm{d}X_t = a(X)\,\mathrm{d}t + b(X)\,\mathrm{d}W_t$$

- d, δ, $+$ imply vector space structure
- This is highly coordinate dependent
- (Analysis + Geometry)

Curved Scheme

Let γ_x be a choice of curve at each point x of M. $\gamma_x(0) = x$.

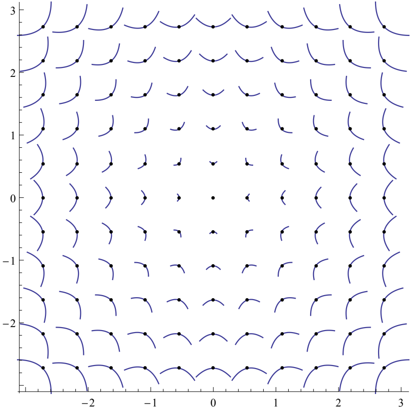


Consider the scheme

$$X_{t+\delta t} = \gamma_{X_t}(\delta W_t) \qquad X_0$$

Concrete example

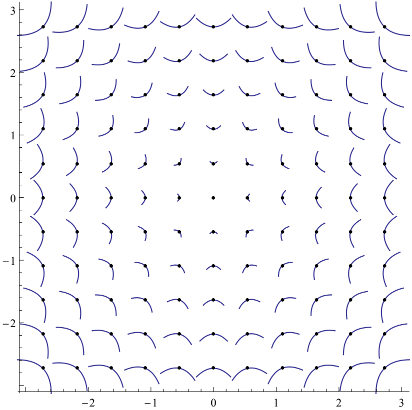
$$\gamma^{E}_{(x_1,x_2)}(t) = (x_1, x_2) + t(-x_2, x_1) + 3t^2(x_1, x_2)$$



- First order term is rotational vector
- Second order term is axial vector

Concrete example

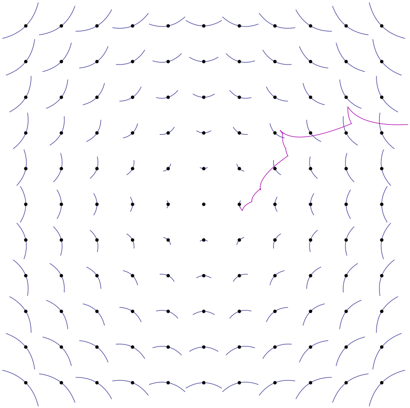
$$\gamma^{E}_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$



- First order term is rotational vector
- Second order term is axial vector

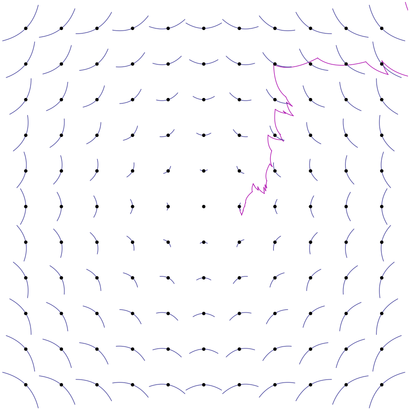
Simulation: Large time step

$$\gamma^{E}_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$



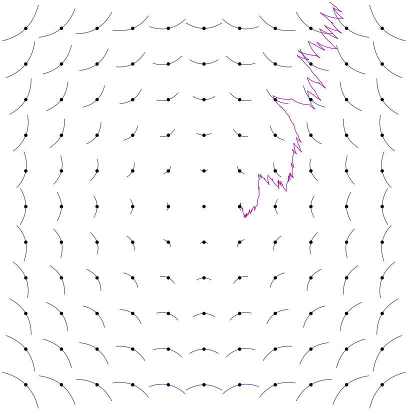
Simulation: Smaller time step

$$\gamma^E_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$



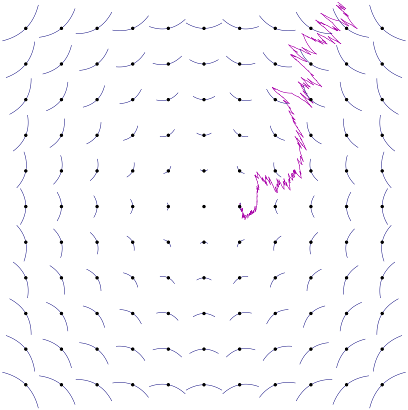
Simulation: Even smaller

$$\gamma^{E}_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$



Simulation: Convergence

$$\gamma^{E}_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$



Formal argument

Write:

$$\gamma_x(s) = x + \gamma_x'(0)s + \frac{1}{2}\gamma_x''(0)s^2 + O(s^3)$$

Then:

$$\begin{aligned} X_{t+\delta t} &= \gamma_t(\delta W_t) \\ &= X_t + \gamma_{X_t}'(0)\delta W_t + \frac{1}{2}\gamma'' X_t(0)(\delta W_t)^2 + O\left((\delta W_t)^3\right) \end{aligned}$$

Rearranging:

$$\delta X_t = X_{t+\delta t} - X_t = \gamma_{X_t}'(0)\delta W_t + \frac{1}{2}\gamma'' X_t(0)(\delta W_t)^2 + O\left((\delta W_t)^3\right)$$

Taking the limit:

$$\begin{aligned} \mathrm{d}X_t &= b(X)\mathrm{d}W_t + a(X)(\mathrm{d}W_t)^2 + O\left((\mathrm{d}W_t)^3\right) \\ &= b(X)\mathrm{d}W_t + a(X)\mathrm{d}t \end{aligned}$$

where

$$\begin{aligned} b(X) &= \gamma_X'(0) \\ a(X) &= \gamma_X''(0)/2 \end{aligned}$$

Comments

- The curved scheme depends only on the 2-jet of the curve
- SDEs driven by 1-d Brownian motion are determined by 2-jets of curves
- The first derivative determines the volatility term
- The second derivative determines the drift term

ODEs correspond to 1-jets of curves
SDEs correspond to 2-jets of curves

- Rigorous proof of convergence of quadratic scheme can be proved using standard results on Euler scheme

$$\begin{aligned} \mathrm{d}X_t &= a(X)\mathrm{d}t + b(X)\mathrm{d}W_t \\ &= a(X)\left(\mathrm{d}(W_t^2) - 2W_t\mathrm{d}(W_t)\right) + b(X)\mathrm{d}W_t \\ &\approx a(X)\left((\delta W_t)^2\right) + b(X)\mathrm{d}W_t \end{aligned}$$

- For general curved schemes some analysis needed.

Itô's lemma

Given a family of curves γ_x we will write:

$$X_t \smile j_2\left(\gamma_x(\mathrm{d}W_t)\right)$$

if X_t is the limit of our scheme.
If

$$X_t \smile j_2\left(\gamma_x(\mathrm{d}W_t)\right)$$

and $f : X \to Y$ then:

$$f(X)_t \smile j_2\left(f \circ \gamma_x(\mathrm{d}W_t)\right)$$

Itô's lemma is simply composition of functions.

Usual formulation

$$X_t \smile j_2\left(\gamma_x(\mathrm{d}W_t)\right)$$

Is equivalent to:

$$\mathrm{d}X_t = a(X)\mathrm{d}t + b(X)\mathrm{d}W_t, \quad a(X) = \frac{1}{2}\gamma_X''(0), \quad b(X) = \gamma_X'(0)$$

We calculate the first two derivatives of $f \circ \gamma_X$:

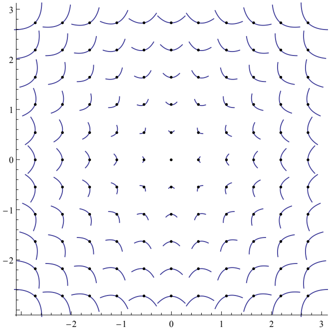
$$(f \circ \gamma_X)'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma_X(t)) \frac{\mathrm{d}\gamma_X}{\mathrm{d}t}$$

$$(f \circ \gamma_X)''(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\gamma_X(t)) \frac{\mathrm{d}\gamma_X^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma_X^j}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma_X(t)) \frac{\mathrm{d}^2\gamma_X}{\mathrm{d}t^2}$$

So $f(X_t) \smile j_2\left(f \circ \gamma_x(\mathrm{d}W_t)\right)$ is equivalent to standard Itô's formula

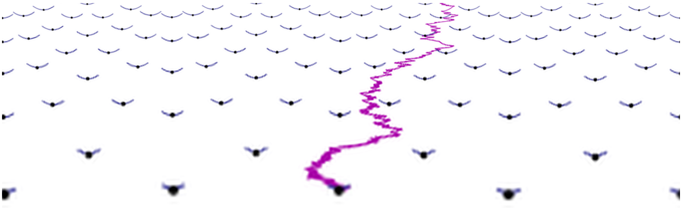
Example

$$\gamma^E_{(x_1,x_2)}(s) = (x_1, x_2) + s(-x_2, x_1) + 3s^2(x_1, x_2)$$

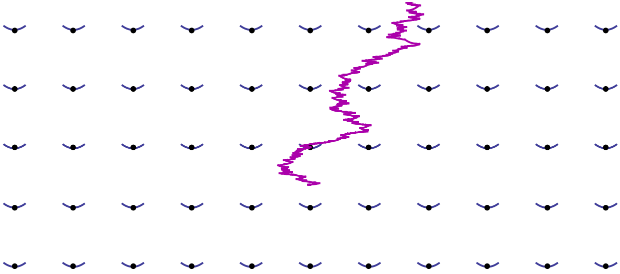


Clearly polar coordinates might be a good idea. So consider the transformation $\phi : \mathbb{R}^2/\{0\} \to [-\pi, \pi] \times \mathbb{R}$ by:

$$\phi(\exp(s)\cos(\theta), \exp(s)\sin(\theta)) = (\theta, s),$$



The process $j_2(\phi \circ \gamma^E)$ plotted using image manipulation software



The process $j_2(\phi \circ \gamma^E)$ plotted by applying Itô's lemma

Drawing SDEs

The following diagram commutes:

$$
\begin{array}{ccc}
\text{SDE for X} & \xrightarrow{\text{Itô's lemma}} & \text{SDE for } f(X) \\
\text{Draw}\big\downarrow & & \text{Draw}\big\downarrow \\
\text{Picture of SDE for X in } \mathbb{R}^n & \xrightarrow{f} & f(\text{Picture of SDE for X})
\end{array}
$$

Outline

- Differential geometry 101 ✓
  - Manifolds ✓
  - Different perspectives on vectors ✓
  - "Coordinate free" geometry ✓
- Drawing SDEs (1) ✓
- Itô's Lemma ✓
- Differential operators
- Drawing SDEs (2)
- Stratonovich calculus
- Drawing SDEs (3)

ODEs vs SDEs

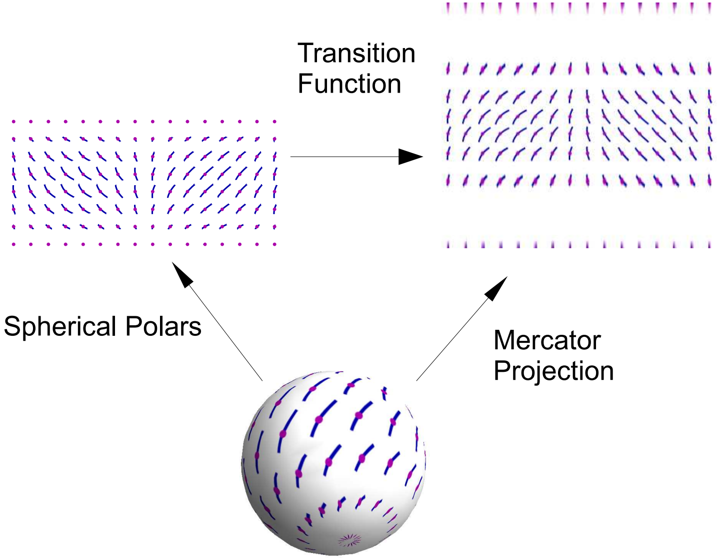
We have the following interpretations of ODEs/Vectors:

1. Vector fields defined as equivalence classes under change of coordinates
2. A smoothly varying choice of a 1-jet at each point of a manifold
3. Linear operators on germs satisfying the Leibniz rule (a.k.a. derivations)

Correspondingly we can understand SDEs as:

1. An equivalence class of coefficients that obey Itô's lemma under change of coordinates
2. A smoothly varying choice of a 2-jet at each point of a manifold
3. Diffusion operators

Local coordinates/2-jets



Operators associated with SDEs

Coordinate free definition of $\mathcal{L}$

- Let γ_x be a field of curves at each point of a manifold. i.e. $j_2(\gamma_x)$ defines an SDE
- Let $f : M \to \mathbb{R}$ be a smooth map
- $f \circ \gamma_x$ defines an SDE on $\mathbb{R}$.
- Let $\mathcal{L}_\gamma f$ be the drift term of this SDE.

$$\mathcal{L}_\gamma f(X) = \frac{1}{2}(f \circ \gamma)''(0)$$

$\mathcal{L}_\gamma f$ determines short time asymptotics of expectation of $f(X)$. If:

$$X_t \smile \gamma(\mathrm{d}W_t)$$

and X_0 is known, then

$$\delta\mathbb{E}(f(X_t)) \approx (\mathcal{L}_\gamma f(X_0))\delta t$$

Generalizing to higher dimensional noise

Coordinate free definition of $\mathcal{L}$

- Let $\gamma_x : \mathbb{R}^k \to M$ at each point x with $\gamma_x(0) = x$. an SDE
- Let $f : M \to \mathbb{R}$ be a smooth map
- $f \circ \gamma_x$ defines an SDE on $\mathbb{R}$.
- Let $\mathcal{L}_\gamma f$ be the drift term of this SDE.

$$\mathcal{L}_\gamma f(X) = \frac{1}{2}\Delta(\gamma \circ f)(0)$$

$\mathcal{L}_\gamma f$ determines short time asymptotics of expectation of $f(X)$. If:

$$X_t \smile \gamma(\mathrm{d}W_t)$$

and X_0 is known, then

$$\delta\mathbb{E}(f(X_t)) \approx (\mathcal{L}_\gamma f(X_0))\delta t$$

Other tensor fields

Recall a vector can be defined as a set of equivalence classes of pairs

$$(v, \phi)$$

where $v \in \mathbb{R}^n$ and ϕ is a chart.

$$(v_1, \phi_1) \sim (v_2, \phi_2) \iff (\phi_1 \circ \phi_2^{-1})_*(v_2) = v_1$$

Note:

$$(\phi_1 \circ \phi_2^{-1})_* \in GL(n, \mathbb{R})$$

Suppose $\tau : GL(n, \mathbb{R}) \to \mathrm{Aut}(V)$ is a group homomorphism. Define associated tensor bundle $\mathbf{V}$ by:

$$(v_1, \phi_1) \sim (v_2, \phi_2) \iff \tau((\phi_1 \circ \phi_2^{-1})_*)(v_2) = v_1$$

where $v \in V$ and ϕ is a chart.

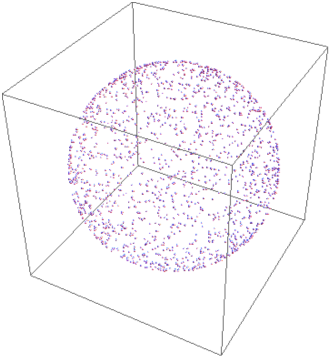
Densities

Definition

A density is a tensor field associated with:

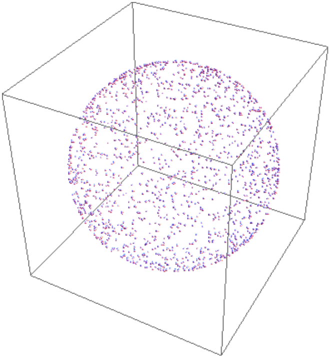
$$\tau(g)v = |\det g|v$$

for $v \in \mathbb{R}$.



Integration

- Probability density functions are densities.
- Integration = Calculation of expectations.
- Integrate f by computing values at each point.



Adjoint operator

- If:

$$X_t \smile \gamma(\mathrm{d}W_t)$$

and X_0 is known, then

$$\delta\mathbb{E}(f(X_t)) \approx (\mathcal{L}_\gamma f(X_0))\delta t$$

- If X_0 is distributed with density ρ then:

$$\frac{\partial}{\partial t}\int f\rho = \int (\mathcal{L}_\gamma f)\rho$$

- So formal adjoint satisfies:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_\gamma^* \rho$$

Remarks

- Functions and densities have different transformation laws
- $\mathcal{L}$ acts on functions and appears e.g. in Feynman–Kac formula
- $\mathcal{L}^*$ acts on densities and appears e.g. in Fokker–Planck equation

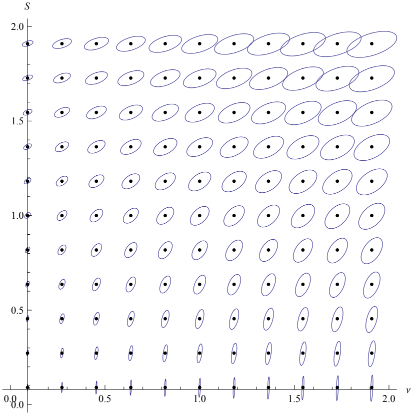
Our treatment of $\mathcal{L}$ has been entirely coordinate free.

Drawing higher dimensional ODEs

- Two jet of map $\gamma_x : \mathbb{R}^k \to M$ at each point x with $\gamma_x(0) = x$.
- $\mathrm{d}W_1^2 = \mathrm{d}W_2^2 = \ldots = \mathrm{d}W_k^2$ so there is some redundancy
- Solutions are the same if 1-jets are the same and $\mathcal{L}$ is the same. i.e. volatility and drift terms match.
- Solutions are weakly equivalent if the paths are rotationally equivalent. Equivalently if $\mathcal{L}$ is the same.

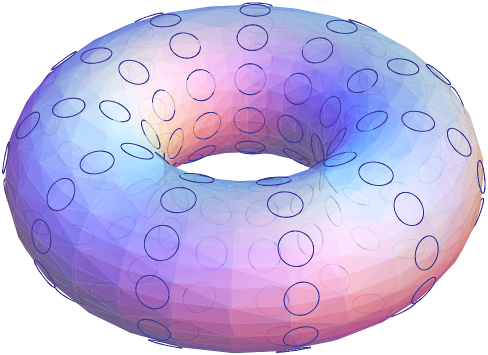
The Heston model

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^1 \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}\left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right)
\end{aligned}
\tag{1}
$$



$\xi = 1,\ \theta = 0.4,\ \kappa = 1,\ \mu = 0.1,\ \rho = 0.5$

Riemannian metrics and Brownian motion



Non degenerate SDE = Riemannian metric + Drift

ODEs vs SDEs

We have four interpretations of ODEs/Vectors:

1. Vector fields defined as equivalence classes under change of coordinates ✓
2. A smoothly varying choice of a 1-jet at each point of a manifold ✓
3. Linear operators on germs satisfying the Leibniz rule (a.k.a. derivations) ✓
4. Infinitesimal diffeomorphisms

Correspondingly we can understand SDEs as:

1. An equivalence class of coefficients that obey Itô's lemma under change of coordinates ✓
2. A smoothly varying choice of a 2-jet at each point of a manifold ✓
3. Diffusion operators ✓
4. Stratonovich drift and volatility vector fields

Flows of vector fields

- Given a vector field X write Φ_X^t for the diffeomorphism at time t associated with the flow.
- Note that defining the flow requires a vector field and not just a vector.

Stratonovich Calculus

- Given two vector fields A and B define a curve at each point by:

$$\gamma_x(s) = \Phi_A^{s^2}(\Phi_B^s(x))$$

- The SDE defined by this field of 2-jets is equivalent to the SDE defined by:

$$\mathrm{d}X_t = A(X)\mathrm{d}t + B(X) \circ \mathrm{d}W_t$$

- In smoothly varying families of n-jets of curves can be described by n vector fields.
- Note that we need the entire vector field for this correspondence.
- Stratonovich and Ito calculus are just alternative coordinate system for the infinite dimensional space of 2-jets of curves.

Drawing 1-d processes

Observations

- Our current diagrams are aesthetically unsatisfying in 1-d.
- The Itô drift is not a coordinate dependent vector because it represents infinitesimal changes of mean.

$$E(f(X)) \neq f(E(X))$$

- On the other hand, for order preserving f:

$$\text{percentile}_p(f(X)) = f(\text{percentile}_p(X))$$

Fan diagram

A fan diagram for a stock price (geometric Brownian motion)

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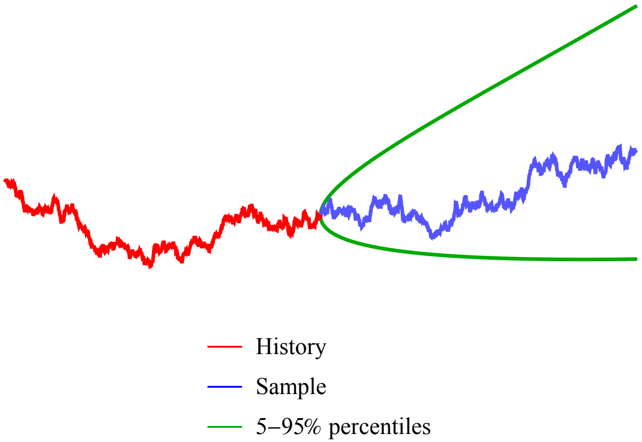

2-jets and fan diagrams

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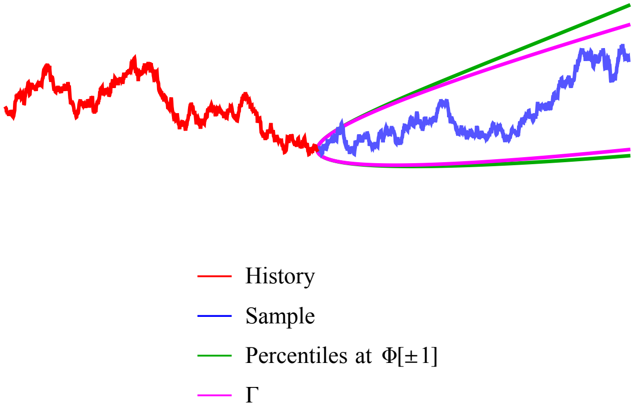
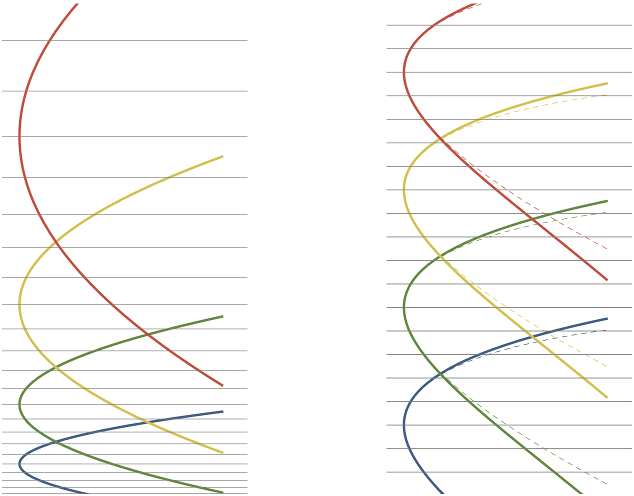
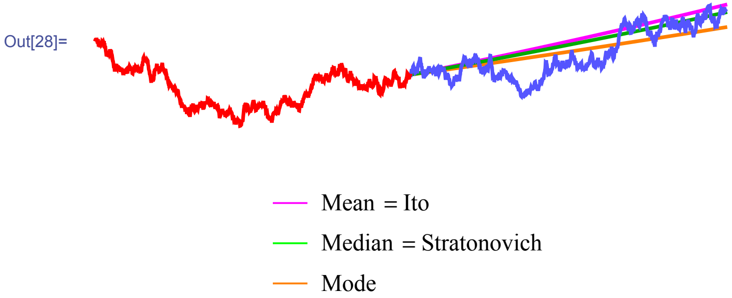


$$\Gamma_x(s) = (s^2, \gamma_x(s))$$

SDE as a fan diagram



Stratonovich calculus and fan diagrams

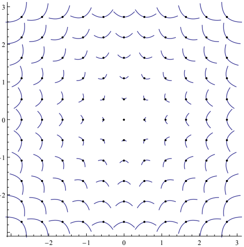


Sketch proof

- All 1-d Riemannian manifolds are isometric
- We cam make a coordinate change such that the volatility is constant (Lamperti transform)
- The SDE is now constant coefficient to second order
- Therefore we can write down first term of asymptotic expansion for solution of Fokker–Planck
- Transform the coordinates back again and read off the result.

This can be generalized since "geodesic normal coordinates" always make a metric constant up to second order.

Summary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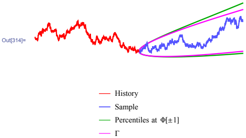
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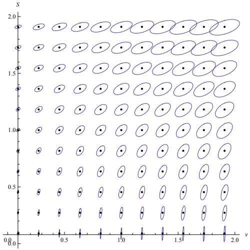
Percentiles at Φ[±1]
Γ


S
2.0
1.5
1.0
0.5
0.0
v
0.5
1.0
1.5
2.0