Stochastic Filtering by Projection

The Example of the Quadratic Sensor

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Motivation

Estimate the current state of a stochastic system from imperfect measurements

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The calculation should be performed online.

Mathematical formulation

$$\begin{aligned} dX_t &= f_t(X_t)\,dt + \sigma_t(X_t)\,dW_t, \quad X_0, \\ dY_t &= b_t(X_t)\,dt + dV_t, \quad Y_0 = 0\ . \end{aligned}$$

- X_t is a process representing the state.
- Y_t is a process representing the measurement.
- W_t and V_t are independent Wiener processes.

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Question

What is the probability distribution for X_t given the values of Y_t up to time t?

The Kushner–Stratonovich equation

With sufficient regularity and bounds, one can show that the probability density p_t satisfies:

$$\mathrm{d}p_t = \mathcal{L}_t^* p_t \mathrm{d}t + p_t[b_t - E_{p_t}\{b_t\}][\mathrm{d}Y_t - E_{p_t}\{b_t\}\mathrm{d}t] \ .$$

where:

- $$\mathcal{L}^* = -f_t \frac{\partial}{\partial x} + \frac{1}{2} a_t \frac{\partial}{\partial x^2}$$
  is the backward diffusion operator
- $a_t^T a = \sigma$ and a is a square root of σ.
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How can we efficiently approximate solutions to the infinite dimensional Kushner–Stratonovich equation?

The geometric idea

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- View the partial differential equation as defining a stochastic vector field.
- Use projection to restrict the vector field to the tangent space.
- Solve the resulting finite dimensional stochastic differential equation.

The linear problem

If:

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One can linearize any filtering problem at each point in time to obtain the *Extended Kalman filter*.

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- A mixture of m Gaussian distributions:

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- The exponential family

$$p_t(x) = \exp(a_0 + a_1 x + a_2 x^2 + \dots a_{2n} x^{2n})$$

  - $a_{2n} < 0$
  - Gives rise to a $2n$ dimensional family.

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- The direct L^2 metric.
  - Works well with mixture families.
  - All integrals that occur can be calculated analytically.

Understanding stochastic differential equations

A stochastic differential equation such as:

$$dX_t = f_t(X_t)\,dt + \sigma_t(X_t)\,dW_t$$

is shorthand for an integral equation such as:

$$X_T = \int_0^T f_t(X_t)\,dt + \int_0^T \sigma_t(X_t)\,dW_t$$

where the right hand integral is defined by the Ito integral:

$$\int_0^T f(t)\,dW_t = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(t_i)(W_{t_{i+1}} - W_{t_i}).$$

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  to get the *Stratonvich integral*. Hence you can define Stratonovich SDE's.
- The difference between the two integrals is an ordinary integral. This allows you to convert between the two formulations.
- Ito SDE's model causality more naturally
- Stratonovich SDE's transform like vector fields.

A recipe for projecting SDE's

To project an SDE onto a submanifold parameterized by $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$:

- Write the SDE as an SDE with vector coefficients in *Stratonovich form*.
- Project all the coefficients onto the tangent space.
- Equate both sides of the projected equations to get an SDE for the θ_i.

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Since Stratonovich SDE's transform like vector fields, this recipe is invariant of the parameterization.

The projected equations

The end result for the case of L^2 projection is:

$$\mathrm{d}\theta^i = \sum_{j=1}^{m} h^{ij} \left\{ \langle p(\theta), \mathcal{L} v_j \rangle \mathrm{d}t - \langle \gamma^0(p(\theta)), v_j \rangle \mathrm{d}t + \langle \gamma^1(p(\theta)), v_j \rangle \circ \mathrm{d}Y \right\}.$$

Where:

- The $v_j = \frac{\partial p}{\partial \theta_j}$ give a basis for the tangent space
- h_{ij} and h^{ij} are the Riemannian metric tensor $\langle v_i, v_j \rangle$.
- $\gamma^0_t(p) := \frac{1}{2} [|b_t|^2 - E_p\{|b_t|^2\}]$
- $\gamma^1_t(p) := [b_t - E_p\{b_t\}]p$
- $\langle \cdot, \cdot \rangle$ is the L^2 inner product.

Note that the inner products and expectations give rise to integrals. We can compute these analytically for the normal mixture family.

Solving the finite system of SDE's

- Approximate the differential equation as a difference equation and solve numerically.
- This is more delicate for stochastic equations than ordinary ones. See Kloeden and Platen. We use the Stratonovich–Heun cheme.
- Note that the resulting difference equation will depend upon the choice of parameterization of the submanifold. Choose coordinates $\phi : \mathbb{R}^n \longrightarrow \mathcal{M}$ so that ϕ is defined on all of $\mathbb{R}^n$.

The quadratic sensor

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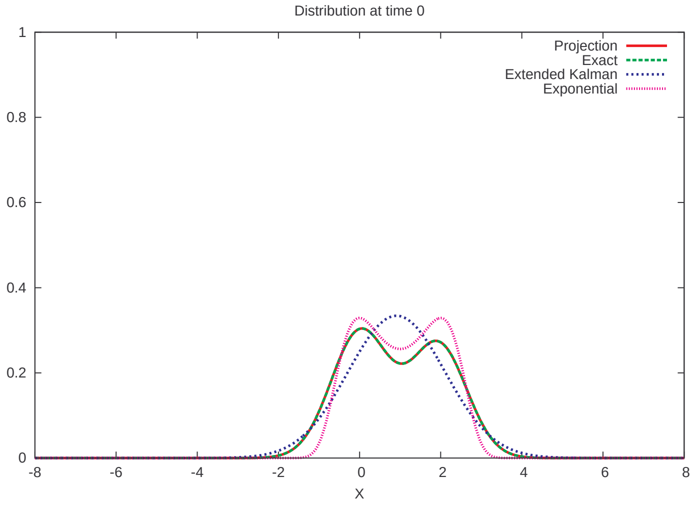
- We do not receive any information on the sign of X.
- We expect that once X has hit the origin, p will be approximately symmetrical.
- We expect a bimodal distribution

Simulation for the Quadratic Sensor



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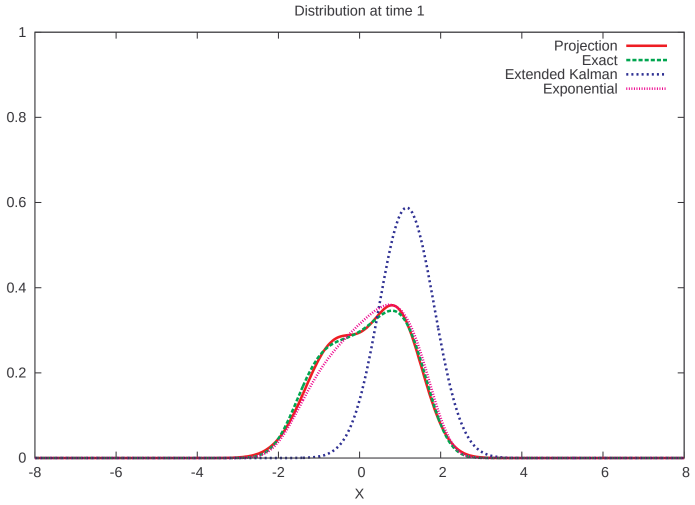

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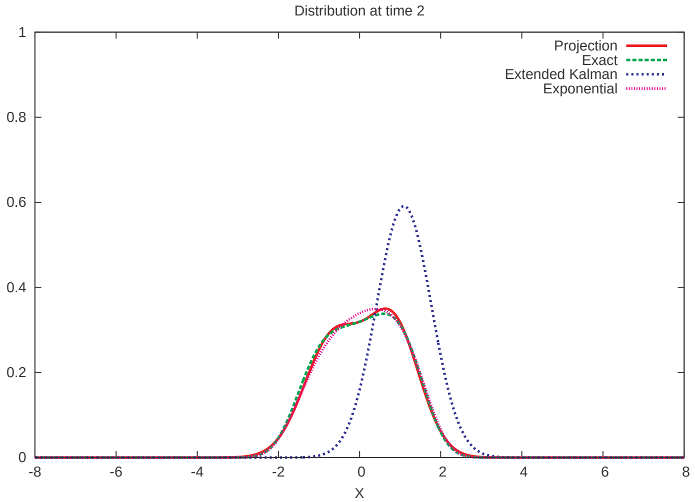

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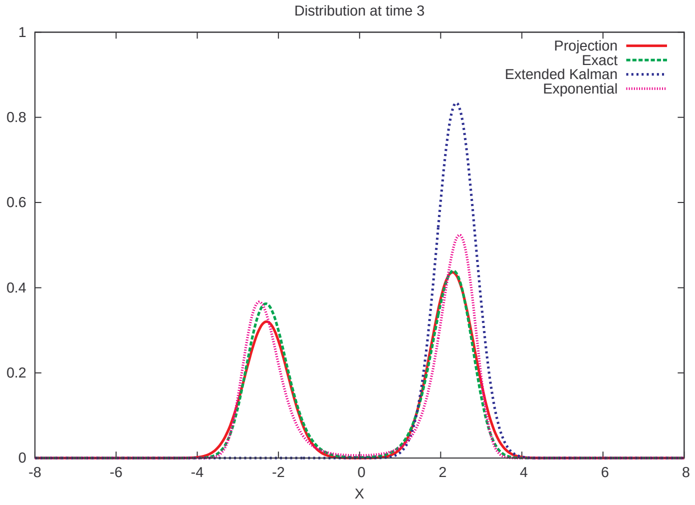

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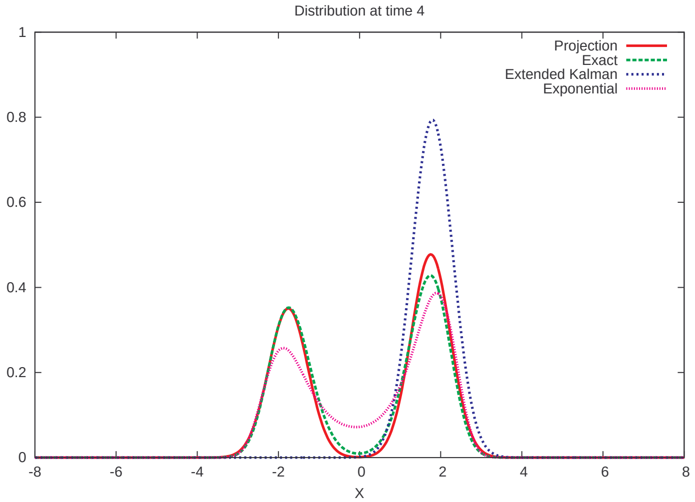

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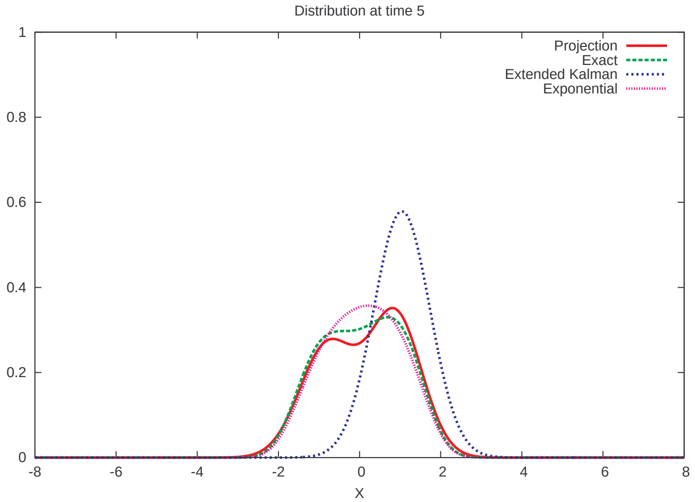

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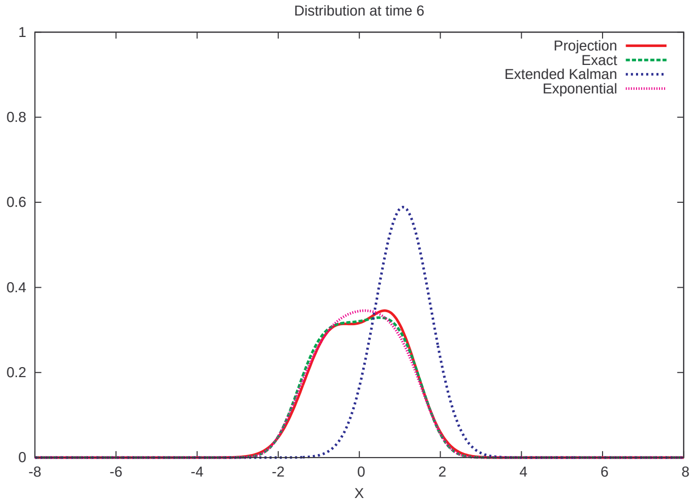

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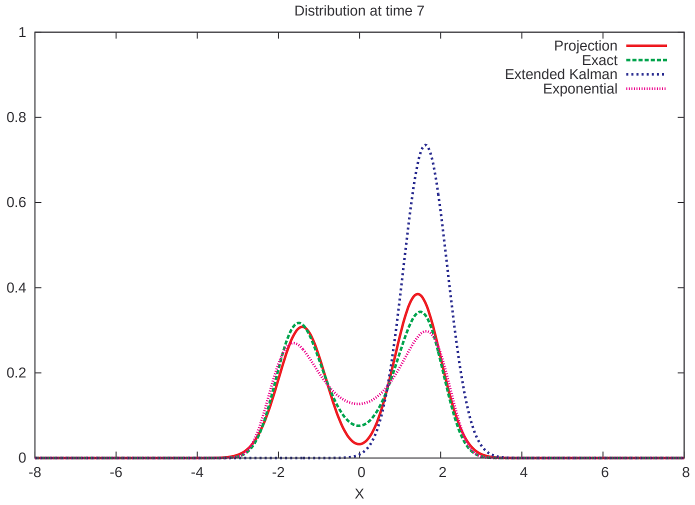

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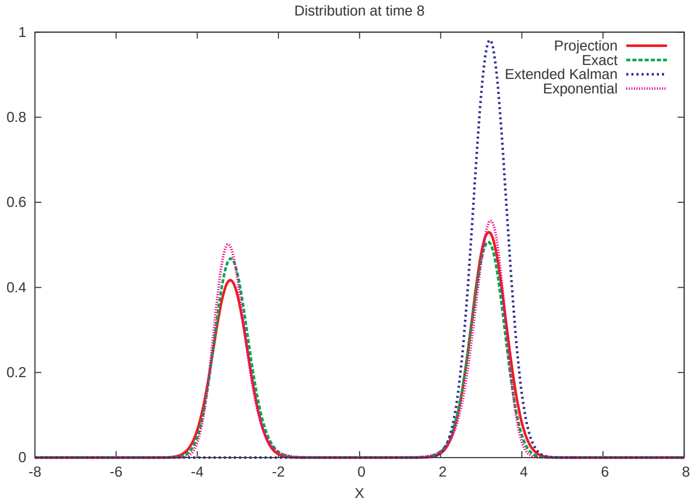

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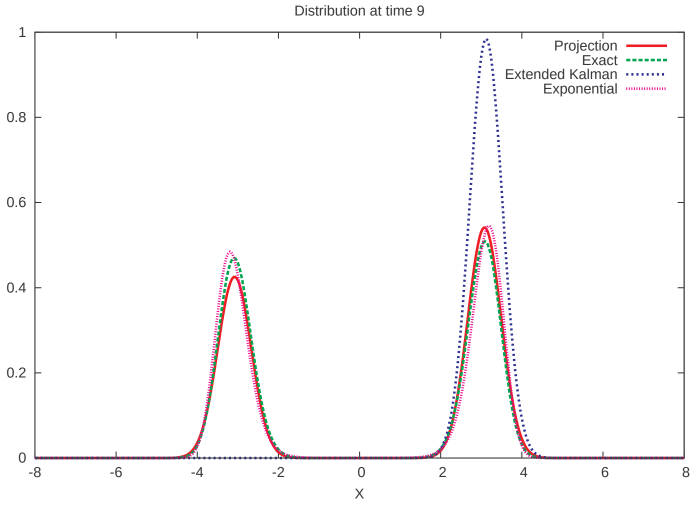

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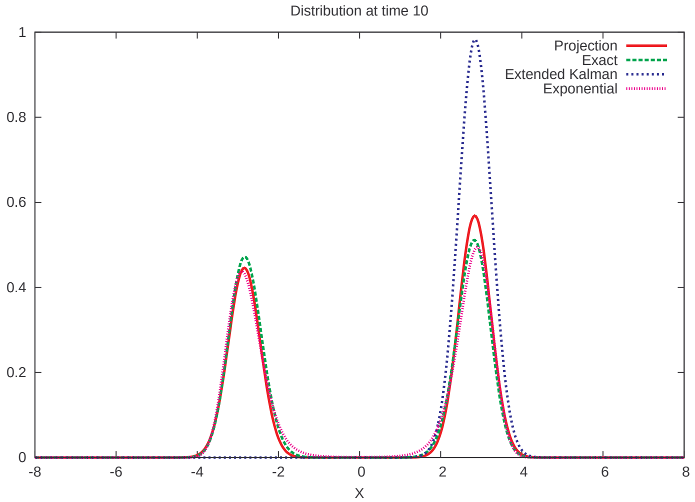
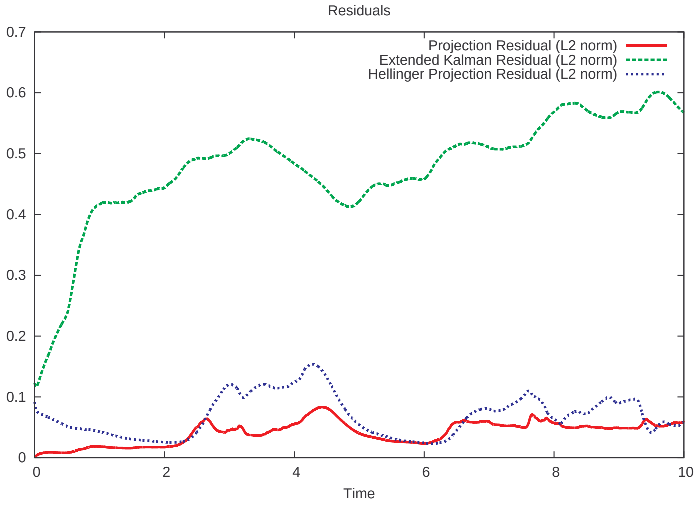
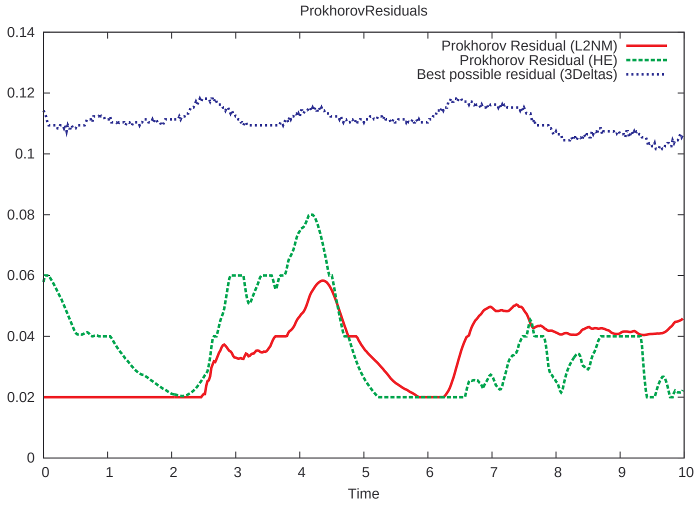


L^2 residuals for the quadratic sensor



Lévy residuals for the quadratic sensor



Conclusions

- Projection methods allow us to approximate the solution to nonlinear problems with surprising accuracy using only low dimensional manifolds.
- This conclusion holds for a variety of projection metrics and manifolds.
- L^2 projection of normal mixtures is particularly promising since all integrals can be computed analytically.