COMPLEX GEOMETRY, RIEMANNIAN GEOMETRY AND THE KÄHLER CONDITION

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Summary:

* $\mathbb{CP}^n$ has a special metric called the Fubini-Study metric
* All complex submanifolds of $\mathbb{CP}^n$ inherit a metric
* These metrics are Kähler
* We know a lot about Kähler manifolds and next to nothing about non-Kähler

COMPLEX AND ALMOST COMPLEX STRUCTURES

A <u>complex manifold</u> is a manifold with charts $U \xrightarrow{z} \mathbb{C}^n \cong \mathbb{R}^{2n}$ with holomorphic transition functions.

Write $z = (z^1, z^2, \dots, z^n)$

then $z^1, \dots, z^n$ are complex functions on M

$\bar{z}^1, \dots, \bar{z}^n$ are also complex valued functions

$dz^1, \dots, dz^n$ are complex valued 1 forms

$d\bar{z}^1, \dots, d\bar{z}^n$ are complex valued 1 forms

Together $dz^1, \dots, dz^n, d\bar{z}^1, \dots, d\bar{z}^n$ span $\Lambda_{\mathbb{R}} \otimes \mathbb{C} =: \Lambda_{\mathbb{C}} = \Lambda$ the space of complex valued 1 forms.

The subspace spanned by $dz^1, \dots, dz^n$ is called $\Lambda^{1,0}$

——————— " ——————— $d\bar{z}^1, \dots, d\bar{z}^n$ is called $\Lambda^{0,1}$

- These definitions are independent of the choice of coordinates.

Write $\quad z^k = x^k + i y^k \quad$ for real functions x^k and y^k

so $\quad dz^k = dx^k + i\, dy^k$

We define $\quad \dfrac{\partial}{\partial z^k} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^k} - i \dfrac{\partial}{\partial y^k}\right) \qquad \dfrac{\partial}{\partial \bar{z}^k} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^k} + i \dfrac{\partial}{\partial y^k}\right)$

these are dual to dz^k, $d\bar{z}^k$

and define $\quad T^{1,0} = \left\langle \dfrac{\partial}{\partial z^1}, \dots, \dfrac{\partial}{\partial z^n} \right\rangle \quad , \quad T^{0,1} = \left\langle \dfrac{\partial}{\partial \bar{z}^1}, \dots, \dfrac{\partial}{\partial \bar{z}^k} \right\rangle$

Define $J : TM \to TM$ by

$$J\left(\frac{\partial}{\partial x^k}\right) = \left(\frac{\partial}{\partial y^k}\right) \qquad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}$$

so $J^2 = -\mathbb{1}$

We can recover $T^{1,0}$, $T^{0,1}$ from J as the $+i$ and $-i$ eigenspaces of J.

Definition: An almost complex manifold (M, J) is a manifold equipped with $J \in \mathrm{End}(TM)$ satisfying $J^2 = -1$.

Note that J is similar to the standard $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$

$$\begin{array}{ccc} \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ i \times \mathrm{id} : \mathbb{C}^n & \to & \mathbb{C}^n \end{array}$$

Observations: * $J^2 = -1 \Rightarrow$ eigenvalues are $\pm i$ and come in pairs

Hence almost complex $\Rightarrow$ even dimensional

* $\det J = 1 > 0$

Hence almost complex $\Rightarrow$ oriented

* Existence of almost complex structure is a question of the global existence of a section of a bundle. This can be understood using theory of <u>characteristic classes</u>.

<u>Example:</u> S^4 does not admit an almost complex structure.

On a general almost complex manifold (M, J) we may define $T^{1,0}$, $T^{0,1}$, $\Lambda^{1,0}$, $\Lambda^{0,1}$

and $\Lambda^{p,q} = \Lambda^p(T^{1,0}) \otimes \Lambda^q(T^{0,1})$

so that $\Lambda^k \otimes \mathbb{C} \cong \Lambda^{k,0} \oplus \Lambda^{k-1,1} \oplus \dots \oplus \Lambda^{1,k-1} \oplus \Lambda^{0,k}$

Example: $dz^1 \wedge dz^2 \wedge \dots \wedge dz^p \wedge d\bar{z}^1 \wedge \dots \wedge d\bar{z}^q \in \Lambda^{p,q}$

Theorem: The following are equivalent

1) $T^{1,0}$ is closed under Lie brackets

2) $d : \Lambda^{1,0} \longrightarrow \Lambda^2 \cong \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$

has image entirely in $\Lambda^{2,0} \oplus \Lambda^{1,1}$

3) $N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X,Y] = 0$

4) (M, J) is a complex manifold

Warning: I lazily talk about $d : \Lambda^1 \to \Lambda^2$ when I should talk about sections, so $d : \Gamma(\Lambda^1) \to \Gamma(\Lambda^2)$.

Pf: (1) $\Longleftrightarrow$ (2) Exercise. [Use $2d\alpha(X,Y) = X(\alpha Y) - Y(\alpha X) - \alpha[X,Y]$]

(2) $\Longleftrightarrow$ (3)

Take $X, Y \in \Gamma(TM)$ so $X - iJX,\ Y - iJY \in \Gamma(T^{1,0})$

$$[X - iJX,\ Y - iJY] \in \Gamma(T^{1,0})$$

$$\Longleftrightarrow\ J[X - iJX,\ Y - iJY] = i[X - iJX,\ Y - iJY]$$

... Exercise: complete this

(4) $\Longrightarrow$ (1)

Take $\sum f_i dz^i \in \Gamma(\Lambda^{1,0})$. $d\left(\sum f_i dz^i\right) = \sum_i df_i \wedge dz^i$

$$= \sum_i \sum_k \left(* dz^k + * d\bar{z}^k\right) \wedge dz^i$$

$$\in \Lambda^{2,0} \oplus \Lambda^{1,1}$$

The implication $N \equiv 0 \implies$ the manifold is complex is called the Newlander - Nirenberg theorem and is hard to prove. (We say that J is integrable.)

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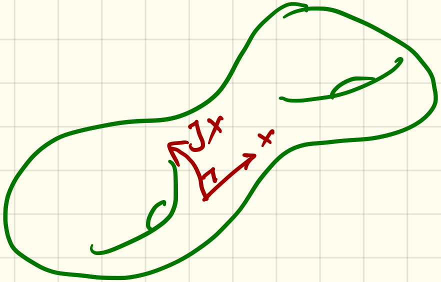
* Ultimately we are looking for a map $(U, J) \xrightarrow{\phi} \mathbb{C}^n$ locally with $\phi_* J = J$. So this is a question of local existence of PDEs.

* In the analytic category you can find out if a PDE has solutions by Cartan - Kähler theory. This is "easy"

* In the smooth category we know some local existence results: Frobenius theorem, closed $\Rightarrow$ exact, elliptic PDEs ...
  Newlander - Nirenberg is an outlier theorem.

Example: Take an oriented Riemannian 2-manifold (M,g)



Define J by rotation through 90° anticlockwise.

$\Lambda^2 \cong \Lambda^{1,1}$ since $dz^1 \wedge dz^1 = 0$ and $d\bar{z}^1 \wedge d\bar{z}^1 = 0$

$\therefore$ all oriented Riemannian 2-manifolds are complex manifolds

$\Longleftrightarrow$ There always exists an isothermal chart in the n'b'd of a point on the surface

$\not\Longrightarrow$ Smooth 2 manifolds have analytic atlases

These are not obvious results.

De Rham cohomology:

$$0 \xrightarrow{d} \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \xrightarrow{d} 0 \qquad d^2 = 0$$

$$H^k = \frac{\operatorname{Ker}\, d : \Lambda^k \to \Lambda^{k+1}}{\operatorname{Im}\, d : \Lambda^{k-1} \to \Lambda^k}$$

Dolbeault cohomology:

On a complex manifold

$$\begin{tikzcd}
 & & \Lambda^{2,0} \\
 & \Lambda^{1,0} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & \\
\Lambda^{0,0} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & & \Lambda^{1,1} \\
 & \Lambda^{0,1} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & \\
 & & \Lambda^{0,2}
\end{tikzcd}$$

$$d^2 = 0$$

$$\Longrightarrow \partial^2 = 0, \quad \partial\bar\partial - \bar\partial\partial = 0, \quad \bar\partial^2 = 0$$

$$H^{p,q}_{\bar\partial} = \text{cohomology of } \bar\partial$$

$$= \frac{\operatorname{Ker}\, \bar\partial : \Lambda^{p,q} \to \Lambda^{p,q+1}}{\operatorname{Im}\, \bar\partial : \Lambda^{p-1,q} \to \Lambda^{p,q}}$$

ALMOST HERMITIAN MANIFOLDS AND THE KÄHLER CONDITION

Definition: An almost Hermitian manifold (M, g, J) is a Riemannian manifold (M, g) an almost complex manifold (M, J) and $J : TM \to TM$ is an isometry so

$$g(JX, JY) = g(X, Y)$$

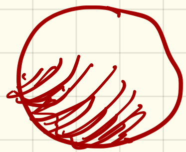
Example: * $\mathbb{C}^n$ or $\mathbb{C}^n/\Lambda$ for a lattice

* Any oriented 2-manifold with J given by rotation through $90°$

* $\mathbb{CP}^n \cong \dfrac{U(n+1)}{U(n) \times U(1)}$ is a symmetric space

Its metric is called the Fubini-Study metric

Example: $\mathbb{CP}^1 = S^2$



Fundamental 2-form: Given an almost Hermitian manifold

define $$\omega(X,Y) = g(JX,Y)$$

ω is non-degenerate i.e. $\omega(X,Y)=0 \ \forall X \implies Y=0$

we can find coordinates so that at a point p

$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}$ are orthonormal

and J is standard

$$\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + \dots + dx^n \wedge dy^n$$
$$= \frac{i}{2}\left(dz^1 \wedge d\bar{z}^1 + \dots + dz^n \wedge d\bar{z}^n\right)$$
$$\in \Lambda^{1,1}$$

$$dz^1 \wedge d\bar{z}^1 = (dx + i\,dy) \wedge (dx - i\,dy)$$
$$= i\,dy \wedge dx - i\,dx \wedge dy$$
$$= -2i\,dx \wedge dy$$

Definition: (M,g,J) is Hermitian if $N=0$

(M,g,J) is almost Kähler if $d\omega=0$

(M,g,J) is Kähler if $d\omega=0$

Definition: (M,ω) is symplectic if ω is a closed non-degenerate two form

Example: * All oriented Riemannian 2-manifolds are Kähler

* The product of two Kähler manifolds is Kähler

* $\mathbb{CP}^n$ is Kähler with the Fubini-Study metric

Proposition: Let (M, g, J) be a compact Kähler manifold then

$$\dim H^{2k}(M) \geq 0 \quad \text{for} \quad k = 1, \dots, n$$

Pf: a) $d(\omega^k) = *(d\omega) \wedge \omega^{k-1} = 0$ $\quad (* = \text{some constant})$

b) ω^n defines the orientation so $\int_M \omega^n > 0$

c) Suppose $\omega^k = d\eta$

then $\omega^n = d\eta \wedge \omega^{n-k} = * d(\eta \wedge \omega^{n-k})$

So ω^k is exact only if ω^n is.

d) Suppose $\omega^n = d\eta$ then by Stokes' theorem

$$\int_M \omega^n = \int_{\partial M} \eta = 0 \quad \text{since } \partial M = 0$$

$\#$

Example: $S^3 \times S^1$ does not admit a Kähler metric (or indeed any symplectic form)

But the quotient of $\mathbb{C}^2 - \{0\}$ by the automorphisms generated by $(z_1, z_2) \to (2z_1, 2z_2)$ is a complex manifold diffeomorphic to $S^3 \times S^1$

This is called the <u>Hopf Surface</u>

* The Hopf surface is complex but has no Kähler metric

Lemma: A complex submanifold of a Kähler manifold is Kähler.

Corollary: The Hopf Surface cannot be embedded in $\mathbb{CP}^n$

Pf: Let (M, g, J) be the large space and let N be a complex submanifold.

Let $\iota : N \to M$ be the inclusion.

We want to show the pull back $i^* \omega$ is the fundamental 2 form on N.

This is obvious for $\mathbb{C}^k \to \mathbb{C}^n$

So it suffices to show we can choose coords so that g, J are standard at a point and i_* is the standard $\mathbb{C}^k \to \mathbb{C}^n$

We know very little about complex manifolds in general but we know a lot about Kähler manifolds

Example: S^6 admits an almost complex structure but does it admit a complex structure?

Theorem: On a compact Kähler manifold

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(M)$$

De Rham — Dolbeault

"Algebraic topology" — "Complex Geometry"

$$H^{p,q}(M) \cong \overline{H^{q,p}(M)}$$

(And indeed much more is true...)

Reference: I've just given a tour of some highlights of Griffiths & Harris chapter 0

Tip: * Section 0.6 on Hodge theory is ludicrously compressed (in my view)

* Compliment Griffiths & Harris Chapter 0 with Donaldson "Rieman surfaces"

* The later chapters of G & H are often easier than Chapter 0.

What I haven't discussed is the meaning of

$$H^{p,q}_{\bar\partial}(M)$$

These cohomology groups can be associated with interesting properties of a manifold such as meromorphic functions, line bundles...

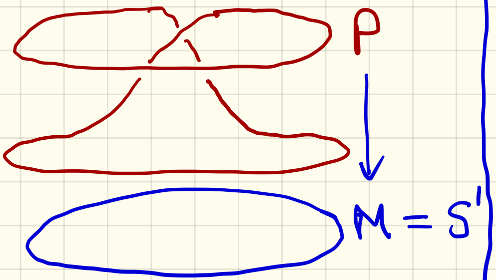
Example: Donaldson's book shows how to deduce the classification of complex tori of 1-d from the results on Hodge theory.

Part II

Bundles & Representations

A <u>principal G bundle</u> for a Lie group G is a fibre bundle $P \to M$ with an action of G on the fibres which is locally isomorphic to the trivial bundle $G \times U \to U \subseteq M$.

<u>Example:</u> $G = \mathbb{Z}_2$



Note that each fibre is topologically equal to G, there is no way to identify the identity element of each fibre (unless it is a trivial bundle)

<u>Important Example</u>: Given a manifold M of dimension n, take the fibre over $p \in M$ to be the set of bases for the tangent space at p. This is called the "frame bundle". It is a principal $GL(n;\mathbb{R})$ bundle.

A <u>representation</u> of a group G is a homomorphism

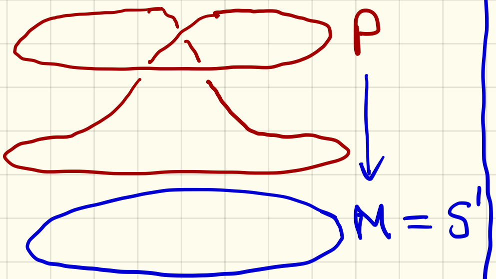
$$\rho : G \longrightarrow \mathrm{Aut}(V)$$

where V is a vector space and $\mathrm{Aut}(V)$ is the group of linear automorphisms of V.

<u>KEY CONSTRUCTION</u>

Given a principal bundle $P \xrightarrow{G} M$ and a representation $\rho : G \longrightarrow \mathrm{Aut}(V)$ we can form a vector bundle $\underline{V} = (P \times V)/G$ where we quotient by the diagonal action of G.

<u>Example:</u> $G = \mathbb{Z}_2$



$\rho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{R})$
by $\rho(0) = \mathrm{id}$, $\rho(1) = -\mathrm{id}$
to get

? ______

Example: Let P be the frame bundle of a manifold

Let $\rho : GL(n,\mathbb{R}) \to Aut(\mathbb{R}^n)$ be the identity

The resulting vector bundle is the ______ bundle of the manifold.

This is the standard representation of $GL(n,\mathbb{R})$

Example: If $\rho : G \to Aut(V)$, define the dual representation

$\rho^* : G \to Aut(V^*)$ by $\rho^*(g) = (\rho(g)^{-1})^*$

check:

$$\begin{aligned} \rho^*(g)\rho^*(h) &= (\rho(g^{-1}))^*(\rho(h^{-1}))^* \\ &= (\rho(h^{-1})\rho(g^{-1}))^* \\ &= (\rho(h^{-1}g^{-1}))^* \\ &= (\rho((gh)^{-1}))^* \\ &= \rho^*(gh) \end{aligned}$$

If P is the frame bundle, the vector bundle associated with the dual of the standard representation is ______

Generalization: Let $\rho_1 : G \to V$ and $\rho_2 : G \to W$ be automorphisms. Define

$$\rho^{Hom} : G \longrightarrow \mathrm{Aut}(\mathrm{Hom}(V, W))$$

by $$\left(\rho^{Hom}(g)T\right)v = \left(\rho_2(g^{-1})\, T\, \rho_1(g)\right)v$$

Exercise (easy): check that ρ^{Hom} is a representation

Lazy Notation: Given representations V, W we get a representation $\mathrm{Hom}(V, W)$

Exercise: Given representations V, W define the tensor product representation $V \otimes W$ the symmetric representation $S^k V$ and the antisymmetric representation $\Lambda^k V$.

Prove that the representation $\mathrm{Hom}(V, W) \cong V^* \otimes W$

Note that it is your job to define what an isomorphism means. You should also note that you've just defined a k-form in a neat way.

Exercise: Given a complex representation ρ, define the conjugate representation $\bar{\rho}$.

Example: Let (M, g) be a Riemannian manifold. Let P be the bundle of orthonormal frames. This is a principal ______ bundle.

Example: Let (M, g) be an oriented Riemannian manifold, the bundle of oriented orthonormal frames is a principal ______ bundle.

These give examples of "reductions of the structure group". If we have more data on our manifold, we'll get smaller and smaller groups of symmetries on the tangent space.

Example: Let $\rho : O(n) \to \mathbb{R}^n$ be the standard representation

Elements $g \in O(n)$ are given by matrices with

$$g^* g = id \qquad (* = \text{transpose})$$

It follows that $\rho^* = \rho$ for the standard representation

Exercise: Prove it

It follows that the metric g defines an isomorphism between the tangent bundle and the cotangent bundle. This is the familiar "raising and lowering of indices" explained in terms of representation theory.

Summary: All your favourite vector bundles can be understood in terms of a principal bundle and the representations of the structure group.

Conclusion: When studying differential geometry it pays to understand the representation theory of the structure group.

Reading: Adams: "Lectures on Lie Groups" (Short)

Fulton & Harris: "Representation Theory" (Reference)

TIP: Representation theory is a tool. If you are in a rush read the results not the proofs.

Definition: A Kähler manifold is an almost Hermitian manifold which is complex and symplectic

Proposition: Let ∇ be the Levi-Civita connection
The following are equivalent
(a) $\nabla\omega = 0$
(b) $\nabla J = 0$
(c) $N = 0$ and $d\omega = 0$
(d) Parallel transport using ∇ gives unitary maps $TM \to TM$

Pf: (a) $\Leftrightarrow$ (b) follows from fact $\nabla g = 0$

(b) $\Rightarrow$ (c) follows from ∇ is torsion free so $[X,Y] = \nabla_X Y - \nabla_Y X$

(a) $\Rightarrow$ (c) follows from ∇ is torsion free + Cartan's formula

(b) $\Leftrightarrow$ (d)

"The Holonomy group is in $U(n)$"

J is an isometry on $TM \implies \nabla J \in T^*M \otimes \mathfrak{so}(2n)$

Differentiating $J^2 = -1 \implies J(\nabla J) + (\nabla J)J = 0$

So $\nabla J \in (T^*M \otimes \mathfrak{so}(2n)) \cap (T^*M \otimes \mathfrak{gl}(n)^\perp)$

$= T^*M \otimes \mathfrak{u}(n)^\perp$

$A \in SO(2n) \iff AA^* = 1$ so $A \in \mathfrak{so}(2n) \iff A + A^* = 0$

Hence $\mathfrak{so}(2n) \cong \Lambda^2$

Under $U(n)$, $SO(2n)$ splits as $[\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}_0] \oplus \langle\omega\rangle$

How can we prove $N=0$ and $d\omega=0 \Rightarrow \nabla\omega=0$?

Option 1: Figure out how to write $\nabla\omega$ in terms of N and $d\omega$

$\Longleftrightarrow$ find the linear map ϕ with $\phi(N, d\omega) = \nabla\omega$

Option 2: Use representation theory of $U(n)$

Idea: Decompose a representation V into irreducibles $V_1 \oplus V_2 \oplus \dots \oplus V_n$

Use Schur's Lemma: if $\phi : V_\rho \to W_{\rho'}$ is an equivariant map and V, W are irreducible then

either • $\phi = 0$

or • $V_\rho \cong W_{\rho'}$ and ϕ is a multiple of the identity

(i.e $V \cong W$ and $\rho \cong \rho'$)

Definition: An irreducible representation is a representation that can't be written as a non-trivial direct sum.

Given your favourite Lie group, you can easily look up the classification of irreducibles. You can also look up how to decompose tensor products, symmetric powers etc into irreducibles. By Schur's lemma you then know all the equivariant maps.

Example: Under $SO(n)$, $T \cong T^*$

$$T^* \otimes T^* \cong \mathrm{End}(TM) \cong T \otimes T \cong S^2_0 T \oplus \Lambda^2 T \oplus \mathbb{R}$$

Symmetric trace free

Alternating

Trace

No other interesting 2 tensors exist that are $SO(n)$ invariant

$$\mathfrak{so}(n) \cong \Lambda^2$$

Write $[\![V]\!]$ for underlying real representation

Write $[V] = W$ if $V = W \otimes \mathbb{C}$ for a real representation W.

$$d\omega \in \Lambda^3_{\mathbb{R}} \cong [\![\Lambda^{3,0} \oplus \Lambda^{2,1}]\!]$$
$$\cong [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda^{2,1}_0]\!] \oplus [\![\Lambda^{1,0}]\!]$$

where "wedge with ω" : $\Lambda^{1,0} \to \Lambda^{2,1}$

and $\Lambda^{2,1}_0$ is the orthogonal complement

$$N \in [\![\Lambda^{0,1} \otimes \Lambda^{0,2}]\!] \cong [\![\Lambda^{0,3}]\!] \oplus [\![A]\!]$$

J is an isometry on T

$$\Longrightarrow \quad \nabla J \in T^*M \otimes \mathfrak{so}(2n) \cong T^*M \otimes \Lambda^2$$

$J^2 = -1$

$$\Longrightarrow \quad J(\nabla_X J) + (\nabla_X J) J = 0$$

$$\Longleftrightarrow \quad \nabla J \in T^*M \otimes \mathfrak{gl}(n)^\perp$$

So $\quad \nabla J \in T^*M \otimes \mathfrak{u}(n)^\perp$

$$\cong [\![\Lambda^{1,0}]\!] \otimes [\![\Lambda^{2,0}]\!]$$

$$\cong [\![\Lambda^{1,0} \otimes \Lambda^{2,0}]\!] \oplus [\![\Lambda^{0,1} \otimes \Lambda^{0,2}]\!]$$

$$\cong [\![\Lambda^{3,0}]\!] \oplus [\![A]\!] \oplus [\![\Lambda^{2,1}_0]\!] \oplus [\![\Lambda^{1,0}]\!]$$

$$\bullet \longleftarrow d\omega \longrightarrow \bullet \longrightarrow \bullet$$

$$\bullet \longleftarrow N \longrightarrow \bullet$$

It is clear that $N = \phi_1(\nabla\omega)$ for some $U(n)$ equivariant map ~~ϕ~~ ϕ_1

Similarly $d\omega = \phi_2(\nabla(\omega))$ for some $U(n)$ equivariant map ϕ_2

So the proof follows from the decomposition into irreducibles + Schur's Lemma.

Moral: Impossibly tedious local coordinate calculations can be done quickly using representation theory.

<u>Conclusion</u>: ∇J has 4 irreducible components so there are $2^4 = 16$ types of almost Hermitian manifold.

The most interesting are Kähler ($\nabla \omega = 0$)

Hermitian ($N = 0$)

almost Kähler ($d\omega = 0$)

Exercises:

1. **Theorem**: The following are equivalent

   1) $T^{1,0}$ is closed under Lie brackets
   2) $d : \Lambda^{1,0} \to \Lambda^2 \cong \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$ has image entirely in $\Lambda^{2,0} \oplus \Lambda^{1,1}$
   3) $N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X,Y] = 0$
   4) (M, J) is a complex manifold

   Prove that $(1) \Longleftrightarrow (2)$

   Complete the proof that $(2) \Longleftrightarrow (3)$

2. Use the fact that $H^{0,0} \cong H^1$ to prove that any holomorphic function $f : M \to \mathbb{C}$ on a compact connected Kähler manifold is constant

3. Find out (eg online) what the explicit formula is for the Fubini-Study metric and convince yourself that it is Kähler

4. Proposition 26 of Donaldson gives an ~~interpretation~~ interpretation of $H^{0,1}$:
"Suppose $H^{0,1}_X$ has finite dimension h, then given any $h+1$ points $p_1, \dots, p_{h+1}$ on X there is a non-holomorphic meromorphic function on X with simple poles at some subset of the $p_1, \dots, p_{h+1}$"
A meromorphic function is a holomorphic map $f : M \to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$
It is a holomorphic function if it has no poles.

Use this proposition plus the relationship of Dolbeault and De Rham cohomology to prove

Corollary 3 of Donaldson: "Any compact Riemann surface of genus 0 is equivalent to the sphere"

5. Check that ρ^{Hom} is a representation

6.

<u>Exercise:</u> Given representations V, W define the tensor product representation $V \otimes W$ the symmetric representation $S^k V$ and the antisymmetric representation $\Lambda^k V$.

Prove that the representation $Hom(V,W) \cong V^* \otimes W$

Note that it is your job to define what an isomorphism means. You should also note that you've just defined a k-form in a neat way.

7. <u>Exercise:</u> Given a complex representation ρ, define the conjugate representation $\bar{\rho}$.

8. Prove that $\rho^* = \rho$ for the standard representation of $O(n)$