Twistor lines on cubic surfaces

The twistor space of S^4

$$\begin{array}{ccc} \mathbb{CP}^3 & \xrightarrow{\pi} & S^4 \\ \cong & & \cong \\ (\mathbb{H}\times\mathbb{H})/\sim_{\mathbb{C}} & \xrightarrow{\pi} & (\mathbb{H}\times\mathbb{H})/\sim_{\mathbb{H}} \end{array}$$

$\sim_{\mathbb{H}}$ defined by $(q_1, q_2) \sim_{\mathbb{H}} (\lambda q_1, \lambda q_2)$ with $\lambda \in \mathbb{H}$
$\sim_{\mathbb{C}}$ defined by $(q_1, q_2) \sim_{\mathbb{C}} (\lambda q_1, \lambda q_2)$ with $\lambda \in \mathbb{C}$

The twistor fibres

Fibre is $\mathbb{H} / \sim_{\mathbb{C}} \cong \mathbb{CP}^1$.

In detail:

$(z_1 + z_2\, j,\; z_3 + z_4\, j) \sim_{\mathbb{H}} (1,\; w_1 + w_2\, j)$

if and only if

$(z_1 + z_2)^{-1} (z_3 + z_4\, j) = w_1 + w_2\, j$

which simplifies to:

$z_3 + z_4\, j \;=\; (z_1 w_1 - z_2 \overline{w_2}) + (z_2 \overline{w_1} + z_1 w_2)\, j$

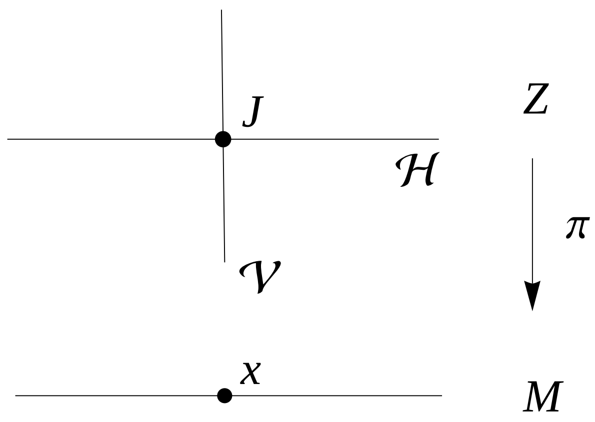
Fibres are skew, holomorphic lines.

The real structure

Multiplication by j defines a map $\mathbb{J} : \mathbb{CP}^3 \longrightarrow \mathbb{CP}^3$.

- $\mathbb{J}$ is anti-holomorphic
- $\mathbb{J}^2 = 1$
- $\mathbb{J}$ has no fixed points
- Fibres/twistor lines correspond to $\mathbb{J}$ invariant lines.

General Riemannian twistor construction



Complex structure of Z is integrable iff $W^{+} \equiv 0$.
Hermitian almost-complex structure
$\Longleftrightarrow$ J-holomorphic section

The discriminant locus

- There are no global sections for topological reasons
- Given complex surface in $\mathbb{CP}^3$ defined by polynomial of degree d, consider polynomial restricted to fibre.

Example

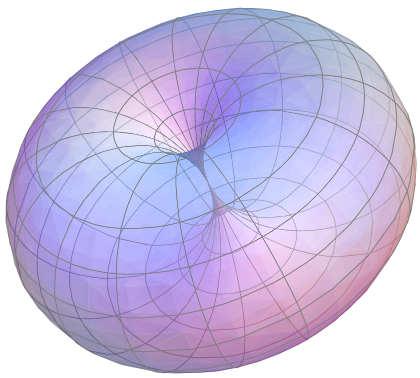
- Plane in $\mathbb{CP}^3$

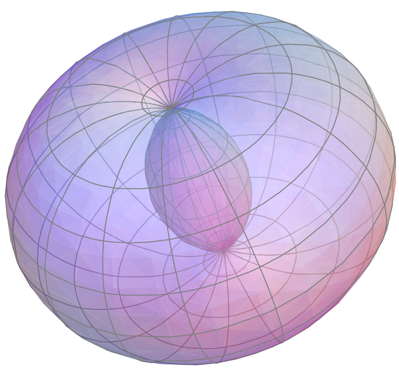
Problems

- Classify the curves of degree d up to conformal transformation.
- Investigate the topology of the discriminant locus for curves of degree d

Progress

- Classify the curves of degree *d* up to conformal transformation.
  - Degree 2 curves classified by Salamon and Viaclovsky 2008
- Investigate the topology of the discriminant locus for curves of degree *d*
  - Degree 2: Either a circle, a torus, a torus pinched at one point or a torus pinched at two points.

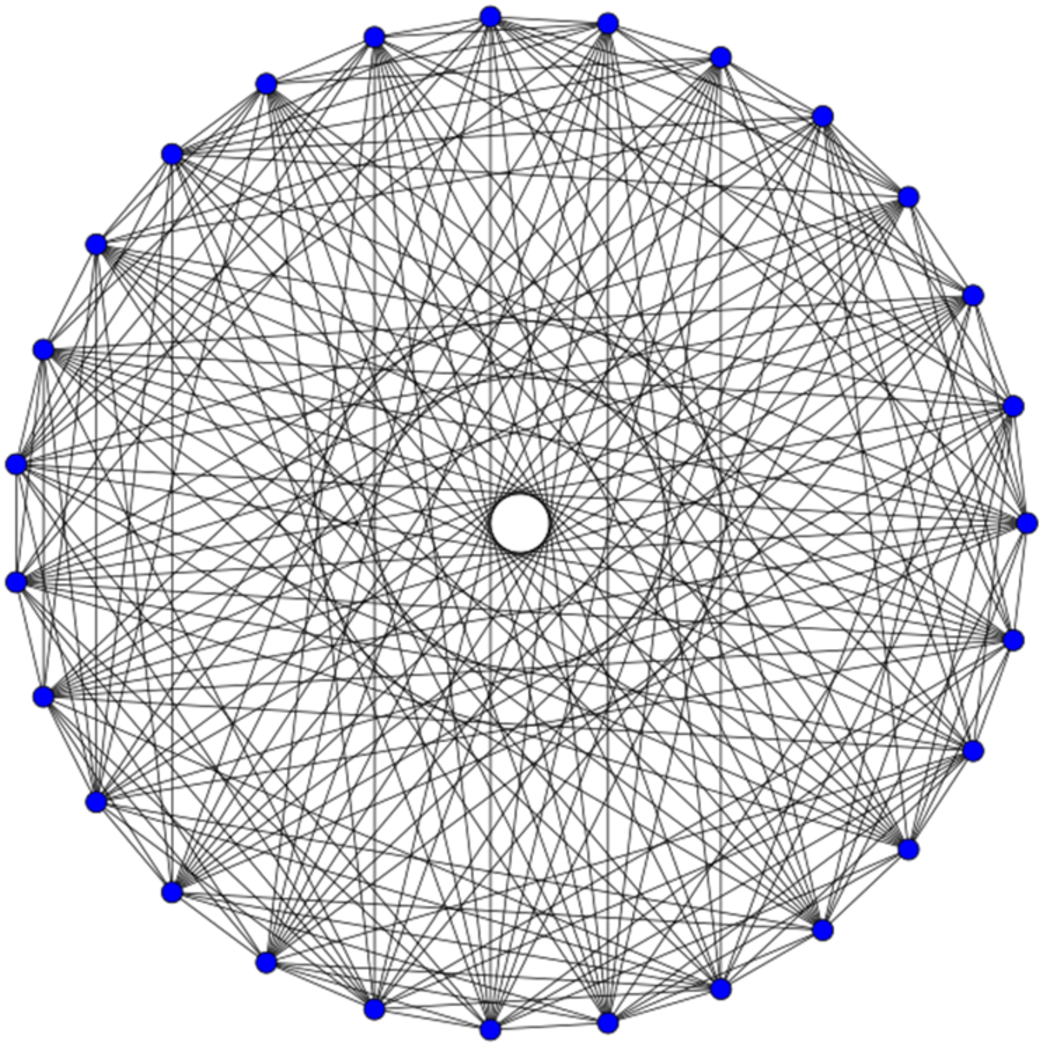




Lines on projective surfaces

- What is the maximum number of twistor lines on a surface of degree d?
- Maximum number of lines on a projective surface of degree d?
  - 27 lines on a cubic
  - Max 64 on a quartic (Segre 43)
  - $N_d \leq (d-2)(11d-6)$
  - $N_d \geq 3d^2$ (Caporaso-Harris-Mazur 95)
  - $N_6 \geq 180$, $N_8 \geq 352$, $N_{12} \geq 864$, $N_{20} = 1600$ (CHM 95, Boissière and Sarti 2006)
- Maximum number of skew lines
  - $\leq 2d(d-2)$ (when $d \geq 4$) (Miyaoka 1984)
  - $\geq d(d-2) + 2$ (Rams 2002)
  - $\geq d(d-2) + 4$ (when $d \geq 7$ and $\gcd(d, d-2) = 1$) (Sarti 2006)

Cubic surfaces and the Schläfli graph



Schläfli's notation for the lines on a cubic

- Label 6 skew lines $a_1, a_2, a_3, a_4, a_5, a_{6.}$
- There are 6 lines $b_1, b_2, b_3, b_4, b_5, b_6$ that each intersect 5 of the a's.
- Such a combination of lines is called a "double six" and written:

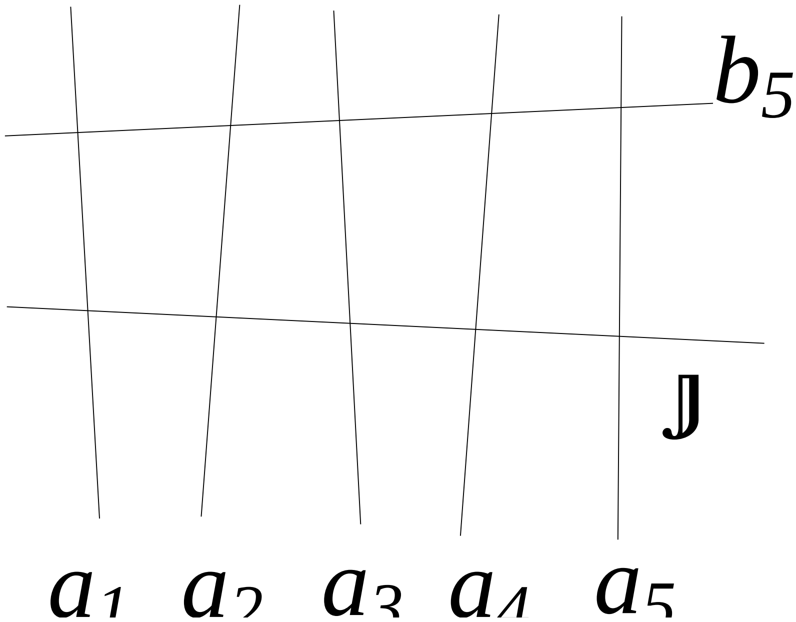
$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

- The remaining 15 lines can be labelled c_{ij}in such a way that c_{ij} intersects a_i and a_j.
- c_{ij} intersects c_{kl} if and only if $\{i, j\} \cap \{k, l\} = \emptyset$

There are 36 double sixes on any cubic. For example:

$$\begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}$$

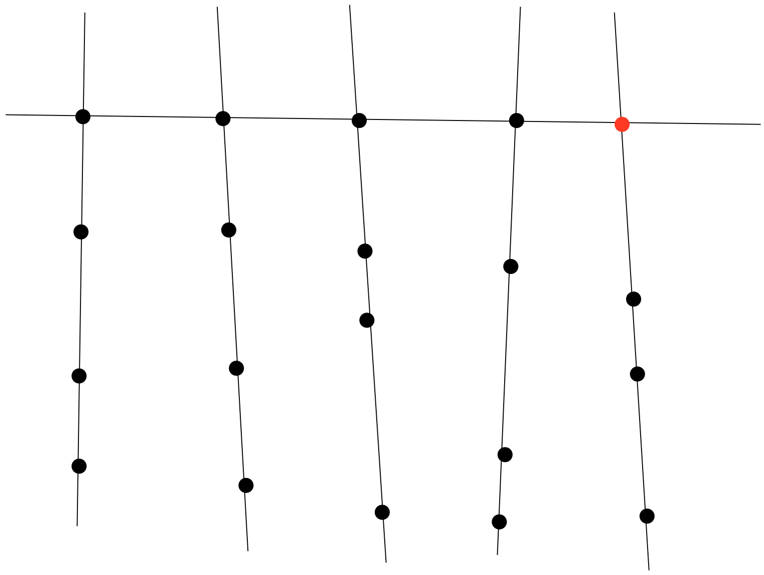
Twistor lines on a cubic



Matching cubics to lines

- dim $S^3(\mathbb{C}^4) = 20$
- Condition that a cubic contains a point defines one linear condition on cubic coefficients.
- Through any four lines there is a cubic surface.
- Generically there are no cubics through five given lines

Cubic through collinear lines



Algebraic formulation

- Line represented as $\omega \in \Lambda^2$ with $\omega \wedge \omega = 0$
- 5 skew lines $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$
- $\omega_i \wedge \omega_j$ defines an element of $\Lambda^4 \cong \mathbb{C}$
- So we can think of $\omega_i \wedge \omega_j$ as defining a 5 by 5 symmetric matrix which will have zeroes on the diagonal.
- Condition that the lines are collinear is $\det(\omega_i \wedge \omega_j) = 0$

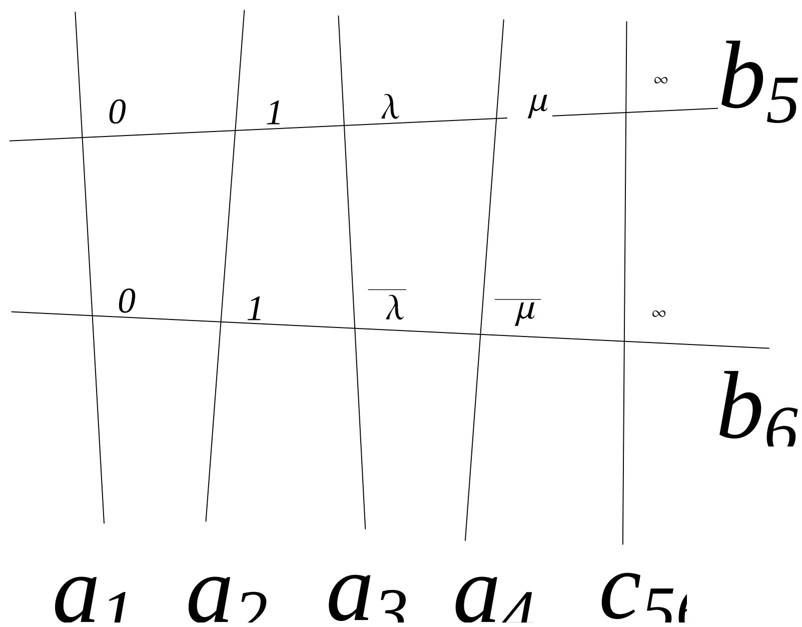
Cubics with 5 twistor lines

- Lines in $\mathbb{CP}^3$ correspond to either:
  - Points
  - Oriented round 2-spheres (and oriented planes)

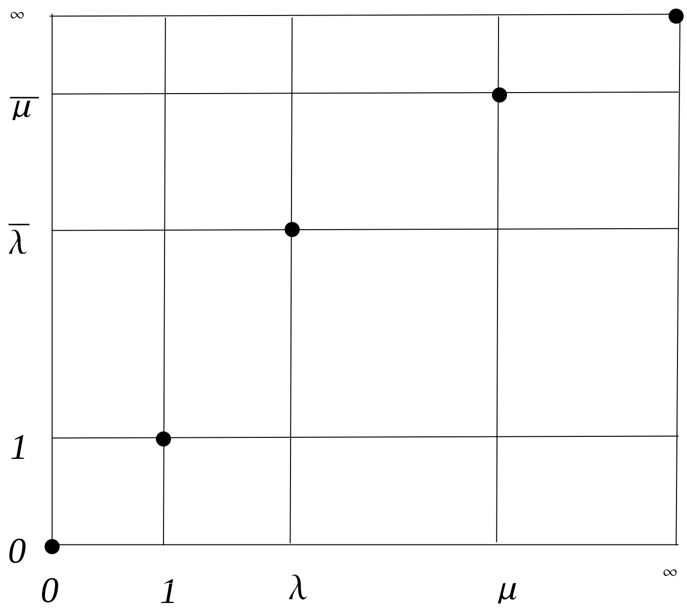
For any 5 points on a round 2-sphere in S^4there is a 1 parameter family of cubic surfaces with these 5 points as twistor lines.

- Generically these cubics will be non-singular
- The family of non-singular cubics over a given set of 5 points are biholomorphic.

Rational map



Classification of cubic surfaces



Cubic surface is $\left(\mathbb{CP}^1 \times \mathbb{CP}^1\right) \sharp\, 5\, \overline{\mathbb{CP}^2}$.

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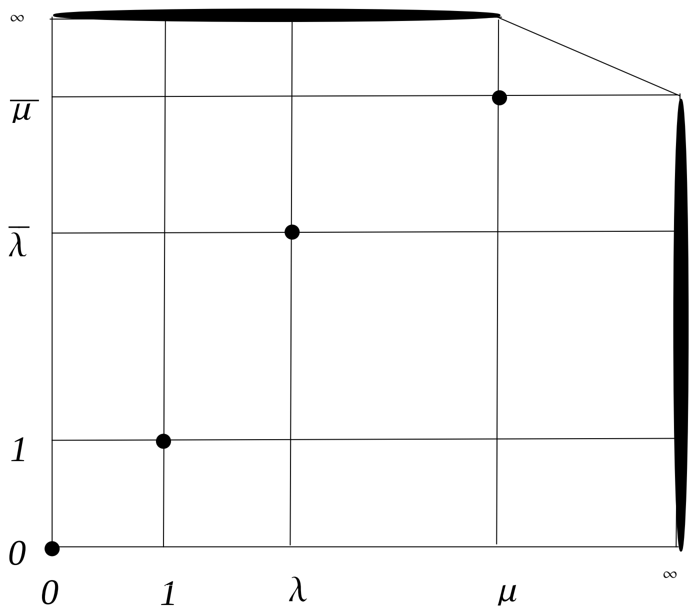

Cubic surface is $(\mathbb{CP}^1 \times \mathbb{CP}^1) \# 5\, \overline{\mathbb{CP}^2}$.