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2023.03.14

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Itô Stochastic Differentials on Manifolds

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Joint work with:

- Andrei Ionescu (stochastic differentials) and
- Damiano Brigo (jets)

Plan:

- I will describe what a classical stochastic differential equation is using integration
- I will motivate the definiton of stochastic differential
- I will show that once the stochastic differential is defined, the key definitions for stochastic processes on manifolds are extremely simple.

ODEs without differentiation

The ODE

$$\frac{dX}{dt} = f(X_t, t), \quad X_0$$

can be written

$$X_u - X_0 = \int_0^u dX_t = \int_0^u f(X_t, t)\, dt$$

which one might write in shorthand as

$$dX_t = f(X_t, t)\, dt, \quad X_0$$

where dX_t and dt are just formal symbols.

Historically stochastic differential equations (SDEs) have been defined entirely in terms of the Itô integral. Differentials have not been defined.

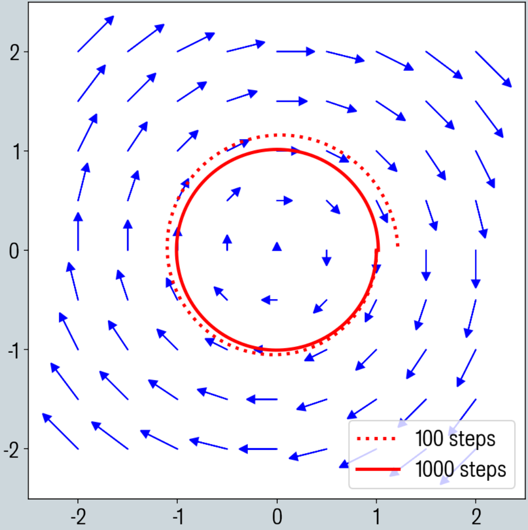
A world without differentiation

- We would not have the notion of two curves being tangent to one another
- We cannot give a meaning to the symbol dX_t at a particular moment in time - ODEs would have to hold in some interval to have any meaning.
- As an integral requires a vector space structure so we are pretty much obliged to define ODEs on manifolds in terms of charts. This is possible, but revolting.

DIfferentiation, visualization and geometry

- Geometrically, ODEs are vector fields. The theory of the tangent space tells us how to think of ODEs in a coordinate free fashion.
- Interpreting a vector as a 1-jet of a curve gives a coordinate free interpretation of Euler-type schemes, just follow the arrows for time δt.

Goal: formally define a stochastic differential and use it to give a coordinate-free treatment of SDEs.



2
1
0
-1
-2
-2
-1
0
1
2
100 steps
1000 steps

History

- Itô was motivated to develop his calculus in order to define Brownian motion on manifolds. He gave a coordinate-based treatment of SDEs on manifolds.
- Itô appears to have been interested in defining a stochastic differential himself
- There is an existing concept called the Nelson derivative which is close to our differential but does not quite achieve what is needed.
- A number of authors have come up with coordinate-free approaches to SDEs including: Stratonovich calculus (Elworthy), Schwarz-Morphisms (Schwarz/Meyer/Emery)

Schwarz: there is nothing “ponctuel” about stochastic differential equations Emery: “existence of the [stochastic differential] is metaphysical and one is free not to believe in it.”

Brownian Motion

In a Stochastic Differential Equation we are interested in the evolution of random processes X_t. The subscript s indicates the value at time t.

The evolution is described in terms of another driving stochastic process, Y_t. For example: Y_t may represent information about the economy and X_t may represent the quantities of different stocks you purchase in response to economic news.

To get started, we will assume that the driving process is a *Brownian motion* W_t.

Definition

W_t is a continuous process in time. The increment $W_{t+\delta t} - W_t$ after time t is independent of any information before time t and is normally distributed with mean 0 and standard deviation $\sqrt{\delta t}$, hence variance δt.

Scaling behaviour

This is the only possible scaling behaviour for independent identical increments with finite variance:

$$\mathrm{Var}(W_{n\delta t} - W_0) = \mathrm{Var}\left(\sum_{i=1}^{n} W_{i\delta t-(i-1)\delta t}\right) = \sum_{i=1}^{n} \mathrm{Var}(W_{i\delta t-(i-1)\delta_t}) = n\,\delta t$$

The Itô Integral

$$\int_0^T a_t \, dW_t \approx \sum_{i=1}^{n} a_{(i-1)\delta t} W_{i\delta t-(i-1)\delta t}, \qquad \delta t = \frac{T}{n}$$

Naively this appears to diverge: we have $T/\delta t$ terms each of size $\sqrt{\delta t}$ so the integral seems to be of the order $\delta t^{-\frac{1}{2}}$. But...

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$$\begin{aligned} E\left(\int_0^T a_t \, dW_t\right)^2 &\approx \sum_{i=1}^{n}\sum_{j=1}^{n} a_{(i-1)\delta t} a_{(j-1)\delta t} E(W_{i\delta t-(i-1)\delta t} W_{j\delta t-(j-1)\delta t}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{(i-1)\delta t} a_{(j-1)\delta t} E(W_{i\delta t-(i-1)\delta t}) E(W_{j\delta t-(j-1)\delta t}) \end{aligned}$$

using the fact that increments are uncorrelated. So terms where $i \neq j$ vanish.

$$E\left(\int_0^T a_t \, dW_t\right)^2 \approx \sum_{i=1}^{n} a^2_{(i-1)\delta t}(\delta t) = \sum_{i=1}^{n} a^2_{(i-1)\delta t} \frac{T}{n}$$

which no longer looks divergent. The Itô integral is therefore defined using mean-square convergence.

Stochastic Differential Equations

The Itô SDE

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t, \quad X_0$$

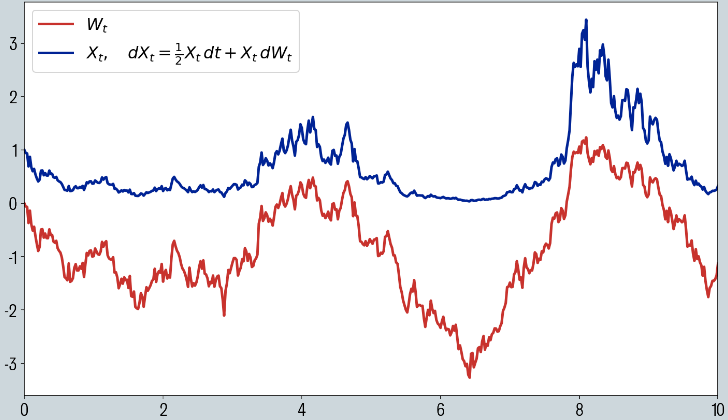
is shorthand for the integral equation

$$X_u - X_0 = \int_0^u a(X_t, t)dt + \int_0^u b(X_t, t)dW_t$$

where the left hand integral is a Riemann integral and the right hand integral is an Itô integral.

An example SDE

$$dX_t = \frac{1}{2} X_t dt + X_t dW_t., \qquad X_0 = 1$$



Idea

The differential is equal to 0 if the process is “small”. For ODEs the correct definition is

$$d_t X = 0$$

$$\text{is defined by } X_{t+\delta t} - X_t = o(\delta t)$$

$$\iff \frac{X_{t+\delta} - X_t}{\delta t} \to 0$$

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$$\Longleftrightarrow \frac{X_{t+\delta} - X_t}{\delta t} \to 0$$

We can then define an equivalence relation $\sim_t$ by $X \sim Y$ if $d(X_t - Y_t) = 0$.

The differential d_t of X is the equivalence class of X.

How should we define small for stochastic processes?

Requirements

- $dX_t = 0$ for all $t \in [0, T]$ should imply X_s is a constant random variable.
- The fundamental theorem of calculus should hold

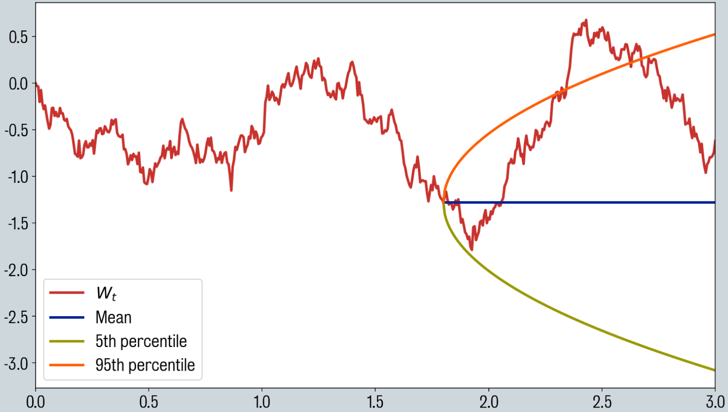
$$d_t \left(\int_0^s dX_u \right) = d_t X$$

  where the integral is an Itô integral.
- The space of differentiable processes X should be as large as possible

Martingales

A martingale is a stochastic process with

- $E(|X_t|)$ finite for all t
- $E(X_{t+\delta t} \,|\, \text{Information up to time t}) = E(X_t)$



Etymology?

Nearly correct definition

e.g. Martingales have uncorrelated increments and so the nice cancellation we saw before occurs for Martingales.

If M_s is a Martingale we might think of defining

$$d_t^E M = 0$$

if

$$E(|M_{t+\delta t} - M_t|^2) = o(\delta t)$$

Modes of convergence

- $X_n \to X$ *in* L^p if $E(|X_n - X|^p) \to 0$ as $n \to \infty$
- $X_n \to X$ *almost surely* if $P(X_n \to X) = 1$
- $X_n \to X$ *in probability* if $P(|X_n - X| > \epsilon) \to 0$ for all $\epsilon > 0$

Also

- $X_n \to X$ in distribution if the distribution function of X_n converges to the distribution of X away from jumps.

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If X_n takes values in a topological space, convergence in expectation depends on metric at infinity, convergence in probability does not.

If we want a diffeomorphism invariant differential we will need convergence in probability. But this is then too weak and $d_t X = 0$ for all t will not imply X is constant.

Quadratic variation

The quadratic covariation of two process is defined by:

$$[X, Y]_t := \underset{|\mathcal{P}|->0}{\text{Plim}} \sum_{k=1}^{m} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k})$$

where $\mathcal{P} = \{x_1, x_2, \ldots x_n\}$ is a partition of $[0, t]$.

The quadratic varation is defined by:

$$[X]_t = [X, X]_t$$

Example:

$$[W]_t = t$$

Birkholder Davis Gundy

Theorem

If X_t is a martingale with $X_0 = 0$ then there are constants c_1, c_2 such that

$$c_1 E([X]_t) \leq E(\sup_{s \in [0,t]} X_s^2) \leq c_2 E([X]_t)$$

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Using the right hand inequality, we can prove uniqueness from a bound on quadratic variation: A martingale with quadratic variation 0 must be constant

Using the left hand inequality, we can get bounds on the quadratic variation from the maximum of the increments

The stochastic differential

Definition

If $X_t = A^1_t - A^2_t + M_t$ where $A^1 + t$ and A^2_t are increasing and M_t is a martingale then we say

$$d_t X = 0$$

iff

$$A^1_{t+\delta t} - A^1_t = o_P(\delta t)$$

$$A^2_{t+\delta t} - A^2_t = o_P(\delta t)$$

$$\sup_{s \in [t, t+\delta t]} (M_s - M_t)^2 = o_P(\delta t)$$

Where $X_h = o_P(f(h))$ if $\frac{X_h}{f(h)}$ converges in probability

Operations on differentials

- Addition

$$d_tX + d_tY := d_t(X + Y)$$

- Multiplication by a random scalar X_t

$$X_t d_t Y = d_t(X_t Y) \quad \text{where } (X_t Y)_s := X_t Y_s$$

- Expectation

$$E_t(d_t X) := d_t(E_t X)$$

Product and quadratic variation

Since

$$X_{t+\delta t}Y_{t+\delta t} - X_tY_t = X_t(Y_{t+\delta t} - Y_t) + Y_t(X_{t+\delta t} - X_t) + (X_{t+\delta t} - X_t)(Y_{t+\delta t} - Y_t)$$

Definition

$$d_t(XY) = X_t d_t Y + Y_t d_t X + d_t X\, d_t Y$$

Lemma

$$d_t[X,Y]_t = d_t X\, d_t Y$$

Stochastic differential on manifolds

Definition

If X, Y are $\mathbb{R}^n$ valued processes $d_t^M X = d_t^M Y$ iff $X_t = Y_t$ almost surely and $d_t X = d_t Y$. If X, Y are manifold valued processes $d_t^M X = d_t^M Y$ iff $d_t^M f(X) = d_t^M f(Y)$ for all smooth $f : M \to \mathbb{R}$.

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Definition

(Pushforward)
If $F : M \to N$ is smooth and X_t is an M valued process we may define

$$F_*(d_t^M X) := d_t^M F(X)$$

it only depends upon the 2-jet of F.

Stochastic differential equations on manifolds

Given a field of 2-jets $\gamma_x : \mathbb{R}^d \to M$ with $\gamma_x(0) = x$ we can associate an SDE by

$$d_t X = (\gamma_x)_* d_t W$$

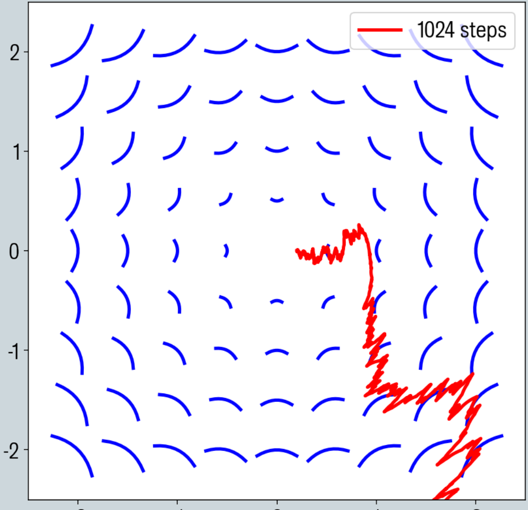
To write this out in classical notation:

- Expand γ_x as a Taylor series to order 2
- Replace the product $d_t W^\alpha \, d_t W^\beta$ with $g^{\alpha\beta} dt$ where $g^{\alpha\beta}$ is the Euclidean metric on $\mathbb{R}^d$

Example

Define a field of 2-jets of curves on $\mathbb{R}^2$ by $\gamma_{(x,y)} : \mathbb{R} \to \mathbb{R}^2$ by

$$\gamma_{(x,y)}(s) = (x, y) + (y, -x)s + 3(x, y)s^2$$



Itô's Lemma

In this notation, the transformation rule for SDEs under a change of coordinates is trivial, but it is opaque when written classically

Lemma

(Itô) If $F : M \to N$ is smooth and X satisfies

$$d_t X = (\gamma_X)_* d_t W$$

then

$$d_t(F \circ X) = (F \circ \gamma_{X_t})_* d_t W$$

Classical Ito's Lemma

Lemma

(Classical Ito's Lemma)
If

$$dX_t^i = a^i(X,t)dt + \sum_{\alpha=1}^{d} b_\alpha^i(X,t)dW_t^\alpha$$

then

$$d(F \circ X)_t^u = \left(\frac{\partial F^u}{\partial x^i} a^i(X,t) + \sum_{\alpha=1}^{d} \frac{1}{2} b_\alpha^i(X,t) b_\alpha^j(X,t) \frac{\partial^2 F^u}{\partial x^i \partial x^j} \right) dt + \sum_{\alpha=1}^{d} \frac{\partial F^u}{\partial x^i} b_\alpha^i dW_t^\alpha$$

Operators associated with stochastic processes

Definition

The *backward diffusion operator* $\mathcal{L}^X$ associated to a stochastic process X is the operator of order 2 acting on 2-jets of functions $f : M \to R$ by:

$$\mathcal{L}^{\mathcal{X}}_{\sqcup} f := \frac{E_t(f_* d_t X)}{dt}$$

The *forward diffusion operator* is its formal adjoint and acts on densities.

Theorem

(Feynman-Kac)
If X is a diffusion and $f : M \to \mathbb{R}$ and $v_t := E_t(f(X_T))$ then

$$\frac{dv}{dt} = \mathcal{L}^{\mathcal{X}}_{\sqcup} v$$

2-jets and connections

Definition

An invariant chart at a point $x \in M$ defined on a neighbourhood $U \ni x$ is a smooth bijection $\phi : T_xM \to U$.

Lemma

Torsion free connections on the tangent bundle of M correspond invariantly to 2-jets of invariant charts.

Since the tangent space has a vector space structure, we can define expectations on the tangent space.

Definition

Given a torsion free connection ∇ define

$$E_t^{\nabla}(d_tX) = (\phi_*)d_t(E\phi_X^{-1}(X))$$

Product structure as a tensor

Let X_s, Y_s be processes on M and suppose that $X_t = Y_t$ almost surely.

Let η^1, η^2 be differential forms defined on M - equivalently fields 1-jets of functions mapping x to 0.

Extend η^1, η^2 arbitrarily to fields of 2-jets and define:

$$(d_t X d_t Y)(\eta^1, \eta^2) := (\eta^1_* d_t X)(\eta^2_* d_t Y)$$

Since the product of differentials only depends on the martingale term, this is well-defined.

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$$\frac{d_t X d_t Y}{dt}$$

is a 2-tensor. For any differentiable X_s

$$\frac{(d_t X)^2}{dt}$$

is a symmetric 2-tensor.

Brownian motion on a manifold

Definition

Brownian motion, X_s, on the Riemannian manifold (M, g) is a stochastic process satisfying:

$$\frac{(d_t X)^2}{dt} = g_{X_t}$$

and

$$E_t^{\nabla^{\mathrm{LC}}}(d_t X) = 0$$

where ∇^{LC} is the Levi-Civita connection.

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It is locally modelled on Brownian motion, with the 2-jets given by the 2-jets of the exponential map.

Thank You!