Dr John Armstrong

Department of Mathematics

Faculty of Natural and Mathematical Sciences

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Department of Mathematics

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The Geometry of Risk

Harry Markowitz, 1927-

- 1990 Nobel Prize for Economics
- ”Modern Portfolio Theory” 1952

Risk and return – Gold vs Bitcoin

Standard deviation and risk

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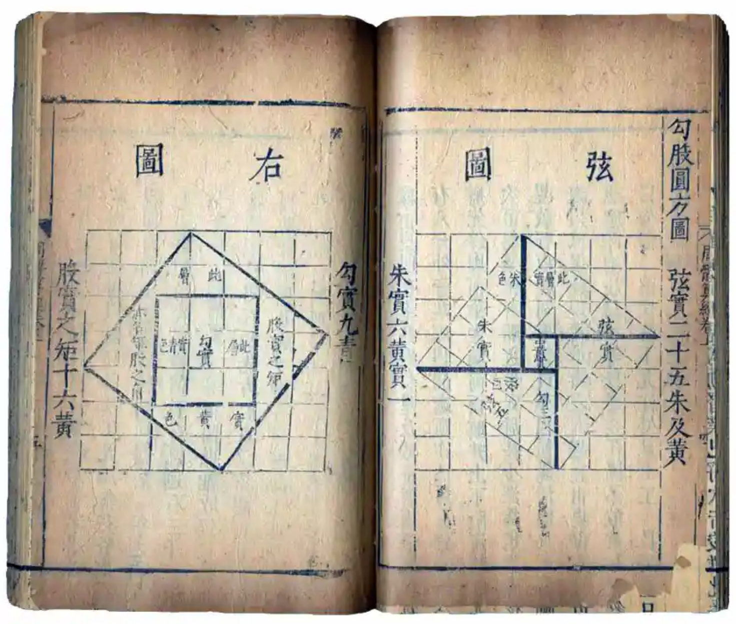
$$|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2$$

Pythagoras of Samos, c570 BC – 495 BC

Pythagoras Advocating Vegetarianism, Sir Peter Paul Rubens 1628-30

Zhoubi Suanjing (周髀算經), 2300 BC – 199 AD



Simple example – a market of identical stocks

In the *market of identical stocks* we have *n* stocks

- Each stock costs \$1
- All the stocks payoffs at time *T* are independent.
- All the stocks are financially indistinguishable: they have mean payoff μ and standard deviation σ.

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We have an amount C to invest. C stands for Capital and it also stands for cost. We will purchase $q_i \in \mathbb{R}$ units of stock i. The letter q stands for quantity. The quantities can be negative. This represents borrowing stocks, just as a negative bank balance represents borrowing money.

$$C = q_1 + q_2 + \ldots + q_n$$

Problem set up

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If X_i is the random variable representing the payoff of stock i, the payoff of our *portfolio* is the random variable

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$$P = \mu q_1 + \mu q_2 + \ldots + \mu q_n = \mu C$$

The variance of our portfolio (i.e. the square of the risk) is

$$\sigma(X^{\text{portfolio}})^2 = q_1^2 \sigma^2 + q_2^2 \sigma^2 + \ldots + q_n^2 \sigma^2 \quad = |\mathbf{q}|^2 \sigma^2$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$

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Problem

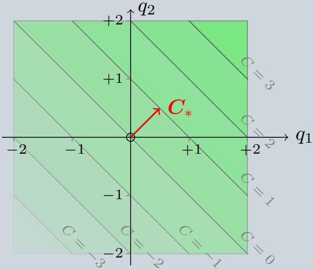
Subject to the condition

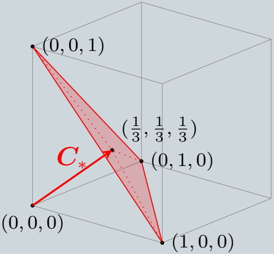
$$C = q_1 + q_2 + \ldots + q_n$$

the expected profit is $C\mu$ for all q_i. What choice of q_i will minimize the risk?

Solution

For the case $C = 1$, we will call the optimal $\mathbf{C}^*$.





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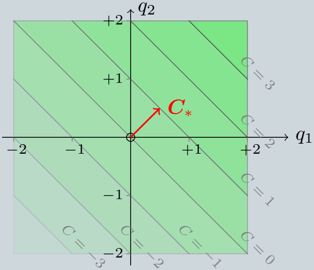

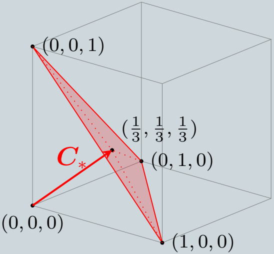

$$\mathbf{C}^* = \left(\tfrac{1}{n}, \tfrac{1}{n}, \ldots, \tfrac{1}{n}\right)$$

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Proof.

By Pythagoras' theorem, if $\mathbf{P}$ is any other point on the hyperplane $C = 1$,

$$|\mathbf{P}|^2 = |\mathbf{C}^*|^2 + |\mathbf{P} - \mathbf{C}^*|^2 \geq |\mathbf{C}^*|^2$$

□

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- This is a central idea in finance and insurance if you can find independent risks you can combine them together to reduce risk.
  - If you run a Casino, you are not taking any risk.
  - If you give insurance to thousands of people, your investment is nearly riskless (unless your model is wrong...)

Markowitz markets

Definition

A Markowitz market is a finite-dimensional real vector space V equipped with

- an inner product $\mathrm{Cov}(X, Y)$;
- a linear function $C : V \to \mathbb{R}$;
- and a linear function $P : V \to \mathbb{R}$.

The points of V are called *portfolios*. The inner product computes the covariance of the payoffs of two portfolios. C represents the cost of the portfolio. P represents the expected payoff.

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Corollary

In a Markowitz market we may assume that the vector space of portfolios is $\mathbb{R}^n$, that the standard deviation of the payoff is given by the distance to the origin and that Cov is the standard inner product.

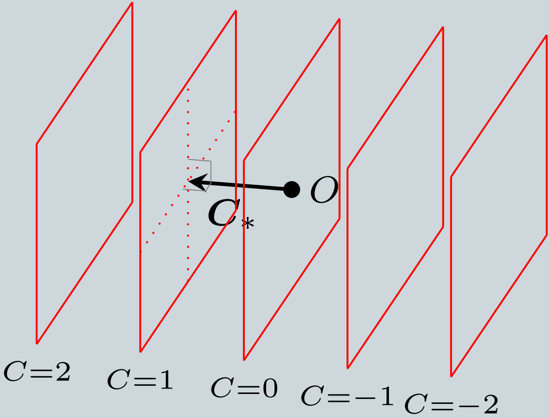
Markowitz's optimization problem

Problem

Find the portfolio q that minimizes the risk for a given cost and expected payoff.

The geometry of the fixed cost

The fixed cost condition means we must lie in a particular hyperplane.



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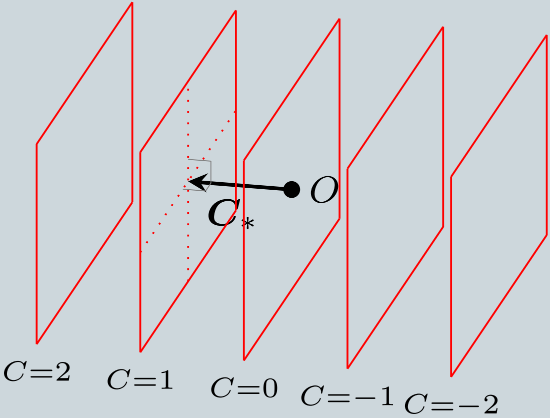

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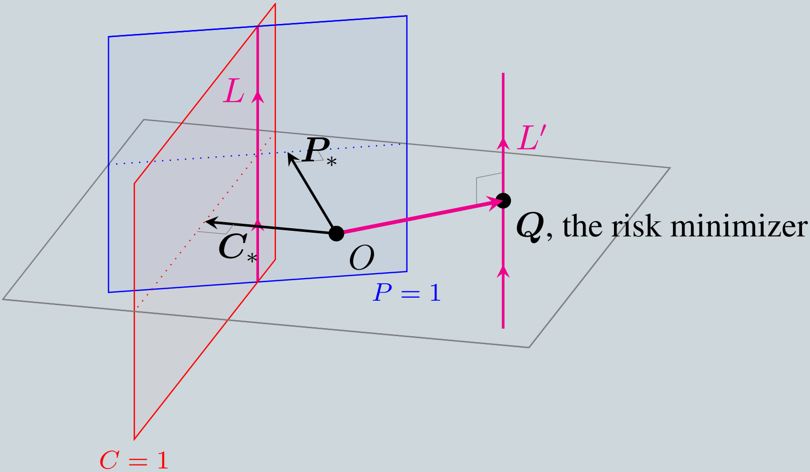
Geometry of the Two Fund Theorem



Theorem

The solution of Markowitz's optimization problem, whatever the cost and expected payoff constraint, lies in the 2-plane spanned by $\mathbf{C}^*$ *and* $\mathbf{P}^*$.

Financial interpretation

- An investment company has lots of customers with different risk and return preferences. They can set up an optimal portfolio for any customer using just a linear combination of two portfolios. Any two portfolios in the plane spanned by $\mathbf{C}^*$ and $\mathbf{P}^*$ will do.

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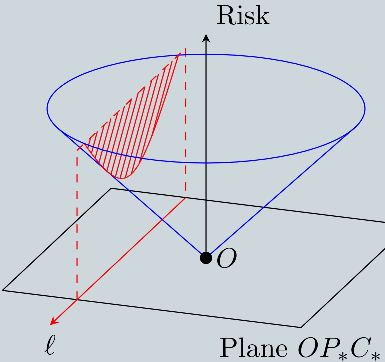
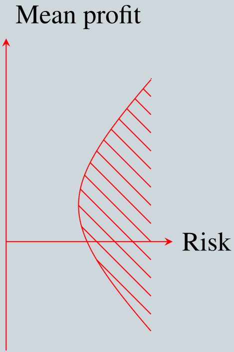
- These two portfolios are the "funds" that give the two fund theorem its name.
- This, together with the idea of diversification, explains why one exchange traded fund stood out as the most highly traded asset in Ryan's example data.

The Efficient Frontier





- For a fixed cost we get a line ℓ of *efficient portfolios*.
- If we plot the risk (standard deviation) of each efficient portfolio against the mean profit we get a picture called the *efficient frontier*.

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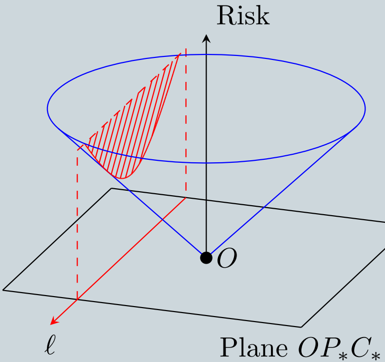

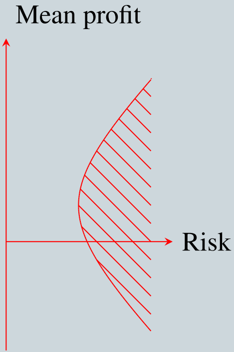

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Theorem

The efficient frontier is a hyperbola.

Isomorphisms of markets

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$$P(x_1, x_2, \ldots x_n) = \alpha x_1, \quad C(x_1, x_2, \ldots x_n) = \beta x_1 + \gamma x_2.$$

We will call these canonical markets*.*

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Rotate $\mathbb{R}^n$ so that P^* lies in the first coordinate direction and C^* lies in the span of the first two coordinate directions. □

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Corollary

A Markowitz market is completely determined up to isomorphism by its dimension and its efficient frontier.

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is a symmetry of a canonical market.
- Therefore only vectors of the form
$$(x_1, x_2, 0, \dots, 0)$$
are invariant under symmetries.

□

Financial interpretation

- Any problem you can write down to identify a specific portfolio in a market that only uses concepts such as risk and expected return that are invariant under isomorphims must yield a portfolio in the plane of efficient portfolios.
- This a substantial generalization of the two fund theorem.
- You can, in effect, solve financial problems about these markets without even knowing what the problem is!

Summary

- You should *diversify* your investments.
- An investment company only needs to create two funds to allow all its customers to invest efficiently.
- Markowitz's theory can be understood geometrically.
- This gives simpler proofs and stronger statements than the classical Lagrange multiplier method. (And don't use weights!)
- Moral: a bit of abstract maths is a good idea.