Introduction to Riemann Surfaces — Lecture 5

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Summary of end of last lecture

Theorem

(Riemann–Hurwitz formula)

Suppose X and Y are compact Riemann surfaces of genus g_X and g_Y respectively and that $f : X \to Y$ is a branched cover. Then

$$(2 - 2g_Y) = d(2 - 2g_x) - \sum_{x \in X}(k_x - 1)$$

Where k_x is the multiplicity of f at x.

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Proof is to triangulate Y with a triangulation so that all critical values are vertices. Then lift the triangulation to Xm and compute Euler characteristics.

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Theorem

(Degree genus formula) A smooth plane curve of degree d has genus $\frac{1}{2}(d-1)(d-2)$.

Bezout's theorem

Definition

Two complex curves in $\mathbb{C}P^2$ intersect transversally at a point p if p is a non-singular point of each curve and if the tangent space of $\mathbb{C}P^2$ at that point is the direct sum of the tangent spaces of the two curves.

Theorem

(Bezout) Two complex curves of degrees p and q that have no common component meet in no more than pq points. If they intersect transversally, they exactly in pq points.

If the polynomial defining a curve factorizes then each factor defines a component of the curve. Smooth curves have only one component because they would clearly not be smooth at ther intersections of the components.

Proof of degree genus formula

- Given a smooth plane curve C of degree d consider the projection from a point p to a line L with p not lieing on C.
- By the fundamental theorem of algebra, the degree of this projection map will be d.
- We can choose coordinates so that the projection of a point (z, w) in affine coordinates is just z. If $P(z, w) = 0$ defines the curve then branch points correspond to points where $P_w = 0$. These have ramification index 1 unless $P_{ww} = 0$.
- By Bezout's theorem we expect there to be $d(d-1)$ branch points and that so long as p does not lie on a line of inflection (i.e. a tangent to the curve through a point of inflection) there will be exactly $d(d-1)$ branch points.
- By Bezout's theorem there are a finite number of lines of inflection (clearly points of inflection will be given by some algebraic condition)
- So for generic p there are exactly $d(d-1)$ branch points of ramification index 1.
- Apply Riemann–Hurwitz formula.

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A map $T : V \longrightarrow \mathbb{C}$ is complex linear if $T(Jv) = iTv$ for all v. It is complex anti-linear if $T(Jv) = -iTv$.

Lemma

Any $\mathbb{R}$-linear map T from V to $\mathbb{C}$ can be written as $T = T' + T''$ where T' is complex linear and T'' is complex anti-linear.

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\begin{aligned}
T' &= \frac{1}{2}(T - iTJv) \\
T'' &= \frac{1}{2}(T + iTJv)
\end{aligned}
$$

Complex structure on a Riemann surface

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Let $T^*_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(T_p, \mathbb{C})$ be the *complex cotangent space*.

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There is a unique complex structure J on T_p such that $\mathrm{d}f$ is a complex linear map with respect to J whenever f is holomorphic.

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Lemma

There is a unique complex structure J on T_p such that $\mathrm{d}f$ is a complex linear map with respect to J whenever f is holomorphic.

Since the definition only depends on the first order terms of f we only need to check that an $\mathbb{R}$-linear map $T : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic if and only if $T(iv) = iT(v)$.

Splitting of one forms

With respect to J we can split the cotangent space into complex linear and complex anti-linear parts. We write the splitting of 1-forms as follows:

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- Note that complex conjugation of $\mathbb{C}$ allows us to define an $\mathbb{R}$ linear map of $\mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{C})$ to itself. We call this complex conjugation too.
- $\Omega^{1,0}$ and $\Omega^{0,1}$ are complex conjugates.

Local coordinates

If $z : U \longrightarrow \mathbb{C}$ is a complex coordinate then writing $z = x + iy$, x and y are real coordinates.

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Equivalently:

$$\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z}), \qquad \mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$$

$\mathrm{d}f$ in complex coordinates

$$
\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\mathrm{d}z + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\mathrm{d}\overline{z} \\
&=: \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}
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- This last line should be seen as defining $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$.

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- This last line should be seen as defining $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$.
- The usual complex analysis definition using limits only makes sense for holomorphic f in which case the two definitions coincide.

∂ and $\overline{\partial}$ in complex coordinates

$$\partial f = \frac{\partial f}{\partial z}\mathrm{d}z, \qquad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

Where by definition:

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\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
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f is holomorphic is equivalent to the statement $\mathrm{d}f = \partial f$ which is equivalent to the statement $\overline{\partial} f = 0$.

Holomorphic and meromorphic 1-forms

It is conventional to write $\mathrm{d} = \overline{\partial}$ when it acts on $(1,0)$ forms and as $\mathrm{d} = \partial$ when it acts on $(0,1)$ forms.

Definition

A $(1,0)$ form $\omega \in \Omega^{1,0}$ is *holomorphic* if $\overline{\partial}\omega = 0$.

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Definition

A meromorphic 1-form is one that can be written as $\omega = f\mathrm{d}z$ with f a meromorphic function.

Contour integration

When you have calculated contour integrals, you have been integrating holomorphic one forms.

Theorem

(Cauchy's theorem) If S is a compact surface with boundary and ω is a holomorphic one form:

$$\int_{\partial S} \omega = 0$$

Definition

(Residue) If p is a pole of a meromorphic 1-form ω then the residue of ω at p is

$$Res_p(\omega) = \frac{1}{2\pi i} \int_C \omega$$

for a small loop C around p.

The Laplace operator

We define Δ by:

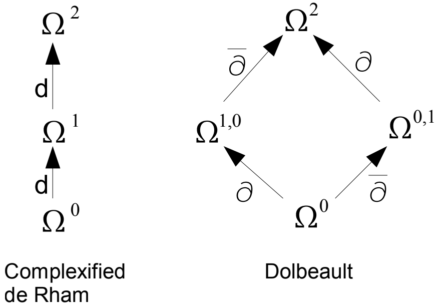
$$\Delta = 2i\bar{\partial}\partial : \Omega^0 \longrightarrow \Omega^2$$

In local coordinates we compute:

$$\begin{aligned}
\Delta f &= \left(2i\frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f\right) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})\right) \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

Dolbeault Cohomology

We have a splitting of the d operator so it is natural to wonder if the De Rham cohomology splits as well.



Dolbeault Cohomology

We define Dolbeault cohomology to be the "cohomology" of $\overline{\partial}$. This means we define:

$$\begin{array}{rcl} H^{0,0} & = & \ker \overline{\partial} \subseteq \Omega^0 = \Omega^0 \\ H^{1,0} & = & \ker \overline{\partial} \subseteq \Omega^{1,0} \\ H^{0,1} & = & \operatorname{coker} \overline{\partial} \subseteq \Omega^{0,1} \\ H^{1,1} & = & \operatorname{coker} \overline{\partial} \subseteq \Omega^{1,1} = \Omega^2 \end{array}$$

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- On an n-manifold, the wedge product of p $(1,0)$-forms and q $(0,1)$-forms defines the notion of a (p,q)-form. So $\Omega^{1,1}$ is just another term for $(\Omega^{1,1})$ and $\Omega^{0,0}$ is just another term for Ω^0.

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- An element of the cokernel is an equivalence class. We will refer to this equivalence class as the cohomology class.

Dolbeault Cohomology

One obvious motivation for considering Dolbeault Cohomology is that the dimensions of these cohomology vector spaces will give us invariants of complex manifolds.

H^2		$H^{1,1}$	
H^1	$H^{1,0}$		$H^{0,1}$
H^0		H^0	
Complexified de Rham		Dolbeault	

Equivalence of De Rham and Dolbeault Cohomology

On a Riemann surface these invariants are trivial. In particular $H^{1,0}$ is isomorphic to $\overline{H}^{0,1}$ and:

$$
\begin{array}{ccccc}
H^2 & \cong & & H^{1,1} & \\
H^1 & \cong & H^{1,0} & \oplus & H^{0,1} \\
H^0 & \cong & & H^0 &
\end{array}
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Complexified
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Key to proof

Theorem

("Main Theorem") If X is a compact and connected Riemann surface then there is a solution f to $\Delta f = \rho$ if and only if $\int_X \rho = 0$. The solution is unique up to the addition of a constant.

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The if is the deep input. Physical arguments suggest solutions to Laplace's equation should always exist. Laplace's equation crops up in gravity, electrostatics, the study of heat etc. It had better have solutions if these theories are going make sense!

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We can represent the cohomology class $[\omega]$ using $\omega + \mathrm{d}f$ for any f.

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The condition that the $1,0$ term of $\omega + \mathrm{d}f$ is holomorphic is equivalent to the requirement $\overline{\partial}\partial f = -\overline{\partial}\omega^{1,0}$ which can easily seen to be equivalent to the condition that the $0,1$ term lies in the kernel of ∂.

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So by the main theorem ω can be written uniquely as an element of $H^{1,0}$ plus an element of $\overline{H}^{1,0}$.

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So by the main theorem ω can be written uniquely as an element of $H^{1,0}$ plus an element of $\overline{H}^{1,0}$.
The results for H^2 and H^0 and the equivalence of $\overline{H}^{1,0}$ and $H^{0,1}$ are similarly easy.

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For the purposes of this course we only really need the result for the sphere and the torus. $H^1(S^2)$ vanishes since the sphere is simply connected. $H^1(T^2) = \mathbb{R}^2$ is a homework exercise.

Interpretation of $H^{0,1}$

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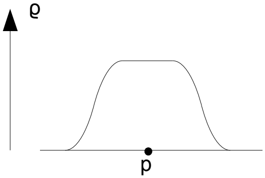
- Define ρ to be a cut-off function equal to 1 in a neighbourhood of p but equal 0 outside of a slightly larger neigbourhood.
- Finding f is equivalent to finding a smooth g on X with $g + \rho\frac{1}{z}$ holomorphic.



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- By definition of $H^{0,1}$ a solution will exist if and only if $[A]$ represents a the trivial element of $H^{0,1}$.
- One can say that $[A] \in H^{0,1}$ is the obstruction to finding a meromorphic function with a simple pole at p and no other poles.

Uniqueness of holomorphic structure on S^2

Corollary

The Riemann sphere is the only Riemann surface of genus 0.

Uniqueness of holomorphic structure on S^2

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The Riemann sphere is the only Riemann surface of genus 0*.*

This is because on a surface of genus 0, $H^{0,1}$ is trivial so one can always find such a meromorphic function. But we have already proved that S^2 is the only Riemann surface that admits a meromorphic function with a simple pole.

Repeating the same argument, if $p_1, p_2, \ldots, p_d$ are d-distinct points then we define $A_1, A_2, \ldots, A_d$ in the same way. We can find a smooth function g satisfying

$$\overline{\partial} g = \lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_d A_d$$

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Since $H^{0,1}$ is g dimensional on a compact Riemann surface of genus g any $g+1$ cohomology classes must be linearly dependent.

Theorem

Given $g+1$ points on a compact Riemann surface X of genus g then exists a non-constant meromorphic function having at worst simple poles at the p_i and no other poles.

Genus 1 surfaces as branched covers

Corollary

A compact Riemann surface of genus 1 *has a meromorphic function with precisely two poles, both of which are simple.*

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Corollary

A compact Riemann surface of genus 1 *is a branched cover of the Riemann sphere of degree* 2 *with four branch points.*

Genus 1 surfaces as branched covers

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Genus 1 surfaces as branched covers

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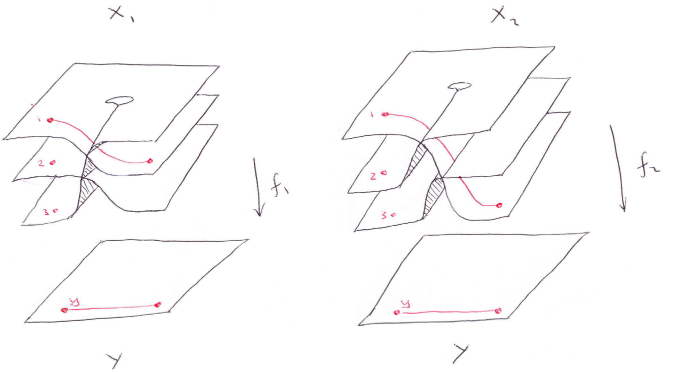
This is a classification theorem for surfaces of genus 1 surfaces because two sheeted branched covers of a sphere are essentially unique.

Uniqueness of branched covers up to monodromy



Uniqueness of branched covers up to monodromy

- Given two branched covers $f_i : X_i \longrightarrow Y$ of degree d with the same critical values, pick a generic point y and label the pre-images of y $x_i^1, x_i^2 \ldots, x_i^d$.

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- This map is well defined if we can choose our labelling such that the "monodromy" around closed loops is the same. The monodromy is defined to be the homomorphism from π_1 to S_d that sends γ to the permuation induced by parallel transport around γ

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- The fundamental group of S_2 with p points removed is generated by loops around the points that have been removed.
- Thus two sheeted branched covers of S_2 are uniquely determined by the critical values.

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- This is obviously a 2 sheeted cover of the sphere with at least 3 branch points using the map $(x, y) \to x$. There must be a branch point at infinity too by the Riemann Hurwitz formula and the fact that a smooth cubic has genus 0. It is easy to check this directly if you prefer.

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- Let $P(w, z) = 0$ be the defining equation of the cubic in inhomogeneous coordinates.
- We have meromorphic functions w and z defined on the cubic. Hence we have a meromorphic forms $\mathrm{d}w$ and $\mathrm{d}z$. By the defining equation for the cubic:

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- So where P_w and P_z are non zero, we have:

$$\frac{\mathrm{d}z}{P_w} = -\frac{\mathrm{d}w}{P_z} =: \omega$$

- Since the cubic is smooth, there are no points where both P_w and P_z vanish. So this defines a single meromorphic 1-form ω that is holomorphic and non-vanishing on $\mathbb{C}^2 \subset \mathbb{C}P^2$.

One can choose coords s.t. ω has no zeros on a cubic

- In general suppose that $P(z,w)$ defines a smooth degree d curve. Let $p(Z_1,Z_2,Z_3)$ be equivalent to P but in homogeneous coordinates, so $p(1,z,w) = P(z,w)$.
- By the fundamental theorem of algebra, there will be d points where the curve intersects the line at infinity (counted with multiplicity). Perturb our coords to ensure that there are exactly d points.
- Suppose that $x = [0,1,0]$ is an intersection point with multiplicity m. Take inhomogeneous coords $[u,1,v] \to (u,v)$. So $u = \frac{1}{z}$, $v = \frac{w}{z}$.
- Define q to be the homogeneous polynomial of degree $d-1$ corresponding to P_w. Since there are exactly d points on the line at infinity, q is non-zero at x and u is a local coordinate.
- $\mathrm{d}z = -u^{-2}\mathrm{d}u$. $P_w(z,w) = q(1,z,w) = z^{d-1}q(u,1,v)$ by homogeneity of q.
- One form is given by: $-\frac{u^{d-3}}{q(u,1,v)}\mathrm{d}u$.

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Any compact Riemann surface X with a non-vanishing holomorphic 1-form ω is biholomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ.

- Define $f : \tilde{X} \longrightarrow \mathbb{C}$ by integrating ω along paths. This is well defined on $\tilde{X}$ the universal cover.
- Show that f is a covering map.
- X is quotient of $\mathbb{C}$ hence equivalent to either $\mathbb{C}/\Lambda$ or a cylinder. Since X is compact, cylinders are ruled out.

Non-vanishing 1-forms implies torus - proof

Definition

A continuous map $F : X \longrightarrow Y$ is a covering map if around each point $y \in Y$ there exists an open neighbourhood V such that $F^{-1}(V)$ is a disjoint union of open sets U_α and F restricted to each U_α is a homeomorphism onto its image.

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- Given $x \in \tilde{X}$ define f_x mapping a neighbourhood of x to some disc $D(f(x), r)$ to be this local homeomorphism — so f_x has a well defined inverse on $D(f(x), r)$.

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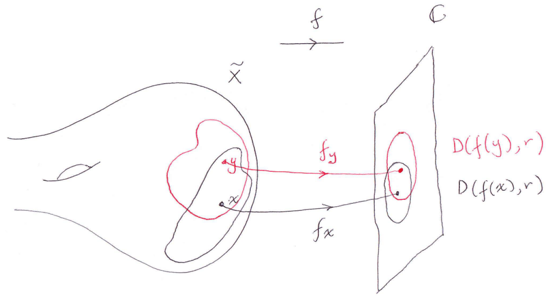
- Given $x \in \tilde{X}$ define f_x mapping a neighbourhood of x to some disc $D(f(x), r)$ to be this local homeomorphism — so f_x has a well defined inverse on $D(f(x), r)$.
- Compactness of X means that we can choose a single value for r that will work for the whole manifold.

Two points less than r apart



Non-vanishing one form implies torus — proof (cont.)

- We say that x and y are *less than r apart* if $y \in f_x^{-1}(D(f(x), r))$. This relationship is symmetric.

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- So $f^{-1}D(y, \frac{r}{2})$ is the disjoint union of Δ_x where $x \in f^{-1}y$.

Summary

- Equivalence of de Rham and Dolbeault cohomology shows that any genus 1 surface is a two sheeted cover with four branch points.
- Monodromy of two sheeted cover of sphere shows that this is a classification result.
- Any two sheeted cover with four branch points can be realised by a non-singular cubic.
- Any non-singular cubic is equivalent to $\mathbb{C}/\Lambda$ because they all have non-vanishing holomorphic one forms.

What have we learned?

- There is only one genus 0 Riemann surface.
- All genus 1 Riemann surfaces are $\mathbb{C}/\Lambda$.
- All genus 1 Riemann surfaces are smooth cubics.
- (Homework) the moduli space of genus 1 Riemann surfaces is $\mathbb{C}$.
- Learning how to prove that Laplace's equation has a unique solution will be a very rewarding pursuit. (Chapter 9 of Donaldson)

Where did we cheat?

- The classification of surfaces assumed lots of Morse theory.
- We have only discussed the fundamental group informally.
- We motivated but didn't prove Bezout's theorem. See Kirwan for details — and perhaps read about "complex quantifier elimination" to understand the handwaving motivation in more detail.
- We didn't prove the existence and uniqueness of solutions to Laplace's equation.