Introduction to Riemann Surfaces — Lecture 4

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KCL

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10. **TODO:** 2 and 3 follow from 4 and 5 given 7 and 8.

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2. We won't prove equivalence of Dolbeault and De Rham cohomology.
3. We *will* show that it is equivalent to the existence and uniqueness of solutions to a certain partial differential equation.
4. In Part II of Donaldson's book he develops enough functional analysis to "solve" this partial differential equation.

Easier reading

1. Our description of the fundamental group has been ultra brief. Any algebraic topology book can fill in the gaps. I learned this from M Armstrong, Basic Topology.
2. Our description of differential forms and calculus on surfaces will proceed at a break-neck pace. Spivak's "Comprehensive introduction to differential geometry" is much much slower.
3. Kirwan's "Complex Algebraic Curves" covers similar ground to this course at a slower pace.

Integration on one manifolds

Suppose $x : U \longrightarrow \mathbb{R}$ and $y : U \longrightarrow \mathbb{R}$ and X are two coordinates on a 1 manifold. Let $\psi = x \circ y^{-1}$ be the transition function. If f is a real valued on $\mathbb{R}$ then:

$$\begin{aligned}\int_{x(U)} f(x)\,\mathrm{d}x &= \int_{y(U)} f(\psi(y))\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y \\ &= \int_{y(U)} f(x(y))\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y\end{aligned}$$

Densities on one manifolds

Definition

A *density at a point p* on a 1-manifold is an equivalence class of a pair (f, x) where f is a number and x is a chart $x \longrightarrow \mathbb{R}$ centered at p. The equivalence relation is given by:

$$(f, x) \sim (g, y) \quad \Leftrightarrow \quad g = f\frac{\mathrm{d}x}{\mathrm{d}y}$$

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We denote the equivalence class $[f, x]$ by $f \, \mathrm{d}x$.

Densities on n-manifolds

If $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism we have:

$$\begin{aligned}\int_U f(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n &= \int_U f(x(y))\partial(\mathbf{x}, \mathbf{y})\mathrm{d}y_1 \dots \mathrm{d}y_n \\ &= \int_{psi(U)} f(\psi^{-1}(y))\partial(\psi, \mathbf{x})^{-1}\mathrm{d}y_1 \dots \mathrm{d}y_n\end{aligned}$$

Where $\partial(\mathbf{x}, \mathbf{y})$ is shorthand for the determinant of the Jacobian matrix.

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We can now define the integral of a density over a manifold. Use a "partition of unity" to define the integral over the entire atlas.

Tangent vectors on 1-manifolds

Definition

A *tangent vector* at a point p on a 1-manifold is an equivalence class of a number v and a chart x with:

$$(v, x) \sim (v\frac{\mathrm{d}y}{\mathrm{d}x}, y)$$

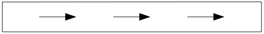
Whereas for a density we had:

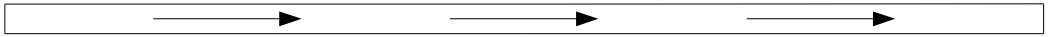
$$(f, x) \sim (v\frac{\mathrm{d}x}{\mathrm{d}y}, y)$$

Transformation of densities and vectors on a 1-manifold

If we change coordinates using $y = 2x$ then, in local coordinates, vectors double in length but densities halve.

On a 1-manifold, densities are dual to vectors. Given a density (p, x) and a vector (v, x) the quantity pv is independent of x. So a density defines an invariant map from the tangent space of p to $\mathbb{R}$. A density is an element of the dual vector space of the tangent space.





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A *vector field* is a smoothly varying choice of vector at each point. The tangent space T_pM at a point p on a manifold M is the set of all tangent vectors at p. It has an obvious vector space structure.

Cotangent vectors on n-manifolds

Definition

A *cotangent vector* p on an n-manifold is an equivalence class of an element $\omega = (\omega_i) \in \mathbb{R}^n$ and a chart $\mathbf{x} = (x^1, \ldots, x^n)$ centered at p with:

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- It is a standard convention to use upper-indices for components of vectors and coordinates and lower-indices for components of forms.
- Equivalently a cotangent vector is an element of $(T_p M)^*$ the dual space of the tangent space. To see this, given a cotangent vector (ω_i) we define a map from the tangent space to $\mathbb{R}$ by $(v^i) \longrightarrow \sum_i \omega_i v^i$. This map does not depend on the choice of coordinates.

The exterior derivative of a function

Given a function f on a manifold and coordinates $\mathbf{x}$ define

$$\mathrm{d}_{\mathbf{x}} f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n})$$

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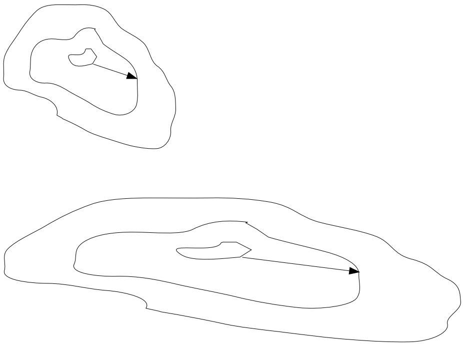
We conclude that $(\mathrm{d}_{\mathbf{x}} f, x)$ and $(\mathrm{d}_{\mathbf{y}} f, y)$ are equivalent cotangent vectors. Hence we have a well defined cotangent vector $\mathrm{d}f$ given independently of our choice of coordinates.

Transformation of covectors and vectors

A good way to draw $\mathrm{d}f$ is to draw its contours. If we rescale by a factor of 2, the terrain becomes shallower by a factor of two as vectors become longer by a factor of 2. The total distance travelled up or down remains constant.



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- We can associated a smooth covector field $\mathrm{d}f$ to a smooth function f. It is somewhat analagous to the gradient of a function, but it is defined independent of coordinates. The standard gradient is only defined up to isometries of $\mathbb{R}^n$ — it depends on the metric.

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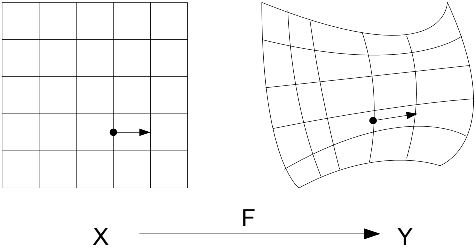
- On 1-manifolds covectors and densities are the same thing — but they're completely different concepts in higher dimensions.

Pushing vectors forward

Given a smooth map $F : X \longrightarrow Y$ betwen smooth manifolds if sending a point $p \in X$ to $q \in Y$ we can define a mapping $F_* : T_pX \longrightarrow T_qY$.



Formal definition of F_*

Given charts **x** for X and **y** for Y. If v^i are the components of a vector V define $F_*(V)$ to have components:

$$(F_*(V))^i = \sum_a \frac{\partial y^i}{\partial x^a} v^a$$

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(Notice that our sums always combine a lower index and an upper index — so long as we think of $\frac{d}{dx^i}$ as having a lower index on account of being the denominator of a fraction. In the *Einstein summation convention*, one drops the $\sum$ symbols and always sums over repeated indices.).

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- Notice that if we have a function $g : Y \longrightarrow \mathbb{R}$ we can define $F^*(g) = g \circ f$ so functions on a manifold "pull back" too.
- Notice that $\mathrm{d}(F^*g) = F^*(\mathrm{d}g)$. You can prove this by a direct calculation, or you can think in terms of contours and say that it is obvious. Both are worth doing!

Areas and volumes in vector spaces

Given a vector space V a good definition of *an area* A for V would be a function that associates an area $A(v_1, v_2)$ to any two vectors v_1 and v_2 that also satisfies:

- Linearity: $A(v_1 + \lambda v_2, v_3) = A(v_1, v_3) + \lambda A(v_2, v_3)$
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In other words we want something that behaves rather like the cross product on 2-vectors. The anti-symmetry condition means that our concept of area detects orientation just as the cross product does.

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With these ideas in mind we define $\Lambda^p V^*$ of a vector space to be the vector space of antisymmetric multi-linear maps from V to $\mathbb{R}$.

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- Divide $\mathbb{R}^p$ into cubes of length ϵ. The edges of each cube correspond to vectors so we can push them forward into M using F. We can then use ω to measure the volume of the cube we have pushed forward.

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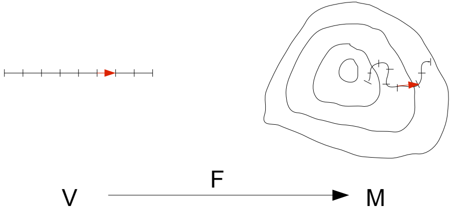
- Define the integral of ω over V by:
$$\int_V \omega = \lim_{\epsilon \to 0} \sum_{\text{cubes}} (p\text{-volume given by } \omega)$$

Integrating $\mathrm{d}f$ on a 1-dimensional submanifold



Fundamental theorem of calculus

The fundamental theorem of calculus is obvious. Given a 1-form ω we write $\omega(X)$ for the length that ω associated to a vector X.

$$\begin{aligned}
\int_V \mathrm{d}f &= \lim_{\epsilon \to 0} \sum_i ((\mathrm{d}f)X_i) \\
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The geometry of the situation is clear. To make the argument rigorous one just needs to use Taylor's theorem to get a bound on the error in the approximation.

Geometric definition of the exterior derivative

Definition

(Non standard) Given a p form ω on a manifold M and vectors $X_1, X_2, \ldots, X_{p+1}$ at a point in M choose a smooth map F from R^{p+1} to M such that F_* sends the coordinate axes to the X_i. Let Δ_ϵ be the the tetrahedron:

$$\Delta_\epsilon = \{(x_1, x_2, \ldots, x_p) : x_i \geq 0, \sum_i (x_i) \leq \epsilon\}$$

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Define $\mathrm{d}\omega$ by:

$$\mathrm{d}\omega(X_1, X_2, \ldots X_{p+1}) = \lim_{\epsilon \to 0} \frac{(p+1)!}{epsilon^{p+1}} \int_{F(\partial \Delta_\epsilon)} (\omega)$$

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- Let $\mathbf{x}$ be a chart centered at a point p on the manifold. Let V be a tangent vector at p and assume that the path $\gamma : \mathbb{R} \to \mathbb{R}$ has tangent vector V at 0.
- Use t to denote the coordinate on $\mathbb{R}$

$$
\begin{aligned}
\mathrm{d}f(X) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\gamma(\partial[0,\epsilon])} f \\
&= lim_{\epsilon \to 0} \frac{1}{\epsilon} (f(\gamma(\epsilon)) - f(\gamma(0)))
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d on 0-forms.

- A 0-form is just a function, f, on a manifold. The integral of 0-form over a 0-dimensional submanifold is just the sum of f over the points in the 0-dimensional submanifold.
- Let $\mathbf{x}$ be a chart centered at a point p on the manifold. Let V be a tangent vector at p and assume that the path $\gamma : \mathbb{R} \to \mathbb{R}$ has tangent vector V at 0.
- Use t to denote the coordinate on $\mathbb{R}$

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- It is now clear from the chain rule that the two definitions we have given for d on 0-forms are equivalent.

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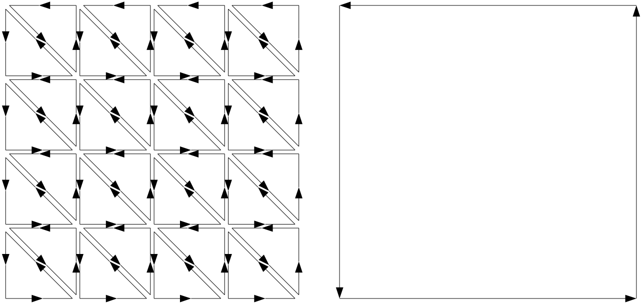
- It satisfies Stokes' theorem $\int_V \mathrm{d}\omega = \int_{\partial V} \omega$.
- It satisfies $\mathrm{d}\mathrm{d}\omega = 0$. This follows from Stoke's theorem because $\partial\partial\Delta_\epsilon$ is empty.

Proof of Stokes' theorem

The definition of d ensures that Stokes' theorem is infinitessimally true.



The wedge product

- Given two 1-forms ω and ν we define $\omega \wedge \nu$ as follows:

$$(\omega \wedge \nu)(X_1, X_2) = \omega(X_1)\nu(X_2) - \omega(X_2)\nu(X_1)$$

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- In general if ω and ν are p and q forms we can define:

$$(\omega \wedge \nu)(X_1, X_2, \dots X_{p+q})) = \frac{1}{p!q!} \sum_{\sigma \in S^n} \mathrm{sgn}(\sigma)\omega(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(p)}) \times \nu(X_{\sigma(p+1)}, X_{\sigma(p+2)}, \dots, X_{\sigma(p+q)})$$

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- Note that $\omega \wedge \nu = (-1)^{pq} \nu \wedge \omega$. So $\wedge$ is anti-commuting on 1-forms.

Formal definition of d

Definition

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For surfaces, this last item simplifies to the special case $\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha$. if f is a function.

Calculating d on a surface

- If (x^1, x^2) are coordinates for S centered at p then $\{\mathrm{d}x^1, \mathrm{d}x^2\}$ gives a basis for $T_p^* M$.

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- We can write any 1-form as $\alpha_1 \mathrm{d}x^1 + \alpha_2 \mathrm{d}x^2$. Using the axioms we compute:

$$\begin{aligned}
\mathrm{d}(\alpha_1 \mathrm{d}x^1 + \alpha_2 \mathrm{d}x^2) &= (\mathrm{d}\alpha_1) \wedge \mathrm{d}x^1 + (\mathrm{d}\alpha_2) \wedge \mathrm{d}x^2 \\
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- Notice that this proves that d is determined by the axioms (on a surface).

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- To check that my non-standard definition is correct, simply check that it satisfies the axioms.
- The standard definition is the more practical choice for most computations.

The Poincaré Lemma

Theorem

On $\mathbb{R}^2$ *if* ω *is a* 1*-form and* $\mathrm{d}\omega = 0$ *there exists a function* f *with* $\mathrm{d}f = \omega$.

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Definition

An *exact* p-form ω is one that can be written $\omega = \mathrm{d}\nu$ for some $(p-1)$-form. exact forms are always closed.

Proof of the Poincaré lemma

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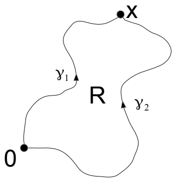
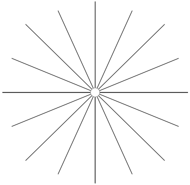
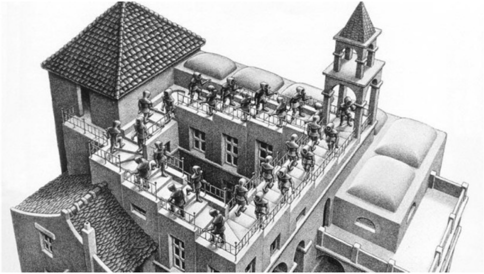
Define $f(x) = \int_{\gamma_1} \omega$. Since $\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_R \mathrm{d}\omega = 0$ we see that f is well defined. By the fundamental theorem of calculus $\mathrm{d}f = \omega$. (Result follows because $\mathbb{R}^2$ is simply connected.)



A closed form ω on $\mathbb{R}^2\backslash\{0\}$ which is not exact





De Rham cohomology

- For clarity write $\mathrm{d}_i = \mathrm{d} : \Omega^{i-1}(M) \longrightarrow \Omega^i(M)$ on an n-manifold M. We have the *exact sequence*:

$$0 \xrightarrow{\mathrm{d}_0} \Omega^0(M) \xrightarrow{\mathrm{d}_1} \Omega^1(M) \xrightarrow{\mathrm{d}_2} \Omega^2(M) \xrightarrow{\mathrm{d}_3} 0$$

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- We have shown that the 1-st betti number is zero for simply connected spaces, but non-zero for $\mathbb{R}^2$.

Bezout's theorem

Definition

Two complex curves in $\mathbb{C}P^2$ intersect transversally at a point p if p is a non-singular point of each curve and if the tangent space of $\mathbb{C}P^2$ at that point is the direct sum of the tangent spaces of the two curves.

Theorem

(Bezout) Two complex curves of degrees p and q that have no common component meet in no more than pq points. If they intersect transversally, they exactly in pq points.

If the polynomial defining a curve factorizes then each factor defines a component of the curve. Smooth curves have only one component because they would clearly not be smooth at ther intersections of the components.

Proof of degree genus formula

- Given a smooth plane curve C of degree d consider the projection from a point p to a line L with p not lieing on C.
- By the fundamental theorem of algebra, the degree of this projection map will be d.
- We can choose coordinates so that the projection of a point (z, w) in affine coordinates is just z. If $P(z, w) = 0$ defines the curve then branch points correspond to points where $P_w = 0$. These have ramification index 1 unless $P_{ww} = 0$.
- By Bezout's theorem we expect there to be $d(d-1)$ branch points and that so long as p does not lie on a line of inflection (i.e. a tangent to the curve through a point of inflection) there will be exactly $d(d-1)$ branch points.
- By Bezout's theorem there are a finite number of lines of inflection (clearly points of inflection will be given by some algebraic condition)
- So for generic p there are exactly $d(d-1)$ branch points of ramification index 1.
- The degree genus formula now follows from the Riemann