MOCK EXAM ANSWERS – "INTRODUCTION TO RIEMANN SURFACES"

1. Suppose to begin with that f is holomorphic, f is ~~[illegible]~~ non constant and $f(0)=0$.

Recall that the zeros of a non-constant holomorphic function are isolated. So we can find a disc D around 0 such that the only point on $\bar{D}$ where f vanishes is the origin.

By compactness of ∂D, there is some $\delta > 0$ such that $|f(z)| > 2\delta$ for all $z \in \partial D$. So if $|w| < \delta$ then $f(z) - w$ does not vanish for z on the boundary of D.

Therefore the number of solutions of $f(z) = w$ in D counted with multiplicity is given by

$$n(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z) - w} \, dz$$

so long as $|w| < \delta$. Since n is ~~It follows that the~~ continuous for ~~all~~ w with $|w| < \delta$ and we know $n(0) > 0$ we see that $n(w) > 0$ $\forall w$ with $|w| < \delta$.

So the ball $B(0;\delta) \subseteq f(D)$.

We have just proved the open mapping theorem for non constant holomorphic functions.

Suppose now that $f'(0) \neq 0$. This means $n(0) = 1$ by definition of multiplicity.

So for every $w \in B(0; \delta)$ we have a unique solution $z \in D$ to the equation $f(z)=w$.

Define $g(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{w f'(z)}{f(z)-w} dz$. Since ~~if~~ $f(z)=w$ has a single root in D it is a standard result in complex analysis that this root is given by $g(w)$.

It is clear from this formula that g is holomorphic.

The result now follows from the open mapping theorem.

2. (i) <u>Liouville's theorem</u>: A bounded holomorphic function defined on the whole of $\mathbb{C}$ is constant.

(ii) $\mathbb{C}$ is diffeomorphic but not biholomorphic to the unit disc D. By Liouville's theorem $\nexists$ any non constant map from $\mathbb{C} \to D$.

iii) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$. f' cannot be equal to zero everywhere as that would imply f is constant. So, after making a transformation of the form $z \to az + b$ we may assume $f(0) = 0$ and $f'(0) = 1$.

So $1/f$ has a pole of order 1 at 0.

Thus $1/f - \beta/z$ is an entire holomorphic function for some $\beta \in \mathbb{C}$.

Let D be the unit disc. Since f is an automorphism, $f^{-1}(\bar{D})$ is compact. ~~so f is~~

~~Since f is 1-1, z ∉ f(D), |f(z)| > 1 ∀z ∈ C ∖ f(D)~~

$\therefore$ $1/f - \beta/z$ is bounded on $f^{-1}(\bar{D})$. On the other hand f is 1-1, so $|f(z)| > 1$ $\forall z$ outside of $f^{-1}(\bar{D})$. $\therefore$ $1/f - \beta/z$ is bounded and holomorphic hence constant.

So $\quad 1/f - \beta/z = \alpha \quad$ for some α

Hence $\quad f = \dfrac{z}{\alpha z + \beta}$

It is easy to see that f is holomorphic iff either $\alpha = 0$ or $\beta = 0$. The condition $f'(0) = 1$ then implies $\alpha = 0$ and $\beta = 1$.

(iii) Let $\Lambda_i = \{ m + \tau_i n \mid m, n \in \mathbb{Z} \}$ be a lattice for $i \in \{1, 2\}$.

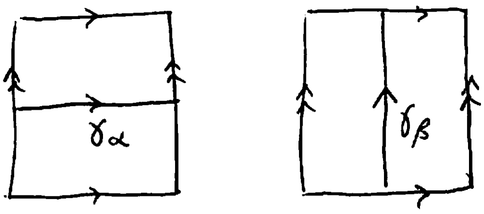
Let $\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a biholomorphism. Then ϕ lifts to the universal cover to give ~~a map~~ an automorphism $\tilde{\phi} : \mathbb{C} \to \mathbb{C}$. By making a translation of $\mathbb{C}$ we may assume that $\tilde{\phi} : \Lambda_1 \to \Lambda_2$.

So $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are biholomorphic if and only if there is a map $z \to az + b$ sending Λ_1 to A_2. As an example take Λ_1 to be a square lattice and Λ_2 to be rectangular. $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are both diffeomorphic to the torus

but $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are not

biholomorphic.

3. Let $\gamma_\alpha, \gamma_\beta$ be the following paths on T^2:



Define $\phi : H^1(T^2) \longrightarrow \mathbb{R}^2$ by

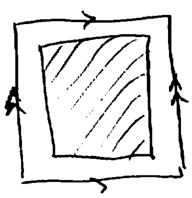
$$\phi([\omega]) = \left(\int_{\gamma_\alpha} \omega , \int_{\gamma_\beta} \omega\right)$$

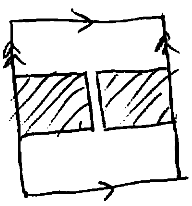
for a closed form ω. This is well defined on cohomology by Stokes' theorem.

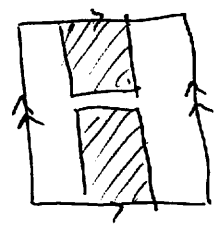
Thinking of T^2 as $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ we have ~~closed forms~~ a coordinate $z = x+iy$ and corresponding closed forms dx, dy. Since these are translation invariant, they define forms on T^2.

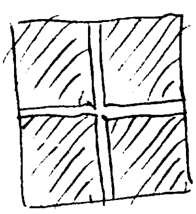
Clearly $\phi(\alpha dx + \beta dy) = (\alpha, \beta)$ so ϕ is onto.

Divide T^2 into four open sets each homeomorphic to $\mathbb{R}^2$ as shown below:

$A^1 =$ 

$A^2 =$ 

$A^3 =$ 

$A^4 =$ 

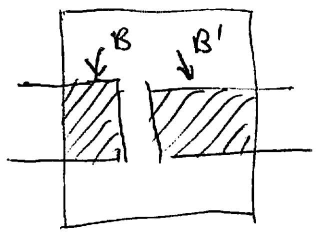
Suppose ω is a closed form with ~~dx~~ $\phi[\omega] = 0$

Since $H^1(\mathbb{R}^2)=0$ we can find functions f_i defined on A_i with

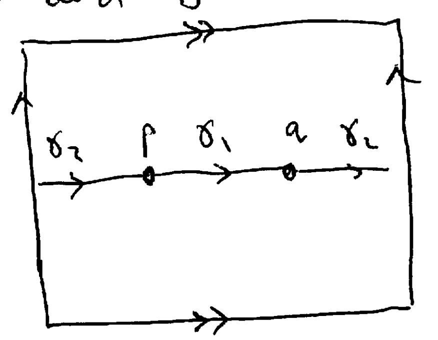
$$df_i = \omega$$

These f_i are unique up to the addition of a constant.

$A^1 \cap A^2$ splits into two connected components which we will call B and B'.



The path γ_α ~~split into two~~ can be written $\gamma = \gamma_1 + \gamma_2$ with ~~and points~~ $\gamma_1 \subset A_1$ and $\gamma_2 \subset A_2$ and end points p and q in B and B':



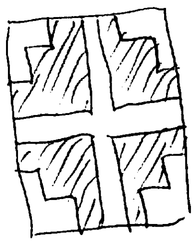
We may assume WLOG that $f_1(p) = f_2(p)$.

~~By Stokes theorem on~~

$$0 = \int_{\gamma_2} \omega = \int_{\gamma_1} df_1 + \int_{\gamma_2} df_2 = f_1(q) - f_1(p) + f_2(p) - f_2(q)$$
$$= f_1(q) - f_2(q).$$

Since B and B' are connected and $d(f_1 - f_2) = 0$ on B, B' we see that $f_1 = f_2$ on B and B'.

Hence we can define f on $A_1 \cup A_2$ with $df = \omega$. We can similarly define f on $A_1 \cup A_2 \cup A_3$ with $df = \omega$.



$(A_1 \cup A_2 \cup A_3) \cap A_4$ is connected so $f - f_4$ is constant on $(A_1 \cup A_2 \cup A_3) \cap A_4$. So without loss of generality $f = f_4$ on this intersection. So we can extend the definition of f to T^2.

So ω is exact.

We deduce that ϕ is 1-1.

4. Let Σ be a compact genus 0 Riemann surface. By the classification theorem for surfaces Σ is diffeomorphic to T^2, hence $H^1(\Sigma) = \mathbb{R}^2$. (De Rham cohomology is a topological invariant and easy to calculate for T^2).

Thus $\dim H^{1,0} = 1$. So there exists a non-zero holomorphic 1 form ω on Σ.

Counted with multiplicity, the number of zeros of a real 1-form on Σ is $2g-2=0$. Since ω is holomorphic all the multiplicities of zeros of $\omega + \bar{\omega}$ are positive. Thus omega is a non-vanishing holomorphic 1-form.

We now define a map $f : \tilde{\Sigma} \to \mathbb{C}$, where $\tilde{\Sigma}$ is the universal cover of Σ. Pick a point p_0 in $\tilde{\Sigma}$ and define

$$f(p) = \int_{\gamma_p} \omega$$

where γ is a contour stating at p_0 and ending at p. f is well defined because $\tilde{\Sigma}$ is simply connected ~~by Stokes~~ ~~theorem~~ – here we are using Stokes theorem and the fact that ω is closed.

Because f is defined as an integral ~~in~~ ~~local coordinates~~ of a non vanishing form, we see that, in local coordinates, f' is non vanishing. Thus by the inverse function theorem f is a local homeomorphism. Using this and the compactness of Σ one can show that f is a covering map (which implies f is onto).

Since $\mathbb{C}$ is simply connected we deduce that f is an isomorphism. Therefore Σ is the quotient of $\mathbb{C}$ by a group of holomorphic automorphisms.

The holomorphic automorphisms of $\mathbb{C}$ are, by Liouville's theorem, just the maps

$$z \to az + b \qquad a, b \in \mathbb{C}$$

It follows from this that the only quotients of $\mathbb{C}$ are $\mathbb{C}/\Lambda$ for some lattice and the cylinder $\mathbb{C} \setminus \{0\}$. Since Σ is compact we must have $\Sigma \cong \mathbb{C}/\Lambda$.