King's College London

University Of London

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc Examination

7CCMFM06 Numerical and Computational Methods in Finance

Summer 2017

Time Allowed: Two Hours

All questions carry equal marks.
Full marks will be awarded for complete answers to FOUR questions.
If more than four questions are attempted, then only the best FOUR will count.

NO calculators are permitted.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) What does it mean to say that "a trader sells a call option on a stock and then hedges their exposure using the discrete-time delta-hedging trading strategy"? **[20%]**

Answer: (Bookwork.) It means that at fixed time points t_i, they invest in stock and a risk free bank account to ensure that the total delta of their portfolio, computed using the Black–Scholes formula, is equal to zero.

(ii) Suppose that a trader does pursue this investment strategy and rehedges at a finite number of time points t_i. Derive the difference equations you would use to simulate the outcome of this strategy.

Answer: (Bookwork.) Let b_i denote the bank balance at time point i, S_i, the stock price at time i, Δ_i the delta. At the initial time:

$$b_0 = P - \Delta_i S_i$$

At intermediate times:

$$b_i = e^{r(t_i - t_{i-1})} b_{i-1} - (\Delta_i - \Delta_{i-1}) S_i$$

At the final time:

$$b_n = e^{r(t_n - t_{n-1})} b_{n-1} + \Delta_{n-1} S_i - (S_n - K)^+$$

[30%]

(iii) Explain what is meant by a utility function and describe how you would estimate the trader's expected utility. **[20%]**

Answer: (Similar to Bookwork.) A utility function is a function $f : \mathrm{R} \to \mathrm{R}$ which assigns to a final portfolio value V, a value $f(V)$ which indicates how happy the investor is with this outcome. The function is usually assumed to be increasing because profit is better than loss and concave to indicate risk-aversion.

To compute the expected utility, simply simulate delta hedging repeatedly and compute the at the final time. The sample mean is an unbiased estimator for the expected utility.

(iv) How would you estimate the accuracy of your calculation of the expected utility? **[10%]**

Answer: (Similar to Bookwork.) The sample standard deviation can be used to estimate the population standard deviation, σ, of the utility. By the central limit theorem, the sample mean will be approximately normally distributed with standard deviation

$$\frac{1}{\sqrt{N}}\sigma.$$

(v) Suppose that the stock follows the Black-Scholes model and that the trader re-hedges at N evenly spaced time points. Is it always true that the trader's expected utility will increase as $n \to \infty$? Justify your answer. [**20%**]

Answer: (Unseen.) No. The delta hedging strategy is risk free in the limit, so one expects the utility to converge to the utility of a bond purchase. The delta hedging strategy with a single time step involves investing in a mixture of the stock and the bond. The expected utility of this depends upon the drift of the stock, but for at least some drifts and utility functions, stock investments will be preferable to bond investments, in which case the utility will be higher when $N = 2$ than it is as $N \to \infty$.

2. (i) A stock price process S_t follows geometric Brownian motion

$$\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t)$$

write a MATLAB function to simulate M price paths at the $N+1$ evenly spaced points $S_0, S_{\delta t}, S_{2\delta t}, \dots S_T$ where $\delta t = \frac{T}{N}$. [**30**%]

Answer: (Bookwork.)

```
function [ S, times ] = generateBSPaths( ...
    T, S0, mu, sigma,nPaths, nSteps  )

dt = T/nSteps;
logS0 = log( S0);
W = randn( nPaths, nSteps );
dlogS = (mu-0.5*sigma^2)*dt + sigma*sqrt(dt)*W;
logS = logS0 + cumsum( dlogS, 2);
S = exp(logS);
times = dt:dt:T;

end
```

(ii) Describe how you would use this code to approximate the risk-neutral price of a discrete-time Knock-out call option with strike K, maturity T and barrier $B > S_0$ in the Black–Scholes model by the Monte Carlo method. (Recall that by definition of this option, if at any time $i\delta t$ where $i \in \{0, 1, 2, \dots, N\}$ the price is above the barrier B the option will have a payoff of zero. Otherwise its payoff is given by $\max\{S_T - K, 0\}$.) [**20**%]

Answer: (Bookwork.) First simulate M price paths in the risk neutral measure (i.e. set $\mu = r$). Compute the payoff for each price path by first computing the maximum along each price path. If this is greater than B the payoff is zero, otherwise the payoff is $\max\{S_T - K, 0\}$. Now compute the mean payoff, M. The price is $e^{-rT}M$.

(iii) How would you estimate the error in your answer? [**10**%]

Answer: (Bookwork.) One would estimate the standard deviation of the payoff using the sample standard deviation of the payoffs $\hat{\sigma}$. The error will be normally distributed with standard deviation $\frac{1}{\sqrt{M}}\hat{\sigma}$.

(iv) How would you apply the control variate method to decrease the error in your answer? [**30**%]

Answer: (Unseen.) I would use a vanilla European call option with the same strike is a control variate as this can be priced analytically. The price of the knock out option can be estimated as the mean of

$$e^{-rT}P_K + c(e^{-rT}P_C - b)$$

where P_K is the payoff of the knockout option in a given scenario, P_C is the payoff of the vanilla call option and b is the Black–Scholes price of the vanilla call option. The value of the parameter c can be chosen to minimize the sample variance of this estimate by taking

$$c = \frac{\mathsf{Cov}(P_K, P_C)}{\mathsf{Var}(P_C)}$$

(v) Suppose that simple Monte Carlo with $100,000$ samples is accurate to within 2 cents and that using the control variate method with the same number of samples the answer is accurate to within 1 cent. Estimate how many samples would be needed to make the simple Monte Carlo method as accurate as the control variate method. Explain your answer. [**10%**]
Answer: (Unseen.) One would need $400,000$ samples as the error is proportional to $\frac{1}{\sqrt{N}}$.

3. (i) Define the term *pseudo square root*. **[10%]**

Answer: (Bookwork.) A pseudo square root of a symmetric matrix M is a matrix L such that $LL^T = M$.

(ii) Define the term *Cholesky decomposition*. **[10%]**

Answer: (Bookwork.) The Cholesky decomposition of a positive definite symmetric matrix M is the unique pseudo square root of M with positive diagonal.

(iii) Let N_1, N_2 and N_3 be independent Gaussian random variables of mean 0 and standard deviation 1. Suppose that X_1, X_2 and X_3 are random variables defined by:

$$\begin{aligned} X_1 &= 2N_1 + N_3 \\ X_2 &= 3N_2 \\ X_3 &= N_1 + 4N_2 + N_3 \quad . \end{aligned}$$

What is the covariance matrix of (X_1, X_2, X_3)? **[20%]**

Answer: (This question and the next is easy if the student realises they should use matrix notation, but this is an unseen question and I expect this will not be obvious to many students.) We calculate the covariance matrix as:

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 9 & 12 \\ 3 & 12 & 18 \end{pmatrix}$$

(iv) Write down two distinct pseudo square roots of this covariance matrix. **[20%]**

Answer: (Unseen.)

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -4 & -1 \end{pmatrix}$$

(v) In the Markowitz model, assets returns over a time period T are assumed to be normally distributed with covariance matrix Σ and mean vector μ. Explain how the Cholesky decomposition could be used to simulate asset returns in this model. **[20%]**

Answer: (Bookwork.) Let L be a pseudo square root of Σ. Simulate a vector n of independent normally distributed random variables of mean 0 and standard deviation 1, then $\mu + Ln$ will have the desired distribution.

(vi) Prove that there is no real valued square matrix L such that

$$LL^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Answer: This is an unseen test that the student understands the relevance of positive definiteness. Let x be any vector and L any matrix then $x^T LL^T x = |L^T x|^2 \geq 0$ for all x. However

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \leq 0.$$

So taking

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we see that

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

cannot be written as LL^T for any L. **[20%]**

4. (i) Let X_t be a stochastic process which solves the SDE

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\,\mathrm{d}t + \left(\sqrt{1+X_t^2}\right)\,\mathrm{d}W_t, \qquad X_0 = 0$$

where W_t is Brownian motion. Write down the Euler scheme for X_t. **[20%]**

Answer: (Similar to Bookwork.)

$$X_{t+\delta t} = X_t + \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\,\delta t + \left(\sqrt{1+X_t^2}\right)\,\sqrt{(\delta t)}\epsilon_t, \qquad X_0 = 0$$

where ϵ_t is normally distributed with mean 0 and variance 1.

(ii) Show that $X_t = \sinh(t + W_t)$. **[20%]**

Answer: (Unseen.) Simply apply Itô's lemma and the uniqueness of solutions to SDEs whose coefficients are smooth with linear growth bounds.

(iii) How would you simulate X_t in practice? Write a MATLAB function that produces a matrix of M simulations of X_t over a time interval T with N time steps. **[30%]**

Answer: (Similar to bookwork.)

```
function X = simulateX( T, N, M )
dt = T/N;
epsilon = sqrt(dt)*randn(N,M);
tPlusW = cumsum(dt + epsilon);
X = sinh(tPlusW);
end
```

(iv) Describe a graph you could plot to test how rapidly the Euler scheme for X_t converges to the true solution of the stochastic differential equation. Briefly describe how you could produce this graph in MATLAB. What result would you expect? **[30%]**

Answer: (Bookwork.) Simulate X_t using the Euler scheme and using the function given above. Take the difference of the simulated values at the final time T and then square this to get an error estimate value e for each simulation. Now compute the root mean squared value of e for a large number of simulations. Plot a log-log plot of the root mean squared error against N. One would expect this graph to look roughly linear and to have a slope of $-\frac{1}{2}$.

5. (i) The backwards heat equation is

$$\frac{\partial u}{\partial t} = -\sigma^2 \frac{\partial^2 u}{\partial x^2}.$$

You are given the condition $u(x, T) = f(x)$ where f is a piecewise smooth bounded real function, and wish to solve this equation numerically using the explicit finite difference method. Derive the difference equations you would use to find the solution to the backward heat equation at time 0. [**30%**]

Answer: (Bookwork.) Students should draw the stencil for the explicit method. Write $u_{i,j}$ as short-hand for $u(i\delta t, j\delta x)$.

$$\frac{\partial u}{\partial t} \approx \frac{u(t,x) - u(t-\delta t, x)}{\delta t} = \frac{u_{i,j} - u_{i-1,j}}{\delta t}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(t, x-\delta x) - 2u(t,x) + u(t, x+\delta x)}{(\delta x)^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\delta x)^2}$$

Hence the difference equations used are:

$$u_{i-1,j} = u_{i,j} + \sigma^2 \frac{\delta t}{\delta x^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})$$

(ii) Explain briefly how solving the heat equation can be used to price derivatives in the Black–Scholes model with an interest rate of 0. [**20%**]

Answer: (Bookwork.) The risk-neutral price process is geometric Brownian motion with drift 0. Hence the log of the stock price s_t obeys the SDE:

$$\mathrm{d}s_t = -\frac{\sigma^2}{2}\mathrm{d}t + \sigma \mathrm{d}W_t$$

Thus the process $s_t - \frac{\sigma^2}{2}t$ is the Brownian motion W_T. By Feynman–Kac, we can calculate the expected value of a function of Brownian motion by solving the bnckward heat equation with this function as final condition. A European derivative whose payoff is a function of S_T can be rewritten to express the payoff in terms of W_T. Hence the expected payoff in the risk neutral measure can be computed by solving the heat equation with the expected payoff as boundary condition.

(iii) When is the explicit finite difference method stable? [**10%**]

Answer: (Bookwork.) When

$$1 - 2\sigma^2 \frac{\delta t}{(\delta x)^2} \geq 0$$

(iv) Give a probabilistic interpretation of the difference equations you derived in the first part of the question. [**20%**]

Answer: (Bookwork.) If one approximates Brownian motion on the same grid with a process that over each time interval δt can move up δx with probability $\sigma^2 \delta t (\delta x)^2$, down with the same probability and remain at the same x value with probability $1 - 2\sigma^2 \frac{\delta t}{(\delta x)^2}$, then our finite difference equation gives the expected value of u at time $t - \delta t$ in terms of the values of u at time t. This probabilistic interpretation is only valid if the stability condition:

$$1 - 2\sigma^2 \frac{\delta t}{(\delta x)^2} \geq 0$$

holds since we require that probabilities are positive.

(v) Let δt be the time step used for finite difference method and δx the space step. Suppose we wish to compute $u(0, T)$ for some fixed time T. Let $u^N(0,0)$ denote the value computed by using the finite difference method with $\delta T = \delta X = \frac{T}{N}$. Show that in this case we can find a function f such that $u^N(0,0)$ does not converge to the correct answer as $N \to \infty$. [**20%**]

Answer: (Bookwork but very unlike any questions asked in previous years.) Suppose that f is equal to 0 on the interval $[-T, T]$ and equal to 1 outside this interval. The expectation of $f(W_t)$ is strictly positive, yet the only terms in the finite difference scheme that affect the value of u at $(0,0)$ are contained in the triangle with vertices $(0,0)$, (T,T), $(-T,T)$. Hence the explicit finite difference scheme will estimate the value of f to be zero for all N. Hence it cannot possibly converge.