King's College London

University Of London

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

PLACE this paper and any answer booklets **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc Examination

7CCMFM06 Numerical and Computational Methods in Finance

August 2019

Time Allowed: Two Hours

All questions carry equal marks. Full marks are awarded for correct answers to FOUR questions. If more than four are attempted then only the best four will count.

You are permitted to use a Calculator.
Only calculators from the Casio FX83 and FX85 range are allowed.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) What is meant by the terms $\mathbb{P}$-measure and $\mathbb{Q}$-measure? **[20%]**

Answer: (Bookwork) A $\mathbb{P}$-measure is a probability measure which models the chance of particular events occurring. A $\mathbb{Q}$-measure is a probability measure which does not model the chance of events but instead allows prices to be determined by computing discounted expectations with respect to this measure.

(ii) In the Black–Scholes model one can simulate a stock price using either the $\mathbb{P}$- or the $\mathbb{Q}$-measure. State which should you use for the following tasks:

(a) Testing the effectiveness of discrete time delta hedging by simulating the use of this strategy in practice.

(b) Computing Value at Risk by the Monte Carlo method.

(c) Computing the Black–Scholes price of an option by the Monte Carlo method.

[10%]

Answer: (Bookwork) (a) $\mathbb{P}$ (b) $\mathbb{P}$ (c) $\mathbb{Q}$.

(iii) What stochastic differential equation does a stock price satisfy in the Black–Scholes model? Give the answer for both the $\mathbb{P}$- and $\mathbb{Q}$-measures. **[10%]**

Answer: (Bookwork) In the $\mathbb{P}$-measure:

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

In the $\mathbb{Q}$-measure:

$$\mathrm{d}S_t = S_t(r \mathrm{d}t + \sigma \mathrm{d}W_t)$$

both have initial condition 0.

(iv) Describe mathematically how you would simulate the stock price in the $\mathbb{P}$-measure model at a final time T given the stock price at time 0, justifying your answer.

Answer: (Bookwork) By Itô's Lemma:

$$\mathrm{d}(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) \mathrm{d}t + \sigma \mathrm{d}W_t.$$

This is Brownian motion with constant coefficients. We integrate to obtain

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

Let ϵ be a normally distributed random variable with mean 0 and standard deviation 1 which we can simulate using `randn`. We can then simulate S_T by computing

$$\log S_T = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{T}\epsilon.$$

Note that only one time step should be used. **[30%]**

(v) Write a MATLAB function to simulate n samples of the stock price at time T. **[20%]**

Answer: (Similar to Bookwork)

```
function ST = simulateStock( S0, T, mu, sigma, n )
ST = exp( log(S0) + (mu - 0.5*sigma^2 ) t + ...
sigma*sqrt(T)*randn( nSamples, 1 ) );
end
```

To get full marks students should observe that they do not need to simulate intermediate times and should adapt their mathematics and code accordingly.

(vi) Use your answer to write the MATLAB code to price a digital call option of strike K and maturity T. **[10%]**

Answer: (Unseen) Use the formula price $= e^{-rT} \times E_{\mathbb{Q}}(payoff)$. In this example this can be computed as follows

```
S = simulateStock(S0,T,r,sigma,n);
price = exp(-r*T)*sum( S>K )/n
```

2. (i) Explain what is meant by a utility function. Give an example and sketch its graph. [**20%**]

Answer: (Bookwork) A utility function is an $\mathbb{R}\cup\{-\infty\}$ valued function mapping the final wealth of an investor to their subjective "happiness". Typically utility functions are concave (indicating risk-aversion) and increasing (indicating a preference for wealth over poverty). An example is the exponential utility $u(x) = 1 - e^{-x}$.

(ii) Trader A writes a call option for a price P price with strike K and maturity T. They hedge the option using the delta hedging trading strategy at the discrete time points $0, \delta t, 2\delta t, \ldots, n\delta t = T$. Derive difference equations that allow their bank balance to be computed at each time point given the stock prices and a risk free interest rate r. [**30%**]

Answer: (Bookwork) Let Δ_i denote the delta. At the initial time they receive the principal P and purchase Δ_i units of stock. Hence

$$b_0 = P - \Delta_i S_i$$

At intermediate times the bank balance gains interest. They purchase $\Delta_i - \Delta_{i-1}$ units of stock to ensure that they own exactly Δ_i units.

$$b_i = e^{r\delta t} b_{i-1} - (\Delta_i - \Delta_{i-1}) S_i$$

At the final time they gain interest, liquidate their stock portfolio and pay off any liability.

$$b_n = e^{r(t_n - t_{n-1})} b_{n-1} + \Delta_{n-1} S_i - (S_n - K)^+$$

(iii) Trader B decides not to trade in options at all. Instead they follow a trading strategy involving only investing in the stock and a risk free bank account. At each discrete time point they rebalance their portfolio to ensure they have invested a fixed proportion, α, of their total wealth in stock. The remainder is placed in the risk free account. Their initial wealth is W_0. Describe how you would numerically compute the expected utility of this trader assuming you have been given a function `simulate-StockPrices` which can simulate the stock price in discrete time in either the $\mathbb{P}$ or $\mathbb{Q}$ measure. [**20%**]

Answer: (Unseen) Let q_i denote the quantity of stock held, b_i the bank balance and w_i the wealth at time i. We have difference equations

$$W_i = q_{i-1} S_i + e^{t\delta t} b_{i-1}$$

$$q_i = \alpha w_i$$
$$b_i = (1-\alpha) w_i$$

which can be used to compute the wealth at subsequent times. We simulate the stock in the $\mathbb{P}$ measure and compute the final wealth and hence the final utility. Repeating this multiple times we can compute the mean final utility and hence estimate the expected utility.

(iv) Suppose that the ask price of the stock at each time point, $i\delta t$, is given by a random variable S_i and the bid price is $(1-\epsilon)S_i$ for some fixed $\epsilon > 0$. How would you change the difference equations for Trader A to account for this (you need only describe the change you would make at intermediate time points)? How would you compute their total wealth W_i at each time point $i\delta t$?

[**20%**]

Answer: (Unseen) At intermediate time points

$$b_i = e^{r\delta t} b_{i-1} - \begin{cases} (\Delta_i - \Delta_{i-1}) S_i & \text{if } (\Delta_i - \Delta_{i-1}) \geq 0 \\ (\Delta_i - \Delta_{i-1})(1-\epsilon) S_i & \text{otherwise} \end{cases}$$

To compute the wealth at each time, I would mark-to-market giving

$$W_i = b_i + \Delta_i (1-\epsilon) \Delta_i S_i.$$

since $\Delta_i \geq 0$ for call options.

(v) Assume $\epsilon > 0$. If the stock prices are simulated in accordance with the Black–Scholes model and Trader A has utility given by an exponential utility function, what will happen to the expected utility of Trader A as δt tends to zero? [**10%**]

Answer: (Unseen) It will tend to minus infinity.

3. (i) Suppose that a stock price follows geometric Brownian motion

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t).$$

Compute an SDE for the log of the stock price, $z_t = \log S_t$. Hence write down the probability density function for the log of the stock price z_T at time T. **[30%]**

Answer: (Bookwork) By Itô's lemma

$$\mathrm{d}z_t = (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W_t.$$

hence

$$p(z_T) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z_T - (z_0 + (\mu - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T}\right)$$

(ii) Suppose that you wish to calculate the price of a call option with strike K and maturity T in the Black–Scholes model. Write down a 1-dimensional integral in terms of z_T which can be used to compute the price of the option.

Answer: (Similar to Bookwork)

$$E_{\mathbb{Q}}(e^{-rT}\text{payoff}) = \int_{\mathrm{R}} e^{-rT}(e^{z_T} - K)^+ \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z_T - (z_0 + (r - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T}\right) \mathrm{d}z_T$$

Note the use of r both to obtain the $\mathbb{Q}$-measure pdf and to discount the expectation **[20%]**

(iii) Name three integration methods which can be used to evaluate an integral of the form

$$\int_a^b f(x)\,\mathrm{d}x.$$

State their rate of convergence (assuming a sufficiently well-behaved function f). **[10%]**

Answer: (Bookwork) Rectangle, $O(N^{-2})$ if f has bounded second derivative. Simpson's, $O(N^{-4})$ if f has bounded fourth derivative. Monte Carlo, $O(N)^{-\frac{1}{2}}$ $\int_a^b f^2(x)\mathrm{d}x \leq \infty$ and $\int_a^b |f(x)|\mathrm{d}x \leq \infty$.

(iv) How would you transform the integral found in part (ii) so that these integration rules can be applied? **[20%]**

Answer: (Similar to Bookwork) Make the substitution

$$z_T = z_0 + (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\Phi^{-1}(\epsilon)$$

$z_0 + (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}N^{-1}(\epsilon)$ so the integral becomes

$$\int_0^1 e^{-rT}(e^{z_T} - K)^+ \mathrm{d}\epsilon.$$

(Many other substitutions could be chosen to change an infinite integral to a finite integral and all valid choices would receive credit, but this is a natural choice in the context.)

(v) Is the integral you have computed sufficiently well-behaved for you to be able to obtain the maximum rate of convergence? Justify your answer and, if necessary, explain what changes you would need to make to achieve the desired rate of convergence. **[20%]**

Answer: (Unseen) The integrand is not differentiable so our convergence estimate for Simpson's rule does not apply (the proof of this convergence rate requires f has a bounded fourth derivative). Split the integral into two pieces: the first from 0 to ϵ_K where ϵ_K satisfies

$$z_0 + (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\Phi^{-1}(\epsilon_K) = K$$

and then from ϵ_K to 1. The integral will be smooth on each piece (indeed it vanishes on the first piece).

4. (i) (a) What is meant by a pseudo square root of a positive definite symmetric matrix A? **[10%]**

Answer: Bookwork. A pseudo square root of A is a matrix B with $BB^T = A$.

(b) What is meant by the Cholesky decomposition of a positive definite symmetric matrix A? **[10%]**

Answer: Bookwork. The Cholesky decomposition of A is the unique lower triangular pseudo squasre root of A with positive diagonal.

(c) Compute the Cholesky decomposition of the matrix:

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

[30%]

Answer: Seen previously. Write

$$B = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$$

with $\alpha > 0$, $\gamma > 0$.

$$BB^T = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 + \gamma^2 \end{pmatrix}$$

Equating BB^T with

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

we find

$$\alpha^2 = 1, \quad \alpha\beta = \rho, \quad \beta^2 + \gamma^2 = 1$$

Solving these equations in sequence, since $\alpha > 0$, $\alpha = 1$. Then $\beta = \rho$ and finally $\gamma = \sqrt{1-\rho^2}$ (we take the positive square root since $\gamma > 0$. Thus the Cholesky decomposition is

$$B = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

(ii) The Box–Muller algorithm is an algorithm to generate independent normally distributed random numbers with mean 0 and standard deviation 1. One first generates uniformly distributed random numbers U_1 and U_2 between 0 and 1. One then defines $Z_1 = R\cos(\theta)$, $Z_2 = R\sin(\theta)$ where $R^2 = -2\log U_1$ and $\theta = 2\pi U_2$.

(a) Write a function `boxMuller` which takes a parameter n and returns an $2 \times n$ sample of independent normally distributed random numbers generated by the Box–Muller algorithm. **[20%]**

Answer: Seen as homework exercise.

```
function ret=boxMuller( n )
U1 = rand(1,n);
U2 = rand(1,n);
R = sqrt( -2 * log( U1 ));
theta = 2 * pi * U2;
ret = [ R.*cos( theta ); R.*sin(theta) ];
end
```

(b) Using your answer to parts (i)(c) and (ii)(a), show how you would write a function which takes a parameter n and returns a $2 \times n$ sample from a two-dimensional multivariate normal distribution with mean 0 and covariance matrix

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

[20%]

Answer: Unseen in this form.

```
function ret=correlatedNormalDistribution( n, rho )
B = [ 1 0; rho sqrt(1-rho^2) ];
eps = boxMuller( n );
ret = B .* eps;
end
```

(c) How would you test this function? **[10%]**

Answer: Unseen. I would generate a large sample with correlation $\rho = 0.5$ and confirm that the mean vector is approximately 0 and the correlation matrix is as required. To test for normality I would compute the 97.5th percentile of the first component of the return vector and check this is equal to 1.96. Many other sensible tests are possible.

5. (i) When the risk-free interest rate, r, is zero, the Black–Scholes partial differential equation simplifies to:

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} = 0.$$

Show that this can be transformed by a change of variables into a constant coefficient equation:

$$C_t = -\frac{1}{2}\sigma^2 C_{xx}.$$

[**20%**]

Answer: (Similar to Bookwork) The equation

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} = 0$$

is associated to

$$\mathrm{d}S_t = S_t \sigma \mathrm{d}W_t$$

by Feynman–Kac. The change of variable $y_t = \log S_t$ yields

$$\mathrm{d}y_t = -\frac{1}{2}\sigma^2 \mathrm{d}t + \sigma \mathrm{d}W_t$$

hence defining $x_t = \log S_t + \frac{1}{2}\sigma^2 \mathrm{d}t$ we have

$$\mathrm{d}x_t = \sigma \mathrm{d}W_t.$$

This is associated by Feynman–Kac to the equation

$$C_t = -\frac{1}{2}\sigma^2 C_{xx}.$$

(ii) Derive the difference equations for the implicit finite difference scheme for solving this equation. [**30%**]

Answer: (Bookwork) We use the forward difference approximation

$$\frac{\partial C}{\partial t} \approx \frac{C(t+\delta t, x) - C(t,x)}{\delta t}$$

and the central difference estimate

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C(t, x+\delta x) - 2C(t,x) + C(t, x-\delta x)}{(\delta x)^2}.$$

Writing $C_{i,j}$ for the value of C at time point $i\delta t$ and space point $j\delta x$ we have

$$\frac{C_{i+1,j} - C_{i,j}}{\delta t} \approx -\frac{1}{2}\sigma^2 \frac{C_{i,j+1} - 2C_{i,j} + C_{i,j-1}}{(\delta x)^2}$$

hence

$$C_{i+1,j} = -\lambda C_{i,j+1} + (1+2\lambda)C_{i,j} + -\lambda C_{i,j-1}$$

where $\lambda = \frac{\sigma^2 \delta t}{2(\delta x)^2}$. Note that the terms on the RHS should be considered the unknowns and the terms on the left should be considered as the knowns.

(iii) What boundary conditions would you use to price a put option in the Black-Scholes model when using this implicit method? Justify your answer. **[30%]**

Answer: (Similar to bookwork) Let $N = \frac{T}{\delta t}$ be the number of time steps and $2M$ be the number of space steps. We choose M so that

$$M\delta x \approx 8\sigma\sqrt{T}$$

as it will be a reasonable approximation that the stock price will not move by eight standard deviations. We can rescale the problem to ensure that the initial stock price is 1. At the final time, the payoff is $(K - S_T)^+ = (K - e^{x-\frac{1}{2}\sigma^2 T})^+$. So

$$C_{N,j} = (K - e^{j\delta x - \frac{1}{2}\sigma^2 T})^+$$

For large values of x, we assume that the option will finish out of the money and so pay off zero. Hence we take

$$C_{i,M} = 0.$$

For small values of x we assume that the option will finish in the money and hence has a value equal to a portfolio of $e^{-rT}K$ units of cash and -1 unit of stock. So

$$C_{i,-M} = e^{r(T-j\delta t)}K - e^{-M\delta x - \frac{1}{2}\sigma^2 j\delta t}.$$

(iv) Explain why it is essential to provide top and bottom boundary conditions to solve the heat equation by the implicit method but only a final condition is required for the explicit method. **[20%]**

Answer: (Unseen) The equation we derived in part (ii) is an equation for three unknowns in terms of one known value. In the finite difference method with M+1 grid points for the x variable, these difference equations yield $M-1$ equations in $M+1$ unknowns. If we add in two boundary conditions we get $M+1$ equations in $M+1$ unknowns which we expect to be able to solve. The explicit method has no such issues as the corresponding equation is an equation for one unkown in terms of three knowns.