**1.** (i) (a) State the Monte Carlo integration rule for a function $f : [a, b] \longrightarrow \mathrm{R}$ defined on a closed interval. **[20%]**

(b) Write the MATLAB code to integrate $e^{-x^2}$ over the interval $[0, 1]$ using Monte Carlo integration. **[30%]**

(ii) The Box–Muller algorithm is an algorithm to generate independent normally distributed random numbers with mean 0 and standard deviation 1. One first generates uniformly distributed random numbers $U_1$ and $U_2$ between 0 and 1. One then defines $Z_1 = R\cos(\theta)$, $Z_2 = R\sin(\theta)$ where $R^2 = -2\log U_1$ and $\theta = 2\pi U_2$.

(a) Write a function `boxMuller` which takes a parameter $n$ and returns an $2 \times n$ sample of independent normally distributed random numbers generated by the Box–Muller algorithm. **[20%]**

(b) How would you test this function? **[10%]**

(c) Using the MATLAB function `chol` or otherwise, show how you would generate a sample from a two dimensional multivariate normal distribution with mean 0 and covariance matrix

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

**[20%]**

**2.** (i) You believe that the 5 stocks will have annual returns that follow a multivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$. You have $1000000 to invest in these stocks and wish to achieve an expected return of 10% over the year. You wish to select a static portfolio, i.e. you must buy and hold. Express the problem of selecting the portfolio that meets these requirements with the minimum standard deviation as a quadratic programming problem [**30%**]

(ii) Explain what is meant by the efficient frontier and sketch its expected shape. Indicate in the same diagram how portfolios consisting of investments in a single stock would perform. [**20%**]

(iii) Suppose that we do not believe that the stocks have normally distributed returns, but that the 5 stocks follow some specific stochastic process. Explain how you could use Monte Carlo simulation to find the optimal static portfolio in terms of a utility function $u$. [**30%**]

(iv) You decide instead to pursue a dynamic investment strategy. Investment strategy $S_1$ is to, once a week, invest all your money in the stock that had the most return in the previous week. Investment strategy $S_2$ is to, once a week invest all your money in the stock that had the least return in the previous week. Assuming the stocks follow a known stochastic process and you have a known utility function $u$, how could you devise a trading strategy that is guaranteed to be at least as good as strategies $S_1$ and $S_2$? [**20%**]

**3.** A trader has $P$ units of cash and wishes to invest in a stock and a risk free bond to maximize their expected utility at time $T$. Their utility function is:

$$u(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$$

The trader believes the stock follows geometric Brownian motion:

$$\mathrm{d}S_t = S_t(\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t)$$

The bond has interest rate $r$. At time 0 the trader invests an amount $Q$ of their wealth in stock and the rest in bonds.

(i) Write the expected utility as an integral **[40%]**

(ii) Write the MATLAB code to compute this integral by a Monte Carlo method **[30%]**

(iii) State a variance reduction technique you could use to improve the rate of convergence of the Monte Carlo method **[10%]**

(iv) $u(x)$ takes the value $-\infty$ when $x$ is negative. What trading constraint does this imply? **[10%]**

(v) How could you use MATLAB to find the optimal value of $Q$? **[10%]**

**4.** (a) What is meant by the Value at Risk of a portfolio? [**20%**]

(b) What is meant by the Expected Shortfall of a portfolio? [**10%**]

(c) If a portfolio has current value $P$ and its value at time $T$ is normally distributed with mean $M$ and standard deviation $\sigma$ write down a formula to compute the Value at Risk of this portfolio over time horizon $T$ in terms of $N^{-1}$, the inverse cumulative distribution function of the standard normal distribution. [**20%**]

(d) Suppose that the market contains $n$ assets. The current price of asset $i$ $(1 \leq i \leq n)$ is $P_i$. At time $T$, the asset prices follow a multivariate normal distribution with means $M_i$ and with covariance matrix $\Omega$. Give a formula for the standard deviation of a portfolio consisting of $\alpha_i$ units of each asset $i$ at time $T$. [**20%**]

(e) An investor has a budget $B$ and wishes to invest this in a portfolio of the $n$ assets in such a way as to make an expected profit of $Q$ at time $T$ while minimizing the portfolio's Value at Risk at time $T$ at confidence level 95%.

  (a) Write down a formula for the Value at risk of the portfolio at time $T$ at confidence level 95%. [**10%**]

  (b) Write down a formula for the cost of the portfolio. [**10%**]

  (c) You can find the portfolio of minimum variance at a given cost using the function quadprog. How would you find the portfolio minimum expected shortfall? [**10%**]