Pathwise superhedging for time-dependent barrier options on càdlàg paths - finite or infinite tradeable European, One-Touch, lookback or forward starting options

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Abstract

We establish pathwise duality using simple predictable trading strategies for the robust hedging problem associated with a barrier option whose payoff depends on the terminal level and the infimum of a càdlàg strictly positive stock price process, given tradeable European options at all strikes at a single maturity. The result allows for a significant dimension reduction in the computation of the superhedging cost, via an alternate lower-dimensional formulation of the primal problem as a convex optimization problem, which is qualitatively similar to the duality which was formally sketched using linear programming arguments in Duembgen&Rogers[DR14] for the case where we only consider continuous sample paths. The proof exploits a simplification of a classical result by Rogers[Rog93] which characterizes the attainable joint laws for the supremum and the drawdown of a uniformly integrable martingale (not necessarily continuous), combined with classical convex duality results from Rockefellar[Roc74] using paired spaces with compatible locally convex topologies and the Hahn-Banach theorem. We later adapt this result to include additional tradeable One-Touch options using the Kertz-Rösler[KR90] condition. We also compute the superhedging cost when in the more realistic situation where there is only finite tradeable European options; for this case we obtain the full duality in the sense of *quantile hedging* as in Soner[Son15]), where the superhedge works with probability $1-\varepsilon$ where ε can be arbitrarily small), and we obtain an upper bound for the true pathwise superhedging cost. In section 5, we extend our analysis to include time-dependent barrier options using martingale coupling arguments, where we now have tradeable European options at both maturities at all strikes and tradeable forward starting options at all strikes. This set up is designed to approximate the more realistic situation where we have a finite number of tradeable Europeans at both maturities plus a finite number of tradeable forward starting options.[1]

1 Introduction

For a martingale M with $M_0=0$, if μ denotes the law of M_T and $\bar{\mu}$ the law of $\bar{M}_T$ where $\bar{M}_t=\sup_{0\le s\le t}M_s$, then a well known result of Blackwell&Dubins[BD63] and Kertz&Rösler[KR90] asserts that

$$\mu\vee\delta_{\{0\}} \quad\preceq\quad \bar{\mu} \quad\preceq\quad \mu^* \tag{1}$$

where $\mu\vee\delta_{\{0\}}$ denotes the law of $\max(X,0)$ where $X\sim\mu$, and $\preceq$ denotes the stochastic ordering and μ^* is the the Hardy-Littlewood transform of μ (see Theorem 2.7 in [Hob98] for details). The Hardy-Littlewood transform of μ is the law of $B_\mu(X)$ if $X\sim\mu$, where $B_\mu(x):=\int_{(x,\infty)}y\mu(dy)/\mu((x,\infty))$ denotes the *Barycentre function*, which is also the law of the terminal maximum for the Azéma-Yor Skorokhod embedding $\tau_{\rm AY}$ with target law μ ($\tau_{\rm AY}:=\inf\{t:W_t\le B_\mu^{-1}(\bar{W}_t)\}$ for a Brownian motion W), which (after a suitable time-change) maximizes the law of the supremum for a continuous martingale X subject to $X_\infty\sim\mu$. Hence the upper bound in (1) is not made sharper if we restrict attention to continuous martingales. [KR90] also prove a converse result, namely that if μ is a probability measure on $\mathbb{R}$ with $\int|x|\mu(dx)<\infty$ and $\bar{\mu}$ is another probability measure which satisfies (1), then there exists a martingale M with $M_T\sim\mu$ and $\bar{M}_T\sim\bar{\mu}$. The lower bound in (1) arises from a trivial 1-step "model" $(X_t)_{0\le t\le T}$ with $X_t=0$ for $t<T$ and $X_T\sim\mu$. If we restrict ourselves to continuous martingales, then this trivial bound is replaced by the non-trivial bound associated with the Perkins[Per86] Skorokhod embedding $\tau_P:=\inf\{t:W_t\in(-\gamma_+(\bar{W}_t),\gamma_-(-\underline{W}_t))\}$ for some functions γ_+,γ_- depending on μ, and for this stopping time we have $\mathbb{P}(W_{\tau_P}\ge b)=\mu[b,\infty)+\inf_{K\le b}\frac{1}{b-K}[\int((x-b)^+-(K-x)^+)d\mu]$, also known as the Minimax-Maximin embedding because it not only minimizes the law of the maximum but also maximizes the law of the minimum (see Cox[Cox04] and Hobson[Hob10] for more on this). Cox&Obloj[CO15] develop these ideas

[1]The author would like to thank Professors Charles Akemann and Teemu Pennanen for useful discussions.

further, and use pathwise inequalities to derive upper and lower bounds on the joint exit probabilities of a continuous U.I. martingale given its terminal law, and by constructing new and explicit solutions to the Skorokhod embedding problem, they show these bounds are sharp.

For a probability measure μ on $[0,\infty)\times[0,\infty)$, the classical article of [Rog93] gives four conditions on μ which (if all four hold) provides a necessary and sufficient condition for μ to be the joint law of the supremum and the drawdown for some càdlàg uniformly integrable martingale. If we restrict attention to continuous uniformly integrable (U.I.) martingales, the result still holds with a simple modification, namely that the non-trivial Rogers condition becomes an equality. [Rog93] also proves that if X is an a.s. convergent continuous local martingale, then X is a U.I. martingale if and only if (i) $\mathbb{E}(|X|_\infty)<\infty$, (ii) $\mathbb{E}(X_\infty)=0$ and (iii) $\lim_{a\to\infty} a\mathbb{P}(\bar{X}_t>a)=0$ (see Azéma,Gundy&Yor[AGY80] and Elworthy,Li&Yor[ELY97] for further results of this nature).

In more recent work, Duembgen&Rogers[DR14] give a similar characterization for the joint law of the terminal level, the minimum, the maximum, and the direction of the final excursion for a simple symmetric random walk stopped at some almost-surely finite stopping time; from this they compute the minimal superhedging cost (and the corresponding superhedge) for an exotic option whose payoff depends only on these four quantities, as the solution to a linear programming problem.

Less work has been done on robust hedging for discontinuous price paths (for continuous paths, see e.g. [BCH16], [GTT16],[GTT17],[GTT16b],[KTT17] et al.) because in this setting we can no longer express the price process as a time-changed Brownian motion and appeal to the extensive literature on Skorokhod embeddings for Brownian motion. [GTT17] consider the robust hedging problem on the Skorokhod space of càdlàg paths; the set of martingale measures $\mathcal{M}(\mu)$ consistent with a finite set of marginals μ here is not tight with respect to the standard topologies, which makes it difficult to adapt known duality results in discrete-time settings to the continuous time case. Dolinsky&Soner[DS15] circumvent this issue by using a limiting argument with a discretization of the price paths in an n-dimensional setting, and imposing that the superhedge work *pathwise* and only allowing trading strategies with bounded variation so we can define the stochastic integral pathwise using the Stieltjes integration-by-parts formula. [GTT17] derive a quasi-sure duality result for càdlàg paths, using the S-topology on $\mathcal{M}(\mu)$ introduced in Jakubowski[Jak97], which is induced by the notion of S-convergence, and we can then define S^*-convergence as the convergence induced by the S-topology. Rather than use the usual weak topology on the space of probability measures, they use another notion of convergence, which allows for a variant of the standard Prokhorov theorem to hold under S-tightness, i.e. where tightness yields sequential compactness (by S-tightness they are just replacing the usual notion of tightness using compact sets with a set which is compact under the S-topology).

In this note, we simplify the Rogers result for the càdlàg case, showing that one of Rogers' conditions is already implied by the other conditions. We then give the corresponding result for μ to be the joint law of the infimum and the drawup for some càdlàg U.I. martingale. Using this simplification, we then establish a pathwise duality result for the robust hedging problem associated with a general type of barrier option on a càdlàg stock price path, subject to tradeable European options at all strikes at a single maturity. By "pathwise" we mean the superhedge works for any càdlàg stock price process (not necessarily a semi-martingale under some probability measure) and thus also works for e.g. rough paths. We show that the primal problem now has an alternate lower-dimensional formulation; this approach allows us to reduce the problem of computing the superhedging cost from a convex minimization problem over martingale measures on the infinite-dimensional space of càdlàg paths to a convex maximization problem over probability measures on a convex subset of $(0,\infty)\times(0,\infty)\times(0,\infty)\times(0,\infty)$. We then modify this result to include the case when we also have tradeable One-Touch options at all barrier levels, for which the range of admissible prices is given by the inequality (1) and the Kertz&Rösler result described above. We later extend these results to deal with so-called *time window barrier* options where the barrier can change level at an intermediate maturity, given tradeable European options at both maturities and tradeable forward-starting options. For this problem we again establish pathwise duality using a conditional version of the simplified Rogers result.

1.1 Notation

- $\mathcal{X}_{x_0}:=(0,x_0]\times(0,\infty)$.
- $\Omega=\mathcal{D}^+_{x_0}[0,T]$ is the space of strictly positive càdlàg paths on $[0,T]$ with $X_0=x_0>0$.
- $X=(X_t)_{0\le t\le T}$ is the canonical process given by $X_t(\omega)=\omega_t$ for all $\omega\in\Omega$ and $\mathcal{F}_t$ is the the right continuous filtration $\mathcal{F}_t:=\cap_{s>t}\mathcal{F}^X_s$.
- A probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F}_T)$ is a martingale measure if the canonical process $(X_t)_{t\ge0}$ is martingale under $\mathbb{Q}$.
- $\mathbb{M}^+$ is the collection of all probability measures $\mathbb{Q}$ on $(\mathcal{D}^+_{x_0}[0,T],\mathcal{F}_T)$ such that the canonical process $X_t(\omega)=\omega_t$ is an $\mathcal{F}_t$-martingale.

- $\mathbb{M}^+(\mu_X)$ is the set of elements $\mathbb{Q} \in \mathbb{M}^+$ for which $X \sim \mu_X$ under $\mathbb{Q}$.
- $\mathbb{M}^+(\mu_1, \mu_2, \mu_3) \subset \mathbb{M}^+$ is the elements of $\mathbb{M}^+$ such that $X_{T_1} \sim \mu_1$, $X_{T_2} \sim \mu_2$ and $X_{T_2}/X_{T_1} \sim \mu_3$.
- For a general measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$, we let $\mathcal{M}(\tilde{\Omega})$ denote the space of signed Radon measures on $\tilde{\Omega}$, $\mathcal{M}^+(\Omega)$ the non-negative measures in $\mathcal{M}(\Omega)$.
- $\mathcal{M}_f^+((0, x_0))$ is the convex cone of finite non-negative measures on $(0, \infty)$ of the form $\nu(da) = \sum_{i=1}^n \alpha_i \delta_{\{a_i\}}(da)$ with $\alpha_i > 0$, $a_i \in [0, x_0)$, i.e. a finite positive linear combination of dirac masses.
- $\mathcal{H}$ is the space of all $\mathcal{F}_t$-simple predictable processes.
- $C_c^+(\mathcal{X}_{x_0})$ is the space of non-negative functions on $\mathcal{X}_{x_0}$ with compact support.
- $\mathcal{X}_1 = (0, 1] \times [0, \infty)$.
- $\tilde{\mathcal{X}} = (0, x_0] \times (0, \infty) \times (0, 1] \times [0, \infty) = \mathcal{X}_{x_0} \times \mathcal{X}_1$.
- $\mathcal{E} = \{g : \mathcal{X}_{x_0} \mapsto \mathbb{R},\ g(\underline{x}, y) = \int (\underline{x} + y - a)\, 1_{\underline{x} < a}\, \nu(da)\ ,\ \nu \in \mathcal{M}_f^+((0, x_0))\}$.

2 The class of admissible laws for the supremum and the drawdown of a càdlàg martingale - simplifying the Rogers condition

We let $\mathcal{X}_{x_0} := (0, x_0] \times (0, \infty)$ throughout. We now recall the classical result from Theorem 2.2 in [Rog93], on which this article is based:

Proposition 2.1 *Fix $T > 0$. A probability measure μ on $[0, \infty) \times [0, \infty)$ is the joint law of $(S_T, Y_T := S_T - X_T)$ for some càdlàg martingale $(X_t)_{t \geq 0}$ (where $S_t := \sup_{0 \leq s \leq t} X_s$ and $X_0 = 0$) if and only if the following four conditions are satisfied*

$$\begin{cases} \int\int |s - y|\, \mu(ds, dy) < \infty & \\ \int\int (s - y - a) 1_{s > a}\, \mu(ds, dy) \quad \geq \quad 0 & \forall a > 0 \\ \int\int (s - y)\, \mu(ds, dy) \quad = \quad 0 & \end{cases} \tag{2}$$

and $c(a) := \mathbb{E}(X 1_{S > a})/\mathbb{P}(S > a) = \mathbb{E}(X | S > a)$ is increasing when $\mathbb{P}(S < a) > 0$, and we use X and S as shorthand for X_T and S_T.

Remark 2.1 If $\mu(S > a) = 0$ for some $a > 0$, then the Rogers[Rog93] U.I. martingale X which embeds μ can never exceed a, because clearly $\bar{X}_t := \sup_{0 \leq s \leq t} X_s \leq \bar{X}_T = S \leq a$ for $0 \leq t \leq T$.

Remark 2.2 If we replace càdlàg martingales with continuous martingales in Proposition 2.1, then the result still holds but now the second condition in (2) is an equality; see [Rog12] and Theorem 3.1 in [Rog93] for details.

2.1 The Rogers condition for the infimum and the drawup

From here on we will refer to $X_t - \underline{X}_t$ as the **drawup** of X at time t. We now adapt Proposition 2.1 to characterize the admissible joint laws of the infimum and the drawup of a nonnegative càdlàg martingale.

Proposition 2.2 *Let $G^a(\underline{x}, y) := 1_{\underline{x} < a}\, (\underline{x} + y - a)$. Then a probability measure μ on $\mathcal{X}_{x_0}$ is the joint law of $(\underline{X}_T, Y_T := X_T - \underline{X}_T)$ for some càdlàg strictly positive martingale $(X_t)_{t \geq 0}$ with $X_0 = x_0$ (where $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$) if and only if the following three conditions are satisfied*

$$\begin{cases} \int\int G^a(\underline{x}, y)\, \mu(d\underline{x}, dy) \quad \leq \quad 0 & \forall a \in (0, x_0) \\ \int\int (\underline{x} + y)\, \mu(d\underline{x}, dy) \quad = \quad x_0 & \end{cases} \tag{3}$$

and $\underline{c}(a) := \int\int (\underline{x} + y) 1_{\underline{x} < a} d\mu / \int\int 1_{\underline{x} < a} d\mu$ is increasing in a whenever $\int\int 1_{\underline{x} < a} d\mu > 0$, where $\int\int$ indicates that we are integrating over $\mathcal{X}_{x_0}$.

Proof. Condition 2.2iii) in [Rog93] can be re-written as

$$\mathbb{E}((X-a)1_{S>a}) \quad \geq \quad 0 \tag{4}$$

for all $a > 0$. But for any càdlàg martingale X with $X_0 = 0$, clearly $-X$ is also a càdlàg martingale and this transformation is one-to-one. Hence (4) is equivalent to

$$\mathbb{E}((-X-a)1_{\overline{(-X)}>a}) \quad = \quad \mathbb{E}((-X-a)1_{-\underline{X}>a}) \quad = \quad \mathbb{E}((-X-a)1_{\underline{X}<-a}) \quad \geq \quad 0$$

and multiplying by -1 and setting $a \mapsto -a$, this is also equivalent to

$$\mathbb{E}((X-a)1_{\underline{X}<a}) \quad \leq \quad 0$$

for all $a < 0$. If we now add x_0 to X so $X_0 = x_0$ and set $a \mapsto a + x_0$, we have

$$\mathbb{E}((X-(a+x_0))1_{\underline{X}<(a+x_0)}) \quad \leq \quad 0$$

for all $a < 0$, or equivalently

$$\mathbb{E}((X-a')1_{\underline{X}<a'}) \quad \leq \quad 0$$

for all $a' < x_0$. But we are also assuming that $\mu(\underline{X} \leq 0) = 0$, so we can restrict attention to $a' \in (0, x_0)$, which can now be re-written in integral form as the first condition in (3). The second equation in (3) is the centering condition Eq 2.4 in [Rog93]. Using that same $X \mapsto -X$ transformation as above, the final monotonicity condition on $c(a)$ in Proposition 2.1 is transformed to the monotonicity condition on $\underline{c}(a)$. ■

Lemma 2.3 *The final monotonicity condition on $\underline{c}(a)$ in Proposition 2.2 is unnecessary, as it is already implied by the other two conditions.*

Proof. Let $a_2 > a_1$. Then we have

$$\begin{aligned}
& \underline{c}(a_2) - \underline{c}(a_1) \\
= \; & \mathbb{E}(X1_{\underline{X}<a_2})/\mathbb{P}(\underline{X}<a_2) - \mathbb{E}(X1_{\underline{X}<a_1})/\mathbb{P}(\underline{X}<a_1) \\
= \; & \frac{\mathbb{E}(X1_{\underline{X}<a_2})\mathbb{P}(\underline{X}<a_1) - \mathbb{E}(X1_{\underline{X}<a_1})\mathbb{P}(\underline{X}<a_2)}{\mathbb{P}(\underline{X}<a_2)\mathbb{P}(\underline{X}<a_1)} \\
= \; & \frac{(\mathbb{E}(X1_{\underline{X}<a_1}) + \mathbb{E}(X1_{\underline{X}\in[a_1,a_2)}))\mathbb{P}(\underline{X}<a_1) - \mathbb{E}(X1_{\underline{X}<a_1})(\mathbb{P}(\underline{X}<a_1) + \mathbb{P}(\underline{X}\in[a_1,a_2))}{\mathbb{P}(\underline{X}<a_2)\mathbb{P}(\underline{X}<a_1)} \\
= \; & \frac{\mathbb{E}(X1_{\underline{X}\in[a_1,a_2)})\mathbb{P}(\underline{X}<a_1) - \mathbb{E}(X1_{\underline{X}<a_1})\mathbb{P}(\underline{X}\in[a_1,a_2))}{\mathbb{P}(\underline{X}<a_2)\mathbb{P}(\underline{X}<a_1))} \\
\geq \; & \frac{a_1\mathbb{P}(\underline{X}\in[a_1,a_2))\mathbb{P}(\underline{X}<a_1) - \mathbb{E}(X1_{\underline{X}<a_1})\mathbb{P}(\underline{X}\in[a_1,a_2))}{\mathbb{P}(\underline{X}<a_2)\mathbb{P}(\underline{X}<a_1))} \\
\geq \; & \frac{\mathbb{E}((a_1-X)1_{\underline{X}<a_1})\mathbb{P}(\underline{X}\in[a_1,a_2))}{\mathbb{P}(\underline{X}<a_2)\mathbb{P}(\underline{X}<a_1))} \\
\geq \; & 0
\end{aligned}$$

where we have used the first condition in (3) to obtain the final inequality. ■

2.2 Description of the Rogers U.I. martingale

(This subsection can be omitted and is just for readers who are curious about the [Rog93] construction). The Rogers[Rog93] martingale for the sufficiency part of Theorem 2.2 in [Rog93] is constructed as follows: take a Brownian motion B and let $T = \inf\{u : B_u \leq h(S_u)\}$ where $h : (0,\infty) \to \mathbb{R}$ is given by $h(s) := c^{-1}(s) - v(c^{-1}(s))$ if c is strictly increasing, where c is defined as in the proof of Proposition 2.1, $v(s) := \int y\mu(dy|s)$ (i.e. $\mathbb{E}(Y|S)$) and $\mu(ds,dy)$ is the joint target law for (S,Y). We then set $A_t := \int_0^t 1_{u:B_u<c^{-1}(S_u)}du$, $\tau_t = \inf\{u : A_u > t\}$, $M_t := B_{\tau_t \wedge T}$, and we see that τ_t is right but not left continuous in general because time is "lost" when $B_u \geq c^{-1}(S_u)$ which means that $\bar{M}_t < \bar{B}_{\tau_t\wedge T}$ (see Figure 1 in [Rog93] for a nice graph of what is going on here). Then M is a U.I. martingale with $\bar{M}_\infty \sim \int_{y\in[0,\infty)} \mu(ds,dy)$ (i.e. the correct target marginal for the supremum), and $\bar{M}_\infty - M_\infty = v(\bar{M}_\infty)$, where $v(s) = \int y\mu(dy|s)$, i.e. M_∞ is a deterministic function of $\bar{M}_\infty$, similar to the well known Azéma-Yor Skorokhod embedding (the fact that $\underline{c}(.)$ is increasing is key to the proof). Finally we set

$$N_t \quad = \quad \begin{cases} M_{t/(1-t)} & 0 \leq t < 1 \\ M_\infty & 1 \leq t < 2 \\ Z & t \geq 2 \end{cases}$$

where Z is such that $\bar{M}_\infty - Z \sim \mu(.|\bar{M}_\infty)$. N is then the desired U.I. càdlàg martingale, i.e. $(\bar{N}_\infty, \bar{N}_\infty - N_\infty) \sim \mu$.

3 The first duality result

Let μ_X be a target probability measure on $(0,\infty)$ with $\int x\mu_X(dx) = x_0$. Using the sup norm on $(0,\infty)$, $C_0((0,\infty))$ is a normed vector space, and the dual space $C_0((0,\infty))^*$ (i.e. the space of linear functionals of $C_0((0,\infty))$ which are continuous under the sup norm) is the space of signed Radon measures on $(0,\infty)$ (see e.g. section 7.3 in Folland[Fol99] for details). Thus, if we know the value of the cost functional $c \in C_0((0,\infty))^*$ given by

$$c(\psi) \quad := \quad \int \psi(x)\mu_X(dx) \tag{5}$$

for all $\psi \in C_0((0,\infty))$ [2], then μ_X is uniquely determined, and vice versa. Thus if we know the value of $c(\psi)$ for all ψ in $C_b((0,\infty))$, then μ_X is also uniquely determined, because clearly $C_0((0,\infty)) \subset C_b((0,\infty))$.

Let $\Omega = \mathcal{D}^+_{x_0}[0,T]$ denote the space of strictly positive càdlàg paths on $[0,T]$ with $X_0 = x_0 > 0$. Let $X = (X_t)_{0\le t\le T}$ be the canonical process given by $X_t(\omega) = \omega_t$ for all $\omega \in \Omega$ and $\mathcal{F}_t$ denote the right continuous filtration $\mathcal{F}_t := \cap_{s>t}\mathcal{F}^X_s$. A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ is a martingale measure if the canonical process $(X_t)_{t\ge 0}$ is martingale under $\mathbb{Q}$.

Let $\mathbb{M}^+$ denote the collection of all probability measures $\mathbb{Q}$ on $(\mathcal{D}^+_{x_0}[0,T], \mathcal{F}_T)$ such that the canonical process $X_t(\omega) = \omega_t$ is an $\mathcal{F}_t$-martingale and $\mathbb{M}^+(\mu_X)$ denotes set of elements $\mathbb{Q} \in \mathbb{M}^+$ for which $X \sim \mu_X$ under $\mathbb{Q}$.

For a general measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$, we let $\mathcal{M}(\tilde{\Omega})$ denote the space of signed Radon measures on $\tilde{\Omega}$, $\mathcal{M}^+(\Omega)$ the non-negative measures in $\mathcal{M}(\Omega)$.

We now state the first duality result.

Proposition 3.1 *Let $\mathcal{P} := \mathcal{P}(\mathcal{X}_{x_0})$ denote the space of probability measures on $\mathcal{X}_{x_0}$, and $F : \mathcal{X}_{x_0} \to (0,\infty)$ be bounded and upper semicontinuous. Then we have*

$$\begin{aligned}
P \quad &:= \sup_{\mathbb{P}\in\mathbb{M}^+(\mu_X)} \mathbb{E}^{\mathbb{P}}(F(\underline{X}_T, Y_T)) \\
&= \quad P_0 \quad := \quad \sup_{\mu\in\mathcal{P}}\Big[\int F(\underline{x},y)\mu(d\underline{x},dy) \,\Big|\, \int(\underline{x}+y-a)1_{y<a}\mu(d\underline{x},dy) \;\le\; 0\,,\; \int \psi(\underline{x}+y)\mu(d\underline{x},dy) \;=\; c(\psi) \\
&\qquad \forall a\in(0,x_0)\,,\; \psi\in C_b((0,\infty))\Big] \\
&= \quad D_0 \quad := \inf_{\nu\in\mathcal{M}^+_f((0,x_0)),\psi\in C_b((0,\infty))} \Big[c(\psi) \mid F(\underline{x},y) \;\le\; \psi(\underline{x}+y) + \int_{(0,\infty)} G^a(\underline{x},y)\nu(da) \quad \forall(\underline{x},y)\in\mathcal{X}_{x_0}\Big]\,.
\end{aligned} \tag{6}$$

where $\mathcal{M}^+_f((0,x_0))$ is the convex cone of finite non-negative measures on $(0,\infty)$ of the form $\nu(da) = \sum_{i=1}^n \alpha_i\delta_{\{a_i\}}(da)$ with $\alpha_i > 0$, $a_i \in [0,x_0)$, i.e. a finite positive linear combination of dirac masses.

Proof. See Appendix A. ■

Remark 3.1 Note that the centering condition on μ_X (i.e. that $\int x d\mu_X = x_0$ which is also the second condition in (3)) do not explicitly appear in the definition of P_0, but it is implicitly imposed by the $\int \psi(\underline{x}+y)\mu(d\underline{x},dy) = c(\psi)$ constraint, because we are assuming that μ_X is centered and integrable.

3.1 The financial model and superhedging the Rogers payoff function with dynamic trading

From here on we let $(X_t)_{t\ge0}$ be a strictly positive càdlàg function which models a stock price process, and we assume zero interest rates throughout. As in [DS15], this is the only assumption that we make on our financial market.

Let $\tau_a := \inf\{t : X_t < a\}$, and define $\underline{X}_t$ and Y_t as in Section 2 as the infimum and the drawup respectively of X at time t. $(-\infty, a)$ is an open set and $\mathcal{F}_t$ is a right continuous filtration, so τ_a is an $\mathcal{F}_t$-stopping time (see e.g. Theorem 3, Chapter I in Protter[Prot04]) so

$$H^a_t \quad := \quad 1_{t\in(\tau_a\wedge T,T]} \tag{7}$$

(defined for $0 \le t \le T$) is a simple predictable process (from the general definition of such a process), and of course H^a is left continuous. Then from the definition of the stochastic integral for simple predictable processes and the right continuity of X, we know that

$$\int_{[0,T]} H^a_t dX_t \quad = \quad (X_T - X_{\tau_a\wedge T}) \quad = \quad (X_T - X_{\tau_a})1_{\underline{X}_T<a} \quad \ge \quad (X_T - a)1_{\underline{X}_T<a}\,.$$

[2] $C_0((0,\infty))$ is the space of continuous functions on $(0,\infty)$ which vanish at 0 and ∞.

Hence we can superhedge the Rogers payoff $(X_T - a)1_{\underline{X}_T<a}$ with the simple predictable trading strategy $(H_t^a)_{t\in[0,T]}$ in (7), and thus trivially we can also superhedge $\int(X_T - a)1_{\underline{X}_T<a}\nu(da)$ with the simple predictable trading strategy $(\int H_t^a\nu(da))_{t\in[0,T]}$ for $\nu \in \mathcal{M}_f^+((0,\infty))$, because elements of $\mathcal{M}_f^+((0,\infty))$ consist of just a finite number of positive dirac masses.

3.2 The alternate formulation of the dual problem

In the previous subsection we saw that the Rogers payoff can be super-replicated by a simple predictable trading strategy. Thus we see that

$$D_0 \quad \geq \quad D \quad := \inf_{H\in\mathcal{H},\psi\in C_b((0,\infty))} \left[c(\psi) \mid F(\underline{X}_T, Y_T) \,\leq\, \psi(X_T) + \int_{[0,T]} H_t dX_t \qquad \forall X \in \mathcal{D}_{x_0}^+[0,T]\right]$$

where $\mathcal{H}$ is the space of all $\mathcal{F}_t$-simple predictable processes, because we are taking the inf over a larger set on the right hand side than the left hand side, since a Rogers trading strategy of the form $\int(X_T-a)1_{\underline{X}_T<a}\nu(da)$ for $\nu \in \mathcal{M}_f^+((0,\infty))$ is simple predictable, and for any strictly positive càdlàg function X with $X_0 = x_0 > 0$, we clearly have that $\underline{X}_T \in (0, x_0]$ and $X_T - \underline{X}_T \in (0,\infty)$.

3.3 The weak duality

Consider an admissible superhedging strategy, i.e. a pair $(\psi, H) \in C_b((0,\infty)) \times \mathcal{H}$ such that

$$F(\underline{X}_T, Y_T) \quad \leq \quad \psi(X_T) + \int_{[0,T]} H_t dX_t \qquad \forall X \in \mathcal{D}_{x_0}^+[0,T]$$

(we know such a pair exists because F is bounded), and let $\mathcal{A}$ denote the space of all admissible strategies. Now take a $\mathbb{P} \in \mathbb{M}^+(\mu_X)$ (we also know such a $\mathbb{P}$ exists - we can just use our favourite Skorokhod embedding to embed μ_X at time T). Then taking expectations under $\mathbb{P}$ we see that

$$\mathbb{E}^{\mathbb{P}}(F(\underline{X}_T, Y_T)) \quad \leq \quad c(\psi)\,.$$

Taking the sup over all $\mathbb{P} \in \mathbb{M}^+(\mu_X)$ on the left hand side, and the inf over $\mathcal{A}$ on the right hand side, we obtain the so-called *weak duality*, i.e.

$$P \quad \leq \quad D\,. \tag{8}$$

3.4 The full duality

Combining Proposition 3.1 with the weak duality in (8) and the fact that $D \leq D_0$, we obtain the first main result:

Theorem 3.2

$$P \quad = \quad P_0 \quad = \quad D_0 \quad = \quad D$$

i.e.

$$\sup_{\mathbb{P}\in\mathbb{M}^+(\mu_X)} \mathbb{E}^{\mathbb{P}}(F(\underline{X}_T, Y_T)) \quad = \inf_{H\in\mathcal{H},\psi\in C_b((0,\infty))} \left[c(\psi) \mid F(\underline{X}_T, Y_T) \,\leq\, \psi(X_T) + \int_{[0,T]} H_t dX_t \qquad \forall X \in \mathcal{D}_{x_0}^+[0,T]\right].$$

Remark 3.2 Recall that we only imposed that F be bounded and USC, so the duality result includes e.g. the case when F is the payoff of a down-and-in **One-Touch** option with upper semicontinuous payoff $1_{\underline{X}_T\leq b}$ at T for some $b \in (0, x_0)$ (i.e. $F(\underline{x}, y) = 1_{\underline{x}\leq b}$) or a standard **down-and-in put** option which pays $(K - X_T)^+ 1_{\underline{X}_T\leq b}$ at T (i.e. $F(\underline{x}, y) = (K - \underline{x} - y)1_{\underline{x}<b}$).

Remark 3.3 Theorem 3.1 shows that the minimal cost D of superhedging F for all càdlàg strictly positive stock price paths, using simple predictable trading strategies plus a static position in the tradeable options is (as we would expect) equal to P, the supremum of the expected value of the claim F over all martingale models which are calibrated to the market prices of the tradeable European options. But from the Rogers result, P is also equal to P_0, which is just a minimization problem over probability measures on a convex subset of $(0,\infty) \times (0,\infty)$. Note that we only assume X to be càdlàg but not necessarily a semimartingale, so the superhedge here still works if e.g. X is the exponential of fractional Brownian motion or (more generally) a rough path with Hölder exponent $H \neq \frac{1}{2}$, or a martingale Lévy process (e.g a compensated Poisson process or a pure jump martingale Lévy process as extreme cases).

Remark 3.4 Note that we can further reduce the cardinality of the constraints for P_0 in (6) by re-writing it as

$$P_0 \;=\; \sup_{\mu\in\mathcal{P}}\Big[\int F(\underline{x},y)\mu(d\underline{x},dy) \,|\, \int(\underline{x}+y-a)1_{y<a}\mu(d\underline{x},dy) \;\le\; 0\,,\; \int e^{ik(\underline{x}+y)}\mu(d\underline{x},dy) \;=\; \int e^{ikx}\mu_X(dx) \;\; \forall a\in(0,x_0)\,,\, k\in\mathbb{R})\Big]\,.$$

This suggest a numerical approximation scheme where we only impose the two constraints on a finite grid of a-values and k-values, which then reduces to a semi-infinite linear programming problem (see Davis,Obloj&Raval[DOR14] for more on this and in particular the Karlin-Isii duality theorem for such problems).

3.5 Adding One-Touch options as additional tradeable instruments and the Kertz-Rösler condition

Consider a One-Touch option on X which pays $1_{\underline{X}_T\le b}$ at time T, for $b<x_0$. Then to preclude arbitrage, the price of this option should be given by

$$\mathcal{O}(b) \;=\; \mathbb{Q}(\underline{X}_T\le b)$$

for some martingale measure $\mathbb{Q}$. Thus if we are given $\mathcal{O}(b)$ for all $b\in[0,x_0]$, we can extract a target marginal law $\mu_{\underline{X}}$ for $\underline{X}_T$, in addition to the target law for X that we can extract from European call option prices using the standard Breeden-Litzenberger argument. Define $\tilde{\mu}(dx):=\mu_X(-(dx-x_0))$, $\bar{\mu}:=\mu_{\underline{X}}(-(dx-x_0))$ as the mirror images of μ_X, $\mu_{\underline{X}}$ after shifting the starting point back to zero. Then if $\tilde{\mu},\bar{\mu}$ satisfy the Kertz-Rösler condition in (1) (or equivalently if μ_X, $\mu_{\underline{X}}$ are the X and $\underline{X}$ marginals respectively of some $\mu\in\mathcal{P}(\mathcal{X}_{x_0})$ which satisfies the two Rogers conditions in (3)) we know there exists a martingale such that $X_1\sim\mu_X$ and $\underline{X}_1\sim\mu_{\underline{X}}$. Then we can trivially amend the proof of the main Theorem 3.2 to show that

$$\sup_{\mathbb{P}\in\mathbb{M}^+(\mu_X):\underline{X}_T\sim\mu_{\underline{X}}}\mathbb{E}^{\mathbb{P}}(F(\underline{X}_T,Y_T)) \;=\; \inf_{H\in\mathcal{H},\,\psi,\phi\in C_b((0,\infty))}\Big[c(\psi)+\underline{c}(\phi)\;|\;F(\underline{X}_T,Y_T)\;\le\;\psi(X_T)\,+\,\phi(\underline{X}_T)\,+\,\int_{[0,T]}H_t dX_t\;,\;\; \forall X\in\mathcal{D}^+_{x_0}[0,T]\Big] \tag{9}$$

where $\underline{c}(\phi):=\int_{(0,x_0]}\phi\,d\mu_{\underline{X}}$, which is the minimial superhedging cost when we include these One-Touch options at all barrier levels as tradeable instruments, in addition to the tradeable European options at all strikes with target law μ_X.

4 Finite tradeable European and lookback options

In this section we assume there is only a finite number of tradeable European put options at strikes $K_1<K_2<...<K_N$ with prices $0\le P_1\le P_2\le...\le P_N$ for $i=1..N$. For this we make the following natural assumption throughout this section.

Assumption 4.1 *There exists a $\mu\in\mathcal{P}((0,\infty))$ with $\int xd\mu=x_0$ and $\int(K_i-x)^+d\mu=P_i$ for $i=1..N$.*

(see Theorem 3.1 in [DH07], and Proposition 2.1 in [DOR14]) for conditions on $(K_i)_{i=1..N}$ which ensure the existence of such a μ).

By adapting the proofs of Proposition 3.1 and Theorem 3.2, we obtain the following duality result.

Theorem 4.2 *Under Assumption 4.1, Let $\mathcal{P}(\mathcal{X}_{x_0})$ denote the space of probability measures on $\mathcal{X}_{x_0}$, and $F:\mathcal{X}_{x_0}\to[0,\infty)$ be bounded and upper semicontinuous and $R>x_0$. Then we have*

$$\begin{aligned}
P^R \;&:=\; \sup_{\mathbb{P}\in\mathbb{M}^+:\,\mathbb{E}^{\mathbb{P}}((K_i-X_T)^+)=P_i,\,i=1..N,\,\mathbb{P}(X_T>R)=0}\mathbb{E}^{\mathbb{P}}(F(\underline{X}_T,Y_T))\\
&=\; P_0^R \;=\; \sup_{\mu\in\mathcal{P}(\mathcal{X}_{x_0})}\Big[\int F(\underline{x},y)\mu(d\underline{x},dy)\;|\;\int(\underline{x}+y-a)1_{y<a}\mu(d\underline{x},dy)\le 0\;,\;\int(K_i-\underline{x}-y)^+\mu(d\underline{x},dy)\;=\;P_i\,,\\
&\qquad\int(\underline{x}+y)\mu(d\underline{x},dy)\;=\;x_0\,,\,\forall a\in(0,x_0)\,,\,i=1..N\,,\,\mu\{\underline{x}+y>R\}=0\Big]\\
=\;D_0^R\;&:=\;\inf_{\nu\in\mathcal{M}^+_f((0,x_0)),\alpha\in\mathbb{R},\beta_0\in\mathbb{R},\beta\in\mathbb{R}^N}\Big[\alpha\,+\,\beta_0x_0\,+\,\sum_{i=1}^N\beta_iP_i\;|\;F(\underline{x},y)\;\le\;\alpha\,+\,\beta_0(\underline{x}+y)\,+\,\sum_{i=1}^N\beta_i(K_i-x)^+\\
&\quad+\;\int_{(0,\infty)\times(0,\infty)}G^a(\underline{x},y)\nu(da)\quad,\quad\forall(\underline{x},y)\in\mathcal{X}_{x_0}:\underline{x}+y\le R\Big]\\
&=\;\inf_{H\in\mathcal{H},\alpha\in\mathbb{R},\beta\in\mathbb{R}^N}\Big[\alpha+\sum_{i=1}^N\beta_iP_i\;|\;F(\underline{X}_T,Y_T)\le\alpha+\sum_{i=1}^N\beta_i(K_i-X_T)^+ +\int_{[0,T]}H_tdX_t\quad\forall X\in\mathcal{D}^+_{x_0}[0,T]:x_T\le R\Big]
\end{aligned}$$

Proof. See Appendix C. ■

Remark 4.1 R can be made arbitrarily large here, and from the Markov inequality, we know that $\mathbb{P}(X_T > R) \leq \frac{x_0}{R}$ for all $\mathbb{P} \in \mathbb{M}^+$, hence by choosing $R = \frac{x_0}{\varepsilon}$, we can ensure that the superhedge works with probability $\geq 1-\varepsilon$. This *quantile hedging* approach (see also [Son15]) still allows for paths where $\bar{X}_T > R$; conversely a trader may wish to choose a particular finite R value which be believes that the stock price cannot exceed at maturity, which then allows him to reduce the superhedging cost.

Remark 4.2 The most common practical situation to apply Theorem 4.2 would be when F is the payoff of a down-and-in **One-Touch** option with USC payoff $1_{\underline{X}_T \leq b}$ at T for some $b \in (0, x_0)$ (i.e. $F(\underline{x}, y) = 1_{\underline{x} \leq b}$) or a standard **down-and-in put** option which pays $(K - X_T)^+ 1_{\underline{X}_T \leq b}$ at T (i.e. $F(\underline{x}, y) = (K - \underline{x} - y) 1_{\underline{x} < b}$ as in Remark 3.2.

Remark 4.3 We can also easily amend this result to include M tradeable **lookback options** with prices L_j and payoff function $(\underline{K}_j - \underline{x})^+$ for $j = 1..M$ if there exists a $\mu \in \mathcal{P}(\mathcal{X}_{x_0})$ with $\int x d\mu = x_0$ and $\int (K_i - x)^+ d\mu = P_i$ for $i = 1..N$ and $\int (\underline{K}_j - \underline{x})^+ d\mu = L_j$ for $j = 1..M$.

4.1 Removing the $X_T \leq R$ restriction - an upper bound for the superhedging cost

If we wish to remove the $\underline{x} + y \leq R$ restriction in Theorem 4.2, we can adapt the duality proof to obtain

$$
\begin{aligned}
P \;&:=\; \sup_{\mathbb{P} \in \mathbb{M}^+ : \, \mathbb{E}^{\mathbb{P}}((K_i - X_T)^+) = P_i, \, i=1..N} \mathbb{E}^{\mathbb{P}}(F(\underline{X}_T, Y_T)) \\
&\leq\; P_0 \;=\; \sup_{\mu \in \mathcal{P}(\mathcal{X}_{x_0})} \Big[\int F(\underline{x}, y) \mu(d\underline{x}, dy) \;\Big|\; \int (\underline{x} + y - a) 1_{y<a} \mu(d\underline{x}, dy) \leq 0 \,,\; \int (K_i - \underline{x} - y)^+ \mu(d\underline{x}, dy) \;=\; P_i \,, \\
&\qquad \int (\underline{x} + y) \mu(d\underline{x}, dy) \;\leq\; x_0 \,,\, \forall a \in (0, x_0) \,,\, i = 1..N \Big] \\
&=\; D_0 \;:=\; \inf_{\nu \in \mathcal{M}_f^+((0,x_0)), \alpha \in \mathbb{R}, \beta_0 \geq 0, \beta \in \mathbb{R}^N} \Big[\alpha + \beta_0 x_0 + \sum_{i=1}^N \beta_i P_i \;\Big|\; F(\underline{x}, y) \;\leq\; \alpha + \beta_0(\underline{x} + y) + \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ \\
&\qquad + \int_{(0,\infty) \times (0,\infty)} G^a(\underline{x}, y) \nu(da) \quad , \quad \forall (\underline{x}, y) \in \mathcal{X}_{x_0} \Big] \,.
\end{aligned}
$$

(proof follows from trivial modifications to Appendix C). We can only assert that $P \leq P_0$ now (as opposed to $P = P_0$) because we are only imposing that $\int x d\mu \leq x_0$ in the definition of P_0; the reason we cannot get equalities here is the Folland lemma in the Appendix requires the superhedging payoff to be bounded from below, which is not the case for a negative position in a forward contract, which we have excluded here. But we also know that $D \leq D_0$ and $P \leq D$ where

$$
D \;:=\; \inf_{H \in \mathcal{H}, \, \alpha \in \mathbb{R}, \, \beta \in \mathbb{R}^N} \Big[\alpha + \sum_{i=1}^N \beta_i P_i \;\Big|\; F(\underline{X}_T, Y_T) \;\leq\; \alpha + \sum_{i=1}^N \beta_i (K_i - X_T)^+ + \int_{[0,T]} H_t dX_t \quad \forall X \in \mathcal{D}_{x_0}^+[0,T] \Big]
$$

so

$$
P \;\leq\; D \;\leq\; D_0 \;=\; P_0
$$

which gives an upper bound for the superhedging cost D.

5 Robust hedging of options with a time-dependent barrier with tradeable European options at two maturities and tradeable forward-starting options

Now let $0 < T_1 < T_2$ and μ_1, μ_2 be probability measures on $(0, \infty)$ which are strictly increasing in the convex order and let μ_3 be an additional probability measure which will be used to incorporate forward-starting options into the calibration set. Let

$$
\tilde{\mathcal{X}} \;=\; (0, x_0] \times (0, \infty) \times (0, 1] \times [0, \infty) \;=\; \mathcal{X}_{x_0} \times \mathcal{X}_1 \tag{10}
$$

where $\mathcal{X}_1 = (0, 1] \times [0, \infty)$.

Assumption 5.1 $\int_{(0,\infty)} x\mu_1(dx) = \int_{(0,\infty)} x\mu_2(dx) = x_0$.

Let $X_{T_1,T_2} := X_{T_2}/X_{T_1}$, $\underline{X}_{T_1,T_2} := \inf_{s\in[T_1,T_2]} X_s/X_{T_1}$ and $Y_{T_1,T_2} = X_{T_1,T_2} - \underline{X}_{T_1,T_2}$. We let $\mathbb{M}^+(\mu_1,\mu_2,\mu_3) \subset \mathbb{M}^+$ denote those elements of $\mathbb{M}^+$ such that $X_{T_1} \sim \mu_1$, $X_{T_2} \sim \mu_2$ and $X_{T_2}/X_{T_1} \sim \mu_3$ and we assume μ_3 is such that $\mathbb{M}^+(\mu_1,\mu_2,\mu_3)$ is non-empty (see Corollary 5.4 below for a precise if and only if statement for when this condition is satisfied).

Assumption 5.2 $\int_{(0,\infty)} x^p\mu_3(dx) < \infty$ *for some* $p \in (1,\infty)$.

Proposition 5.3 *Let* $(\underline{X}, Y, V, W)$ *be a random variable defined on* $(\tilde{\mathcal{X}}, \mathcal{B}(\tilde{\mathcal{X}}), \mathbb{Q})$ *for some probability measure* $\mathbb{Q}$*. Then the law of* $(\underline{X}, Y, V, W)$ *is the law of* $(\underline{X}_{T_1}, Y_{T_1}, \underline{X}_{T_1,T_2}, Y_{T_1,T_2})$ *for some càdlàg martingale with* $X_0 = x_0 > 0$ *if and only if the following four conditions are satisfied*

$$\begin{cases} \mathbb{E}((X-a)1_{\underline{X}<a}) \quad \leq \quad 0 \qquad \forall a \in (0,x_0) \\ \mathbb{E}((V+W-b)1_{V<b}\, 1_{(\underline{X},Y)\in A}) \quad \leq \quad 0 \qquad \forall b \in (0,1)\,,\ A \in \mathcal{B}(\mathcal{X}_{x_0}) \\ \mathbb{E}(X) \quad = \quad x_0 \\ \mathbb{E}((V+W-1)1_{(\underline{X},Y)\in B}) \quad = \quad 0 \qquad \forall B \in \mathcal{B}(\mathcal{X}_{x_0}) \end{cases} \tag{11}$$

where $X := \underline{X} + Y$*, and all expectations here are taken under* $\mathbb{Q}$*.*

Proof. See Appendix D. ■

Corollary 5.4 $(\mu_1,\mu_2,\mu_3) \in \mathbb{M}^+(\mu_1,\mu_2,\mu_3)$ *if and only if there exists a probability measure* μ *on* $\tilde{\mathcal{X}}$ *such that if* $(\underline{X}, Y, V, W) \sim \mu$*, then the four conditions in* (11) *are satisfied and* $\underline{X} + Y \sim \mu_1, X(V+W) \sim \mu_2$ *and* $V + W \sim \mu_3$*, where* $X = \underline{X} + Y$*.*

Proof. This is a direct consequence of the previous proposition. ■

Lemma 5.5 *If* $\mathbb{E}(|V+W|^p) < \infty$ *for some* $p \in (1,\infty)$ *(which corresponds to the final condition in Assumption 5.2), the second and fourth conditions in* (11) *are satisfied if and only if*

$$\begin{cases} \mathbb{E}((V+W-b)1_{V<b}\,\varphi_1(\underline{X},Y)) \quad \leq \quad 0 \qquad \forall b \in (0,1),\ \varphi_1 \in C_c^+(\mathcal{X}_{x_0}) \\ \mathbb{E}((V+W-1)\varphi_2(\underline{X},Y)) \quad = \quad 0 \qquad \forall \varphi_2 \in C_c^+(\mathcal{X}_{x_0}) \end{cases} \tag{12}$$

where $C_c^+(\mathcal{X}_{x_0})$ *denotes the space of non-negative functions on* $\mathcal{X}_{x_0}$ *with compact support.*

Proof. $C_c(\mathcal{X}_{x_0})$ is dense in $L^q(\mathcal{X}_{x_0}, \mathbb{Q})$ for $1 \leq q < \infty$ (see e.g. Proposition 7.7.9 in [Fol99]); hence for any $A \in \mathcal{B}(\mathcal{X}_{x_0})$ (without loss of generality) we can find a sequence $\varphi_n \in C_c^+(\mathcal{X}_{x_0})$ such that $\mathbb{E}(|1_A - \varphi_n(\underline{X},Y)|^q)^{\frac{1}{q}} \leq \frac{1}{n}$ and hence

$$\begin{aligned} \mathbb{E}((V+W-b)1_{V<b}(1_A - \varphi_n(\underline{X},Y))) \quad &\leq \quad \mathbb{E}(|V+W-b|^p)^{\frac{1}{p}} \cdot \mathbb{E}(|1_A - \varphi_n(\underline{X},Y)^q|^{\frac{1}{q}} \\ \text{(where } \frac{1}{p} + \frac{1}{q} = 1, \text{using Holder's inequality} \quad & \\ &\leq \quad (\mathbb{E}(|V+W|^p)^{\frac{1}{p}} + \mathbb{E}(b^p)^{\frac{1}{p}}) \cdot \frac{1}{n} \text{ (by the Minkowski inequality)} \\ &\leq \quad C_1 \cdot \frac{1}{n} \end{aligned}$$

and some finite constant C_1, dealing with the second equation in (12) follows similarly. ■

Using this lemma, we obtain the main duality result for this section:

Theorem 5.6 *Let $F : \tilde{\mathcal{X}} \to (0, \infty)$ be bounded and upper semicontinuous. Then we have the following duality result:*

$$
\begin{aligned}
P &= \sup_{\mathbb{P}\in\mathbb{M}^+(\mu_1,\mu_2,\mu_3)} \mathbb{E}^{\mathbb{P}}(F(\underline{X}_{T_1}, Y_{T_1}, \underline{X}_{T_1,T_2}, Y_{T_1,T_2})) \\
&= P_0 \;:=\; \sup_{\mu\in\mathcal{P}(\tilde{\mathcal{X}})} \Big[\int F(\underline{x}, y)\mu(d\underline{x}, dy, dv, dw) \,|\, \int G^a(\underline{x}, y)\mu(d\underline{x}, dy, dv, dw) \le 0, \int G^b(v, w)\varphi(\underline{x}, y)\, \mu(d\underline{x}, dy, dv, dw) \le 0\,, \\
&\quad \int \psi_1(\underline{x} + y)\mu(d\underline{x}, dy, dv, dw) = c_1(\psi_1)\,, \int \psi_2((\underline{x} + y)(v + w))\mu(d\underline{x}, dy, dv, dw) = c_2(\psi_2), \\
&\quad \int \psi_3(v + w)\mu(d\underline{x}, dy, dv, dw) = c_3(\psi_3)\ ,\ \forall a \in (0, x_0)\,, b \in (0, 1),\ \varphi \in C_c^+(\mathcal{X}_{x_0})\,,\ \psi_1, \psi_2, \psi_3 \in C_b((0, \infty))\Big] \\
&= \inf_{\nu_1\in\mathcal{M}_f^+((0,x_0))\,,\,\nu_2\in\mathcal{M}_f^+((0,1)),\varphi\in C_c^+(\mathcal{X}_{x_0}),\,\psi_1,\psi_2,\psi_3\in C_b((0,\infty))} [c_1(\psi_1) + c_2(\psi_2) + c_3(\psi_3) \mid F(\underline{x}, y, v, w) \le \psi_1(\underline{x} + y) \\
&\quad + \psi_2((\underline{x} + y)(v + w)) + \psi_3(v + w) + \int_{(0,x_0)} G^a(\underline{x}, y)\nu_1(da) + \int_{(0,1)} G^b(v, w)\varphi(\underline{x}, y)\nu_2(db) \qquad \forall(\underline{x}, y, v, w) \in \tilde{\mathcal{X}}] \\
&= \inf_{H\in\mathcal{H},\psi_1,\psi_2,\psi_3\in C_b((0,\infty))} \big[c_1(\psi_1) + c_2(\psi_2) + c_3(\psi_3) \mid F(\underline{X}_{T_1}, Y_{T_1}, \underline{X}_{T_1,T_2}, Y_{T_1,T_2}) \le \psi_1(X_{T_1}) + \psi_2(X_{T_2}) + \psi_3(X_{T_2}/X_{T_1}) \\
&\quad + \int_{[0,T_2]} H_t dX_t \qquad \forall X \in \mathcal{D}^+_{x_0}[0, T_2]\big]
\end{aligned}
\tag{13}
$$

where $c_i(\psi_i) := \int \psi_i(x)\mu_i(dx)$ for $i = 1, 2, 3$.

Proof. See Appendix C. ■

Remark 5.1 This result includes the case when F is a general barrier option with upper semicontinuous payoff $(K - X_{T_2})^+ 1_{\underline{X}_{T_1}\le b_1} 1_{\inf_{T_1\le t\le T_2} X_t\le b_2}$ for $K, b_1, b_2 \in (0, \infty)$, which is a down-and-in put option with a **time-dependent barrier** level.

P is the minimal cost of superhedging a (possibly) time-dependent barrier option with payoff $F(\underline{X}_{T_1}, Y_{T_1}, \underline{X}_{T_1,T_2}, Y_{T_1,T_2})$ using dynamic trading in the underlying and European options with all strikes at maturities T_1 and T_2 and forward-starting options which pay $(X_{T_2}/X_{T_1} - K)^+$ for all strikes $K > 0$.

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A Proof of Proposition 3.1

Throughout we let $z = (\underline{x}, y)$. We first define the following class of functions

$$\mathcal{E} \quad = \quad \{g : \mathcal{X}_{x_0} \mapsto \mathbb{R},\, g(\underline{x}, y) \;=\; \int (\underline{x} + y - a)\, 1_{\underline{x} < a}\, \nu(da) \;\;,\;\; \nu \in \mathcal{M}_f^+((0, x_0))\}\,.$$

Then all elements of $\mathcal{E}$ are bounded from below (because $\underline{x} + y \in (0, x_0]$) and ν is just a finite sum of non-negative dirac masses) and lower semicontinuous in the usual sup norm topology. Hence throughout all Appendices, all infs or sups over $g \in \mathcal{E}$ are in fact infs/sups over $\mathcal{M}_f^+([x_0, 0))$.

We now recall Theorem 1 from Rockafellar[Roc74]:

Theorem A.1 *Let X and U be real linear spaces and let $F : X \times U \to [-\infty, \infty]$ be convex. Then*

$$\Phi(u) \quad = \quad \inf_{x \in X} F(x, u)$$

is convex.

To apply the conjugate duality framework in [Roc74], we choose the X space at the start of section 3 in [Roc74] to be $C_0(\mathcal{X}_{x_0})$ paired with its dual space $V = X^* = C_0(\mathcal{X}_{x_0})^*$ (the space of signed Radon measures on $\mathcal{X}_{x_0}$)) with the bilinear form $\langle x, v\rangle = \int_{\mathcal{X}_{x_0}} x dv$. We use the sup norm topology on X and the weak* topology on V, which are consistent topologies in the sense of section 3 in [Roc74] i.e. both these topologies are locally convex and any continuous linear functional F on V under the weak* topology can be represented by a unique element of $x \in X$ as $F(\mu) = \langle x, \mu\rangle$ (see e.g. Corollary 4.8, page 123 in Weaver[Wea13]).

We define the value function $\varphi : C_0(\mathcal{X}_{x_0}) \to \mathbb{R}$ as

$$\begin{aligned}
\varphi(u) &= \inf_{\psi\in C_b((0,\infty)),\, g\in\mathcal{E}} [c(\psi) \mid F + u \le \psi + g] \\
&= \inf_{\psi\in C_b((0,\infty)),\nu\in\mathcal{M}_f^+((0,x_0))} [c(\psi) \mid F + u \le \psi + \int (\underline{x} + y - a)\, 1_{\underline{x}<a}\nu(da)] \\
&= \inf_{\psi\in C_b((0,\infty)),\nu\in\mathcal{M}_f((0,x_0))} [c(\psi) \mid F + u \le \psi + g_\nu \,,\, \nu^-((0,x_0]) = 0] \\
&= \inf_{\psi\in C_b((0,\infty)),\,\nu\in\mathcal{M}_f((0,x_0))} [c(\psi) + \infty\cdot 1_{(\psi,\nu,u)\notin C}] \qquad \text{(A-1)}
\end{aligned}$$

where ν^- is the negative part of ν, $C = \{(\psi,\nu,u) : \psi \in C_b((0,\infty)), \nu \in \mathcal{M}_f^+((0,x_0)), F + u \le \psi + g_\nu\}$ and $F + u \le \psi + g_\nu$ means that $F(z) + u(z) \le \psi(x) + g_\nu(z)$ for all $z \in \mathcal{X}_{x_0}$. We note that C is a convex subset of $C_b((0,\infty)) \times \mathcal{M}_f((0,x_0)) \times C_0(\mathcal{X}_{x_0})$.

We let $z = (\underline{x}, y)$ throughout Appendix A. We first need to verify that φ is convex. To this end, proceeding as on page 1 in [Roc74] we first note that the function f defined by $f(\psi,\nu,u) = c(\psi)$ is affine in ψ, and thus is classically convex on C. Thus the function $\hat{f}$ defined by $\hat{f} = f$ on C and $\hat{f} = +\infty$ otherwise is convex as an extended real-valued function, i.e. the epigraph epi $\hat{f} := \{(\psi,\nu,u,\alpha) : \psi \in C_b((0,\infty)), \nu \in \mathcal{M}_f((0,x_0)), u \in C_0(\mathcal{X}_{x_0}), \alpha \in \mathbb{R}, \alpha \ge \hat{f}(\psi,\nu,,u)\}$ is convex as a subset of $C_b((0,\infty)) \times \mathcal{M}_f((0,x_0)) \times C_0(\mathcal{X}_{x_0}) \times \mathbb{R}$. $C_b((0,\infty))$ and $\mathcal{M}_f((0,x_0))$ are linear spaces, and thus so is $C_b((0,\infty)) \times \mathcal{M}_f((0,x_0))$, so convexity follows from Theorem A.1.

By the Riesz representation theorem, the dual of $C_0(\mathcal{X}_{x_0})$ may be identified with $\mathcal{M}(\mathcal{X}_{x_0})$. Thus the conjugate of φ can be written as

$$\begin{aligned}
\varphi^*(\mu) &= \sup_{u\in C_0(\mathcal{X}_{x_0})} [\int u d\mu - \varphi(u)] \\
&= \sup_{u\in C_0(\mathcal{X}_{x_0}),\psi\in C_b((0,\infty)),g\in\mathcal{E}} [\int u d\mu - c(\psi) \mid F + u \le \psi + g]
\end{aligned}$$

for $\mu \in \mathcal{M}(\mathcal{X}_{x_0})$. The following lemma will be needed:

Lemma A.2 *If $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$, then there exists a $u \in C_0(\mathcal{X}_{x_0})$ such that $u \le 0$ and $\int u d\mu > 0$.*

Proof. See Appendix B. ■

From this we obtain the trivial corollary:

Corollary A.3 *For $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$, we have $\sup_{u\in C_0(\mathcal{X}_{x_0}),u\le 0} \int u d\mu = \infty$.*

Lemma A.4 *For $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$ we have*

$$\sup_{u\in C_0(\mathcal{X}_{x_0})} [\int u d\mu \mid F + u \le \psi + g] = +\infty \,. \qquad \text{(A-2)}$$

Proof. Let $\bar{u} \in C_0(\mathcal{X}_{x_0})$ satisfy the constraint $F + \bar{u} \le \psi + g$. Then for $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$ we have

$$\begin{aligned}
\sup_{u\in C_0(\mathcal{X}_{x_0})} [\int u d\mu \mid F + u \le \psi + g] &= \sup_{u\in C_0(\mathcal{X}_{x_0})} [\int (u + \bar{u}) d\mu \mid F + u + \bar{u} \le \psi + g] \\
&\ge \sup_{u\in C_0(\mathcal{X}_{x_0}),u\le 0} [\int (u + \bar{u}) d\mu \mid F + u + \bar{u} \le \psi + g] \\
&= \int \bar{u} d\mu + \sup_{u\in C_0(\mathcal{X}_{x_0}),u\le 0} \int u d\mu = \infty
\end{aligned}$$

from Corollary A.3. ■

Corollary A.5 *$\varphi^*(\mu) = +\infty$ for $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$.*

Intuitively, for $\mu \in \mathcal{M}_+(\mathcal{X}_{x_0})$ we want to make u as large as possible whilst ensuring that the constraint is satisfied, by setting $u = u_1 = \psi + g - F$. However g is only known to be bounded from below and LSC and ψ is only known to be in $C_b((0,\infty))$, so we may not have $u_1 \in C_0(\mathcal{X}_{x_0})$. To deal with this issue, we recall the following lemma from Folland[Fol99]:

Lemma A.6 *(Corollary 7.13 in [Fol99]). If μ is a non-negative Radon measure and f is bounded from below and lower semicontinuous, then*

$$\int f d\mu \quad = \sup_{h \in C_0(\mathcal{X}_{x_0})} [\int h d\mu \mid h \le f]\,. \tag{A-3}$$

Thus, using this lemma and using that ψ, g and $-F$ are lower semincontinuous and bounded from below, we have the following corollary:

Corollary A.7 *For $\mu \in \mathcal{M}^+(\mathcal{X}_{x_0})$ we have*

$$\sup_{u \in C_0(\mathcal{X}_{x_0})} [\int u d\mu \ \mid \ F + u \ \le \ \psi + g] \quad = \quad \int [\psi + g - F] d\mu$$

where $\int \psi d\mu$ is shorthand for $\int \psi(x)\mu(d\underline{x}, dy)$.

Thus we can re-write $\varphi^*(\mu)$ as

$$\varphi^*(\mu) \quad = \quad \sup_{\psi \in C_b((0,\infty)), g \in \mathcal{E}} [\int (\psi \ + \ g \ - \ F)\, d\mu \ - \ c(\psi)]$$

for $\mu \in \mathcal{M}^+(\mathcal{X}_{x_0})$. Using (5) we can re-write $\int \psi d\mu - c(\psi)$ as

$$\int \psi d\mu - c(\psi) \quad = \quad \int \psi d\mu \ - \ \int \psi(x) \mu_X(dx)\,.$$

If there exists a $\psi \in C_b((0,\infty))$ such that $c(\psi) \ne \int \psi d\mu$ then (by linearity) we see that

$$\sup_{\psi \in C_b((0,\infty))} [\int \psi d\mu \ - \ \int \psi(x) \mu_X(dx)] \quad = \quad +\infty\,.$$

Thus we have

$$\varphi^*(\mu) \quad = \quad \begin{cases} \sup_{g \in \mathcal{E}} [\int (g \ - \ F)\, d\mu] & \text{if} \quad \mu \in \mathcal{P}(\mathcal{X}_{x_0})\ ,\ \int \psi d\mu \ = \ c(\psi) \qquad \forall \psi \in C_b((0,\infty)), \\ +\infty & \text{otherwise}\,. \end{cases}$$

We now take the supremum over $g \in \mathcal{E}$, and let $\mathcal{R}$ denote the set of probability measures on $\mathcal{X}_{x_0}$ which satisfy the first condition in (3). If $\mu \notin \mathcal{R}$ then we can find a $g \in \mathcal{E}$ such that $\int g d\mu > 0$ and thus (by linearity) $\varphi^*(\mu) = \infty$; otherwise we have that $\int g\, d\mu = 0$ for all $g \in \mathcal{E}$. Putting this together we have

$$\varphi^*(\mu) \quad = \quad \begin{cases} -\int F\, d\mu & \text{if} \quad \mu \in \mathcal{R},\ \int \psi d\mu \ = \ c(\psi) \qquad \forall \psi \in C_b((0,\infty)), \\ +\infty & \text{otherwise} \end{cases}$$

By the bi-conjugate theorem (cf. Theorem 5 in [Roc74]), we have that $\varphi^{**} = \operatorname{cl} \operatorname{co} \varphi$, where cl denotes the closure operator and co denotes the convex hull (see [Roc74] for definitions). Since φ is convex, $\operatorname{co} \varphi = \varphi$, and (setting $\psi = \alpha$ a constant so $c(\psi) = \alpha$, and $g \equiv 0$) we also note that

$$\varphi(u) \quad = \inf_{\psi \in C_b((0,\infty)),\, g \in \mathcal{E}} [c(\psi) \ \mid \ F + u \ \le \ \psi + g] \quad \le \quad \inf_{\alpha \in \mathbb{R}} [\,\alpha \mid \ F + u \ \le \ \alpha] \quad \le \quad \|F\| + \|u\| \quad < \quad \infty \tag{A-4}$$

so $\operatorname{dom} \varphi$ is the whole space $C_0(\mathcal{X}_{x_0})$, i.e. φ is finite on $C_0(\mathcal{X}_{x_0})$ and in particular is bounded on any ball around the origin by $\|F\| + |\delta|$ where δ is the size of the ball. But a proper convex function f is continuous on $\operatorname{int} \operatorname{dom} f$ if and only if it is bounded from above on a neighborhood of an interior point of $\operatorname{dom} f$ (Theorem 2.14 in [BP12]), thus φ is continuous on $C_0(\mathcal{X}_{x_0})$ and hence $\operatorname{cl} \varphi = \varphi$, and

$$\varphi(u) \quad = \quad \varphi^{**}(u) \quad = \quad \sup_{\mu} (\int u d\mu - \varphi^*(\mu)) \quad = \quad \sup_{\mu \in \mathcal{R}} (\int (u + F)\, d\mu \,|\, \int \psi d\mu \ = \ c(\psi) \quad , \quad \forall \psi \in C_b((0,\infty))$$

so in particular

$$\varphi(0) \quad = \quad \varphi^{**}(0) \quad = \quad \sup_{\mu \in \mathcal{R}} [\int F d\mu \,|\, \int \psi d\mu \ = \ c(\psi) \quad \forall \psi \in C_b((0,\infty))\,] \quad = \quad P_0\,.$$

But $\varphi(0) = \inf_{\psi \in C_b((0,\infty)), g \in \mathcal{E}} [\,c(\psi) \ \mid \ F(z) \le \psi(x) + g(z) \quad \forall z \in \mathcal{X}_{x_0}] = D_0$. Finally the fact that $P = P_0$ just follows from Proposition 2.2.

B Proof of Lemma A.2

$\mathcal{M}^+(\mathcal{X}_{x_0})$ is a closed convex subset of $\mathcal{M}(\mathcal{X}_{x_0})$, and any $\mu \in \mathcal{M}(\mathcal{X}_{x_0}) \setminus \mathcal{M}^+(\mathcal{X}_{x_0})$ is a closed compact subset of $\mathbf{M}$. Thus, by the Hahn-Banach separation theorem, there exists a $\lambda \in C_0(\mathcal{X}_{x_0})$ such that

$$\langle \lambda, \mu \rangle \quad > \sup_{\nu \in \mathcal{M}^+(\mathcal{X}_{x_0})} \langle \lambda, \nu \rangle \tag{B-1}$$

We now need to verify that λ is non-positive. If there exists a $z \in \mathcal{X}_{x_0}$ such that $\lambda(z) > 0$ then $\langle \lambda, \alpha\, \delta_{\{z\}} \rangle = \alpha \lambda(z) > 0$ for every $\alpha > 0$, so the sup on the right hand side of (B-1) can be made infinitely large, which violates the inequality. Thus, we must have $\lambda \leq 0$. In that case, the sup on the right equals zero (setting $\nu = 0$). Thus $\langle \lambda, \mu \rangle > 0$ which completes the proof.

C Proof of Theorem 4.2

Proceeding along similar lines to Appendix A, we define $\varphi : C_0(\mathcal{X}_{x_0}) \to \mathbb{R}$ as

$$\varphi(u) \;=\; \inf_{\alpha, \beta_0 \in \mathbb{R},\, \beta \in \mathbb{R}^N,\, g \in \mathcal{E}} [\, \alpha \,+\, \beta_0 x_0 \,+\, \sum_{i=1}^N \beta_i P_i \;|\; F + u \;\leq\; \alpha \,+\, \beta_0(\underline{x} + y) \,+\, \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ \;+\; g \;+\; \infty \cdot 1_{\underline{x}+y>R}]$$

We can easily verify that φ is convex using the same arguments as Appendix A and

$$\begin{aligned} \varphi^*(\mu) \;&=\; \sup_{u \in C_0(\mathcal{X}_{x_0})} [\int u d\mu \,-\, \varphi(u)] \\ &=\; \sup_{u \in C_0(\mathcal{X}_{x_0}), \alpha, \beta_0 \in \mathbb{R}, \beta \in \mathbb{R}^N,\, g \in \mathcal{E}} [\int u d\mu - \alpha \,-\, \beta_0 x_0 - \sum_{i=1}^N \beta_i P_i \;|\; F + u \;\leq\; \alpha \,+\, \beta_0(\underline{x} + y) \\ &\quad +\; \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ + g \,+\, \infty \cdot 1_{\underline{x}+y>R}] \end{aligned}$$

for $\mu \in \mathcal{M}(\mathcal{X}_{x_0})$, and using almost identical arguments to Appendix A, we find that $\varphi^*(\mu) = +\infty$ for $\mu \notin \mathcal{M}^+(\mathcal{X}_{x_0})$.

Corollary C.1 *For $\mu \in \mathcal{M}^+(\mathcal{X}_{x_0})$ we have*

$$\begin{aligned} &\sup_{u \in C_0(\mathcal{X}_{x_0})} [\int u d\mu \;\;|\;\; F + u \;\leq\; \alpha \,+\, \beta_0(\underline{x} + y) \,+\, \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ \,+ g + \infty \cdot 1_{\underline{x}+y>R}\,] \\ =\; &\int [\alpha \,+\, \beta_0(\underline{x} + y) \,+\, \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ \,+ g \,+\, \infty \cdot 1_{\underline{x}+y>R}]) - F\,]\, d\mu\,. \end{aligned}$$

Proof. $\alpha + \beta_0(\underline{x}+y) + \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ g + \infty \cdot 1_{\underline{x}+y>R} - F$ is LSC and bounded from below, so the result follows from again from Corollay 7.13 in [Fol99]. Note that without the $\infty \cdot 1_{\underline{x}+y>R}$ term, this function is not bounded from below if $\beta_0 < 0$. ■

Thus we can re-write $\varphi^*(\mu)$ as

$$\varphi^*(\mu) = \sup_{\alpha \in \mathbb{R}, \beta_0 \in \mathbb{R}, \beta \in \mathbb{R}^N, g \in \mathcal{E}} [\int (\alpha \,+\, \beta_0(\underline{x} + y) \,+\, \sum_{i=1}^N \beta_i (K_i - \underline{x} - y)^+ \,+\, g \,+\, \infty \cdot 1_{\underline{x}+y>R} \,-\, F)\, d\mu \,-\, \alpha \,-\, \beta_0 \underline{x}_0 \,-\, \sum_{i=1}^N \beta_i P_i]$$

for $\mu \in \mathcal{M}^+(\mathcal{X}_{x_0})$.

If there exists a $i = 1..N$ such that $\int (K_i - \underline{x} - y)^+ d\mu \neq P_i$ then (by linearity) we see that $\varphi^*(\mu) = +\infty$. Similarly, if μ is not a probability measure, then $\int \alpha d\mu - \alpha \neq 0$, so we also see that $\varphi^*(\mu) = +\infty$. Likewise, if $\int (\underline{x} + y) d\mu \neq x_0$ then $\varphi^*(\mu) = +\infty$. And if $\mu\{\underline{x} + y > R\} > 0$ then clearly we also have that $\varphi^*(\mu) = +\infty$. Putting these facts together, we see that

$$\varphi^*(\mu) \;=\; \begin{cases} \sup_{g \in \mathcal{E}} \int (g \,-\, F)\, d\mu & \mu \in \mathcal{P}(\mathcal{X}_{x_0})\,,\ \int (\underline{x} + y) d\mu \;=\; x_0\,,\ \int (K_i - \underline{x} - y)^+ d\mu \;=\; P_i \ \forall i = 1..N,\ \mu(\underline{x} + y > R) = 0 \\ +\infty & \text{otherwise} \end{cases}$$

We now consider the supping over $g \in \mathcal{E}$, and let $\mathcal{R}$ denote the set of probability measures on $\mathcal{X}_{x_0}$ which satisfy the first condition in (3). If $\mu \notin \mathcal{R}$ then we can find an $a \in (0, x_0)$ such that $\int (\underline{x} + y - a) 1_{\underline{x}<a} d\mu > 0$ and hence there exists a $g \in \mathcal{E}$ such that $\int g d\mu > 0$ and thus (by linearity) $\varphi^*(\mu) = \infty$; otherwise we have that $\int g\, d\mu \leq 0$ for all $g \in \mathcal{E}$. Putting this together we have

$$\varphi^*(\mu) \;=\; \begin{cases} -\int F\, d\mu & \text{if} \quad \mu \in \mathcal{R}\,,\, \mu\{\underline{x} + y > R\} = 0\,,\, \mu \in \mathcal{P}(\mathcal{X}_{x_0})\ ,\ \int (\underline{x} + y) d\mu \;\leq\; x_0\ ,\ \int (K_i - \underline{x} - y)^+ d\mu \;=\; P_i \quad \forall i = 1..N \\ +\infty & \qquad\qquad \text{otherwise} \end{cases}$$

By the bi-conjugate theorem (cf. Theorem 5 in [Roc74]), we have that $\varphi^{**} = \mathrm{cl}\,\mathrm{co}\,\varphi$, where cl denotes the closure operator and co denotes the convex hull (see [Roc74] for definitions). We wish to show that $\mathrm{cl}\,\mathrm{co}\,\varphi = \varphi$.

Since φ is convex, $\mathrm{co}\,\varphi = \varphi$ and φ is continuous at the origin if and only if it is bounded from above on a neighborhood of the origin (Theorem 8 in [Roc74]). If we now consider a neighborhood of $u = 0$ with $\|u\| \leq \delta$ with $\delta > 0$, then we have

$$\varphi(u) \;\leq\; \inf_{\alpha}[\, \alpha \;|\; F + u \;\leq\; \alpha] \;<\; \infty$$

where the second line follows by setting $\beta_0 = 0$, $\beta = 0$, $g \equiv 0$, and using that F is bounded. Thus φ is finite on a neighborhood of $u = 0$, so φ is continuous at the origin. But a convex function that is continuous at a point is continuous throughout the interior of its domain (so if the domain is the whole space, it is continuous everywhere). Thus we have $\mathrm{cl}\,\varphi = \varphi$,

$$\varphi(u) \;=\; \varphi^{**}(u) \;=\; \sup_{\mu}\Big(\int u d\mu - \varphi^*(\mu)\Big) \;=\; \sup_{\mu\in\mathcal{R}} \int (u + F)\, d\mu \,|\, \int \alpha \;+\; \beta_0(\underline{x} + y) \;+\; \sum_{i=1}^{N} \beta_i (K_i - \underline{x} - y)^+ d\mu \;=\; \sum_{i=1}^{N} \beta_i P_i$$
$$\forall \alpha \;+\; \beta_0(\underline{x} + y) \;+\; \sum_{i=1}^{N} \beta_i (K_i - \underline{x} - y)^+ \in C_0((0,\infty) \times (0,\infty))$$

so in particular

$$\varphi(0) \;=\; \varphi^{**}(0) \;=\; \sup_{\mu\in\mathcal{R}}\Big[\int F d\mu \;|\; \int (\underline{x} + y - a) 1_{y<a} \mu(d\underline{x}, dy) \leq 0\ ,\ \int (K_i - \underline{x} - y)^+ \mu(d\underline{x}, dy) \;=\; P_i\,,$$
$$\int (\underline{x} + y) \mu(d\underline{x}, dy) \;\leq\; x_0\,,\, \forall a \in (0, x_0)\,,\, i = 1..N\,,\, \mu\{\underline{x} + y > R\}\Big] \;=\; P_0\,.$$

But $\varphi(0) = D_0$. Finally the fact that $P = P_0$ just follows from Proposition 2.2.

D Proof of Theorem 5.6

We now let $z = (\underline{x}, y, v, w)$ and we first define the following classes of functions

$$\mathcal{E}_1 \;=\; \{g_1 : \mathcal{X}_{x_0} \to \mathbb{R} :\; g_1(\underline{x}, y) \;=\; \int (\underline{x} + y - a)\, 1_{\underline{x}<a} \nu_1(da)\ \ ,\ \ \nu_1 \in \mathcal{M}_f^+((0, x_0))\}\,,$$
$$\mathcal{E}_2 \;=\; \{g_2 : \tilde{\mathcal{X}} \to \mathbb{R} :\; g_2(\underline{x}, y, v, w) \;=\; \int (v + w - b)\, \varphi(\underline{x}, y) 1_{v<b}\, \nu_2(db)\ \ ,\ \ \nu_2 \in \mathcal{M}_f^+((0,1)),\, \varphi \in C_0^+(\mathcal{X}_{x_0})\}\ .$$

Then all elements of $\mathcal{E}_1, \mathcal{E}_2$ are bounded from below and LSC in the usual sup norm topology.

Similar to Appendix A, we define the value function $\varphi : C_0(\tilde{\mathcal{X}}) \to \mathbb{R}$ as

$$\varphi(u) \;=\; \inf_{\psi_1,\psi_2,\psi_3 \in C_b((0,\infty)), g_1\in\mathcal{E}_1, g_2\in\mathcal{E}_2} [c_1(\psi_1) + c_2(\psi_2) + c_3(\psi_3) \;|\; F + u \;\leq\; \psi_1 + \psi_2 + \psi_3 + g_1 + g_2] \qquad \text{(D-1)}$$

where $\tilde{\mathcal{X}}$ is defined in (10), and again we can trivially show that φ is convex using Theorem 1 in [Roc74]. The conjugate of φ (which we denote by $\varphi^* : \mathcal{M}(\tilde{\mathcal{X}}) \to \mathbb{R}$) can be written as

$$\varphi^*(\mu) \;=\; \sup_{u\in C_0(\tilde{\mathcal{X}})} \Big[\int u d\mu \;-\; \varphi(u)\Big]$$
$$\;=\; \sup_{u\in C_0(\tilde{\mathcal{X}}),\psi_1,\psi_2,\psi_3\in C_b((0,\infty)),g_1\in\mathcal{E}_1,g_2\in\mathcal{E}_2} \Big[\int u d\mu - c_1(\psi_1) - c_2(\psi_2) - c_3(\psi_3) \;|\; F + u \;\leq\; \psi_1 + \psi_2 + \psi_3 + g_1 + g_2\Big]$$

and by repeating the steps in Appendix A we can again verify that $\varphi^*(\mu) = \infty$ for $\mu \notin \mathcal{M}^+(\tilde{\mathcal{X}})$.

For $\mu \in \mathcal{M}^+(\tilde{\mathcal{X}})$ we want to make u as large as possible whilst ensuring that the constraint is satisfied, by setting $u = \psi_1 + \psi_2 + \psi_3 + g_1 + g_2 - F$. Hence for $\mu \in \mathcal{M}^+(\mathcal{X})$, using Corollary 7.13 in [Fol99] as before we have

$$\sup_{u \in C_0(\tilde{\mathcal{X}})} [\int u d\mu \mid F + u \leq \psi_1 + \psi_2 + \psi_3 + g_1 + g_2] = \int [\psi_1 + \psi_2 + \psi_3 + g_1 + g_2 - F] d\mu .$$

Thus we can re-write $\varphi^*(\mu)$ as

$$\varphi^*(\mu) = \sup_{\psi_1, \psi_2 \in C_b((0,\infty)), g_1, g_2 \in \mathcal{E}} [\int (\psi_1 + \psi_2 + \psi_3 + g_1 + g_2 - F) \, d\mu - c_1(\psi_1) - c_2(\psi_2) - c_3(\psi_3)] .$$

We can re-write $\int (\psi_1 + \psi_2 + \psi_3) d\mu - c_1(\psi_1) - c_2(\psi_2) - c_3(\psi_3)$ as

$$\begin{aligned} & \int (\psi_1 + \psi_2 + \psi_3) d\mu - c_1(\psi_1) - c_2(\psi_2) - c_3(\psi_3) \\ = & \int (\psi_1 + \psi_2 + \psi_3) d\mu - \int \psi_1(x) \mu_1(dx) - \int \psi_2(x) \mu_2(dx) - \int \psi_3(x) \mu_3(dx) . \end{aligned}$$

If there exists a ψ_1 or a ψ_2 or a $\psi_3 \in C_b((0,\infty))$ such that $c_1(\psi_1) \neq \int \psi_1 d\mu$ or $c_2(\psi_2) \neq \int \psi_2 d\mu$ or $c_3(\psi_3) \neq \int \psi_3 d\mu$ then (by linearity) we see that

$$\sup_{C_b((0,\infty))} [\int (\psi_1 + \psi_2 + \psi_3) d\mu - \int \psi_1(x) \mu_1(dx) - \int \psi_2(x) \mu_2(dx) - \int \psi_3(x) \mu_3(dx)] = +\infty .$$

Thus

$$\varphi^*(\mu) = \sup_{g_1 \in \mathcal{E}_1, g_2 \in \mathcal{E}_2} \int (g_1 + g_2 - F) \, d\mu$$

if $\int \psi_1 d\mu = c_1(\psi_1)$, $\int \psi_2 d\mu = c_2(\psi_2)$ and $\int \psi_3 d\mu = c_3(\psi_3)$ for all $\psi_1, \psi_2, \psi_3 \in C_b((0,\infty))$ (which implies that μ is also a probability measure), otherwise $\varphi^*(\mu) = +\infty$.

We now consider the supping over $g_1 \in \mathcal{E}_1$. If there exists a $a \in (0, x_0)$ such that $\int G^a(\underline{x}, y) \mu(d\underline{x}, dy, dv, dw) > 0$ then we can find a $g_1 \in \mathcal{E}_1$ such that $\int g_1 d\mu > 0$ and thus (by linearity) $\varphi^*(\mu) = +\infty$. Similarly if there exists a $b \in (0,1)$ such that $\int G^b(v, w) \mu(d\underline{x}, dy, dv, dw) > 0$ then we can find a $g_2 \in \mathcal{E}_2$ such that $\int g_2 d\mu > 0$ and $\varphi^*(\mu) = +\infty$; otherwise we have that $\int g \, d\mu \leq 0$ for all $g \in \mathcal{E}_1 \cap \mathcal{E}_2$ and we see that

$$\varphi^*(\mu) = \begin{cases} -\int F \, d\mu \quad \text{if} \quad \int G^a(\underline{x}, y) \mu(d\underline{x}, dy, dv, dw) \leq 0 \, , \int G^b(v, w) \mu(d\underline{x}, dy, dv, dw) \leq 0, \\ \int \psi_1 d\mu = c_1(\psi_1) \, , \int \psi_2 d\mu = c_2(\psi_2) \, , \int \psi_3 d\mu = c_3(\psi_3) \qquad \forall \psi_1, \psi_2, \psi_3 \in C_b((0,\infty)), \\ +\infty \qquad \text{otherwise} \end{cases}$$

By the bi-conjugate theorem (cf. Theorem 5 in [Roc74]), we have that $\varphi^{**} = \mathrm{cl}\,\mathrm{co}\,\varphi$, where cl denotes the closure operator and co denotes the convex hull (see [Roc74] for definitions). We wish to show that $\mathrm{cl}\,\mathrm{co}\,\varphi = \varphi$.

Since φ is convex, $\mathrm{co}\,\varphi = \varphi$ and φ is continuous at the origin if and only if it is bounded from above on a neighborhood of the origin (Theorem 8 in [Roc74]). If we now consider a neighborhood of $u = 0$ with $\|u\| \leq \delta$ with $\delta > 0$, then by considering $g_1 = g_2 = 0$, $\psi_2 = \psi_3 = 0$ and ψ_1 equal to a constant, we see that

$$\begin{aligned} \varphi(u) &= \inf_{\psi_1, \psi_2, \psi_3 \in C_b((0,\infty)), g_1 \in \mathcal{E}_1, g_2 \in \mathcal{E}_2} [c_1(\psi_1) + c_2(\psi_2) + c_3(\psi_3) \mid F + u \leq \psi_1 + \psi_2 + \psi_3 + g_1 + g_2] \\ &\leq \inf_{\alpha \in \mathbb{R}} [\alpha \mid F + u \leq \alpha] < \infty \end{aligned}$$

because F is bounded by assumption and $\|u\| \leq \delta$. Thus φ is finite on a neighborhood of $u = 0$, so φ is continuous at the origin. But a convex function that is continuous at a point is continuous throughout the interior of its domain (so if the domain is the whole space, it is continuous everywhere). Thus $\mathrm{cl}\,\varphi = \varphi$, and in particular

$$\begin{aligned} \varphi(0) &= \varphi^{**}(0) \\ &= \sup_{\mu \in \mathcal{P}(\tilde{\mathcal{X}})} [\int F d\mu \mid \int \psi_1 d\mu = c_1(\psi_1) \, , \int \psi_2 d\mu = c_2(\psi_2) \, , \int \psi_3 d\mu = c_3(\psi_3) \\ & \quad \int g_1(\underline{x}, y) \mu(d\underline{x}, dy, dv, dw) \leq 0, \int g_2(\underline{x}, y, v, w) \mu(d\underline{x}, dy, dv, dw) \leq 0 \quad \forall \psi_1, \psi_2, \psi_3 \in C_b((0,\infty)), g_1 \in \mathcal{E}_1, g_2 \in \mathcal{E}_2] . \end{aligned}$$

But $\varphi(0) = \inf_{\psi_1, \psi_2, \psi_3 \in C_b((0,\infty)), g_1, g_2 \in \mathcal{E}} [c_1(\psi_1) + c_2(\psi_2) + c_3(\psi_3) \mid F \leq \psi_1 + \psi_2 + \psi_3 + g_1 + g_2]$ as required.

E Proof of Proposition 5.3

Necessity. The first and third conditions in (11) are clearly necessary from Proposition 2.2, and the fourth condition in (11) just follows from the martingale property.

Now let $\mathbb{P} \in \mathbb{M}^+$ (defined at the start of section 4), and recall the definition of $G^a(.,.)$ from Proposition 2.2. Then we have

$$\mathbb{E}^{\mathbb{P}}(G^a(\underline{X}_{T_1,T_2}, Y_{T_1,T_2})1_A) \quad = \quad \mathbb{E}^{\mathbb{P}}(\mathbb{E}^{\mathbb{P}}(G^a(\underline{X}_{T_1,T_2}, Y_{T_1,T_2}) \,|\, \mathcal{F}_{T_1})\, 1_A) \quad \leq \quad 0$$

for all $A \in \mathcal{F}_{T_1}$ and all $a \in (0,1)$, where the inequality follows because the conditional law of $(\underline{X}_{T_1,T_2}, Y_{T_1,T_2})$ given $(\underline{X}_1, Y_1)$ must satisfy both Rogers conditions in (3). Setting $b = a$ and$A = \{(\underline{X}_{T_1}, Y_{T_1}) \in B\} \in \mathcal{F}_{T_1}$ for any $B \in \mathcal{B}(\mathcal{X}_{x_0})$, we see that the second condition in Eq (11) is also a necessary condition.

Sufficiency. From the first condition and Proposition 2.2 we can clearly construct a càdlàg martingale X for which $(\underline{X}_{T_1}, X_{T_1} - \underline{X}_{T_1})$ has the same law as $(\underline{X}, Y)$. Moreover, the second condition in (11) states that

$$0 \quad \geq \quad \mathbb{E}^{\mathbb{Q}}(G^b(V,W)1_{(\underline{X},Y)\in A}) \tag{D-1}$$

for all $A \in \mathcal{B}(\mathcal{X})$, but we can re-write the right hand here side as

$$\mathbb{E}^{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}}(G^b(V,W) \,|\, \sigma(\underline{X}, Y))\, 1_{(\underline{X},Y)\in A})$$

which implies that $\mathbb{E}^{\mathbb{Q}}(G^b(V,W) \,|\, \sigma(\underline{X}, Y)) \leq 0$ $\mathbb{Q}$-a.s. (using e.g. the lemma at the top of page 51 in [Will91]), i.e. the conditional law of (V,W) given $\underline{X}, Y$ satisfies the first condition in (3) with $x_0 = 1$ (and from the fourth condition in (11), we know that $\mathbb{E}(V + W \,|\, \underline{X}, Y) = 1$, so we can construct X so that $(\underline{X}_{T_1,T_2}, Y_{T_1,T_2})$ given $(\underline{X}_{T_1}, Y_{T_1})$ has the same law as the conditional law of (V,W) given $\underline{X}, Y$, as required.