

# Optimal trade execution with unknown drift

Martin Forde

31st Oct 2023

## Abstract

We show how existing results for optimal trading strategies under temporary/resilient price impact or proportional transaction costs can be easily adapted for the more realistic situation when the drift of the asset is unknown, so we have to project to the observable filtration generated by the asset price process, using results from non-linear filtering theory. In particular, we observe that a semimartingale with unknown (constant) drift is the continuation of a generalized bridge process when the true drift is replaced with its unbiased estimate over a fixed time window<sup>1</sup>.

## 1 Model setup

Many price impact articles (and textbooks on the continuous-time Kalman filter) consider a semimartingale price process with a drift process which is an OU process, but since the drift process is not directly observable, we cannot easily estimate its parameters, and even if the drift were observable, we can still e.g. only compute MLE or GMM estimates for its parameters which will typically have non-small sample variance unless the time window under consideration is large (i.e. years in practice) and the model is well specified over this large time window (which will seldom be the case in practice). One can use the Kalman filter combined with the E-M algorithm to do this<sup>2</sup> (for which there are in-built functions in Python for example), but from practical experience, we do not recommend since the sample variance of the estimate for the mean reversion speed of the OU process will be too large.

The alternate approach to this kind of problem (which we do not pursue here) is to use limit order book imbalance to predict mid-price moves (see e.g. [CDJ18], [CDO23], [PRS23] and references therein).

In this note, we consider a financial market living on a stochastic basis  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ , where the filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t_0 \leq t \leq T}$  satisfies the usual conditions.  $\mathbb{P}$  is the objective probability measure, and we assume the basis carries a one-dimensional  $\mathbb{P}$ -Brownian motion  $W$ . We consider a financial market with a single asset with  $\mathbb{P}$ -dynamics

$$P_t = P_0 + \mu t + \sigma W_t \tag{1}$$

and  $\mathcal{F}^P$  will denote the filtration generated by  $P$  (augmented by  $\mathbb{P}$ -null sets). We assume that  $\sigma$  is known and  $\mu$  is unknown to a financial agent and that  $P$  has been observed continuously since  $t_0 < 0$ . The assumption that  $\sigma$  is known is natural since it can be computed from the observed quadratic variation of  $P$  over any subset of  $[t_0, 0]$  which can be estimated from the realized variance of  $P$  (see e.g. Ait-Sahalia&Jacod[AJ14] for details and convergence results in this vein).

We specify an initial distribution  $f(\mu)$  for  $\mu$  at  $t = 0$ , and assume that  $W$  is independent of  $\mu$ . The natural/canonical choice here (which we will henceforth assume unless stated otherwise) is that  $\mu \sim N(\frac{P_0 - P_{t_0}}{0 - t_0}, \frac{\sigma^2}{|t_0|})$ , since this is what we obtain when applying the usual confidence interval approach to estimate  $\mu$  using that  $P_0 - P_{t_0} \sim N(\mu t, \sigma^2 |t_0|)$ , or from a Bayesian standpoint,  $f(\mu)$  is the posterior  $f(\mu|P_0)$  for  $\mu$  using Bayes' theorem if the initial (prior) distribution for  $\mu$  at  $t_0$  is  $U([-n, n])$ , and we then let  $n \rightarrow \infty$  (i.e. we have a flat prior for  $\mu$  at  $t_0$ , see Appendix A for details).

---

<sup>1</sup>We thank Faycal Drissi and Leandro Sanchez Betancourt for many interesting discussions

<sup>2</sup>We thank Leandro Sanchez Betancourt for pointing this out

From the final part of the Appendix, we know that  $\mathbb{E}(\mu|\mathcal{F}_t^P) = \frac{P_t - P_{t_0}}{t - t_0}$ . Hence the process  $\bar{W}_t$  defined by

$$P_t = P_0 + \int_0^t \frac{P_u - P_{t_0}}{u - t_0} du + \sigma \bar{W}_t \quad (2)$$

is an  $\mathcal{F}_t^P$ -Brownian motion, see e.g. Eq 4.5 in Björk et al.[BDL10] or Theorem 6.1 in [Chi] (or Eq 28 in Liptser&Shiryaev[LS04] with  $h_t = P_t$  and  $A_t = \mu$  so  $x_t = W_t$  and  $D_t = d\langle x_t, W_t \rangle = 1$  in their notation). From Ito's formula, we see that

$$dP_t = \frac{P_t - P_{t_0}}{t - t_0} dt + \sigma d\bar{W}_t = \hat{\mu}_t dt + \sigma d\bar{W}_t \quad (3)$$

where  $\hat{\mu}_t = \frac{P_t - P_{t_0}}{t - t_0}$ .

## 1.1 Basic properties of $P$ under $\mathcal{F}^P$

The SDE for  $P$  in (2) does not depend on  $\mu$  at all, and under  $\mathcal{F}^P$ ,  $P$  satisfies the same (linear) SDE as the continuation of a Brownian bridge process constrained to be at  $P_{t_0}$  at  $t = t_0$ , but here  $t \geq 0$  and  $t_0 < 0$ , and has the explicit solution:

$$P_t = P_0 \left(1 - \frac{t}{t_0}\right) + P_{t_0} \frac{t}{t_0} + (t_0 - t) \sigma \int_0^t \frac{d\bar{W}_s}{t_0 - s} \quad (4)$$

(see discussion below Eq 5.6.23 in [KS91]). From (1) we of course know that  $\mathbb{E}(P_t^2) < \infty$  hence  $P_t$  is finite a.s., so  $P$  cannot explode in finite time, despite the apparent mean-fleeing behaviour of  $P$  around  $P_{t_0}$  under  $\mathcal{F}^P$  in (3).

In particular,  $P$  is a Markov process with respect to  $\mathcal{F}^P$ , and we see that

$$\frac{d}{dt} \mathbb{E}(P_t | P_s) = \frac{\mathbb{E}(P_t | P_s) - P_{t_0}}{t - t_0}.$$

Solving this ODE we find that

$$\mathbb{E}(P_t | P_s) = P_{t_0} + \frac{P_s - P_{t_0}}{s - t_0} (t - t_0) = P_{t_0} + \hat{\mu}_s (t - t_0) \quad (5)$$

for  $t_0 \leq 0 \leq s \leq t$ . We also note that  $\hat{\mu}_t$  can be re-written as

$$\hat{\mu}_t = \frac{P_t - P_{t_0}}{t - t_0} = \frac{\mu(t - t_0) + \sigma(W_t - W_{t_0})}{t - t_0} = \mu + \frac{\sigma(W_t - W_{t_0})}{t - t_0} \sim N\left(\mu, \frac{\sigma^2}{t - t_0}\right)$$

so we can view  $\hat{\mu}_t$  as a noisy (but unbiased) estimate for  $\mu$  at time  $t$ . Moreover

$$d\hat{\mu}_t = \frac{dP_t}{t - t_0} - \frac{P_t - P_{t_0}}{(t - t_0)^2} dt = \frac{\mu dt + \sigma dW_t}{t - t_0} - \frac{P_t - P_{t_0}}{(t - t_0)^2} dt$$

so  $\hat{\mu}_t$  is not an  $\mathcal{F}_t$ -martingale, but using (3) we can also re-write the middle expression as

$$d\hat{\mu}_t = \frac{\hat{\mu}_t dt + \sigma d\bar{W}_t}{t - t_0} - \frac{P_t - P_{t_0}}{(t - t_0)^2} dt = \frac{\sigma}{t - t_0} d\bar{W}_t$$

so  $\hat{\mu}_t$  is an  $\mathcal{F}_t^P$ -martingale, as we would expect.

**Remark 1.1**  $\hat{\mu}_t$  changes sign infinitely many times over  $(0, \varepsilon]$  if  $t_0 = 0$ .

## 2 Application to price impact problems

### 2.1 Unconstrained problem

We can now apply many well known price impact methods/results to  $P$  but working under  $\mathcal{F}^P$  - e.g., for an agent subject to linear temporary price impact with no liquidation penalty where the price paid per share at time  $t$  is  $S_t = P_t + kv_t$  and  $v_t$  is the trading speed, using the same pointwise optimization argument with optional projection as in section 4.1 of [FSS22], we know that the optimal buying speed with no liquidation penalty is

$$v_t^* = \frac{\xi_t}{2k}$$

where

$$\xi_t = \mathbb{E}(P_T - P_t | \mathcal{F}_t^P) = \hat{\mu}_t (T - t)$$

and with a non-zero transaction cost of size  $\varepsilon$

$$v_t^* = \frac{1}{2k} (\xi_t - \varepsilon \operatorname{sgn}(\xi_t)) 1_{|\xi_t| \geq \varepsilon}.$$

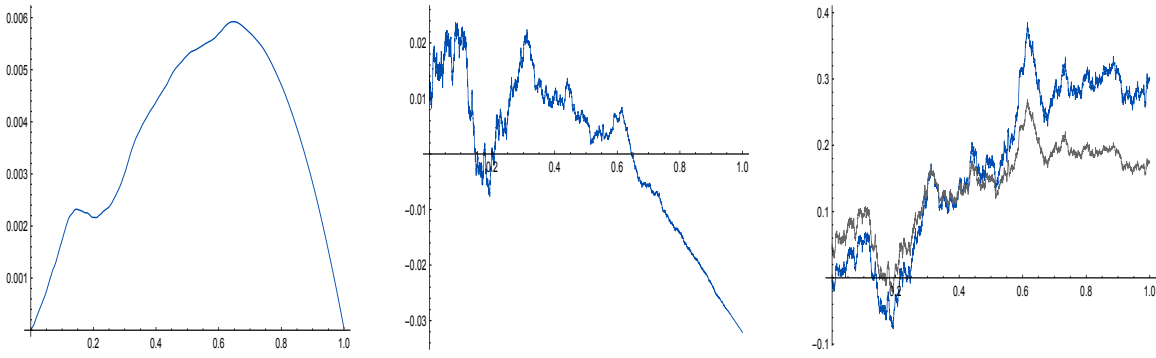


Figure 1: Here we have simulated the optimal stock holding  $X_t$  (left) for a round trip (i.e.  $X_0 = 0$ ) and the corresponding buying speed  $v_t = -u_t$  (middle) for the problem in [NV22] but with  $\mu$  unknown for parameters  $\mu = 0.05$ ,  $\sigma = .2$ ,  $\kappa = 1$ ,  $\rho = 1$ ,  $\lambda = 1$ ,  $\phi = 0$ ,  $\varrho = 1000$ ,  $Y_0 = 0$ ,  $P_0 = 1$  and  $t_0 = -1$  with  $P_{t_0} = P_0 - \mu|t_0| = .95$ . Since the stock went up over  $[t_0, 0]$  by assumption,  $\hat{\mu}_t$  is initially positive at  $t = 0$ , so the agent is initially buying at  $t = 0$  before the non-liquidation penalty really kicks in. On the right we have plotted the true  $P_t$  process (blue) and  $\hat{\mu}_t$  (in grey).

## 2.2 Temporary price impact and exponential resilience with liquidation and running inventory penalties

If an agent is subject to temporary price impact plus transient price impact under the propagator model with exponential resilience as in [NV22] with a running inventory penalty and finite liquidation penalty, then the standard variational and optional projection argument used to derive the main Theorem 3.2 in [NV22] still works under the filtration  $\mathcal{F}^P$ , so we just need to compute

$$\frac{1}{ds} \mathbb{E}(dA_s | \mathcal{F}_t^P) = \mathbb{E}(\hat{\mu}_s | \mathcal{F}_t^P) = \mathbb{E}\left(\frac{P_s - P_{t_0}}{s - t_0} | P_t\right) = \frac{P_{t_0} + \hat{\mu}_t(s - t_0) - P_{t_0}}{s - t_0} = \hat{\mu}_t = \frac{P_t - P_{t_0}}{t - t_0}$$

for  $s \geq t$  (where the third equality follows from (5) with  $s$  and  $t$  swapped round), and this expression is needed for Eq 3.6 in [NV22]. A similar (but simpler) formula (also just requiring  $\mathbb{E}(dA_s | \mathcal{F}_t^P)$ ) appears in Theorem 3.1 in [BMO20] for the case when there is no resilience, and the aforementioned formulae in [BMO20] and [NV22] both require computing a matrix exponential.

**Remark 2.1** For all the price impact problems considered, all that matters ultimately is  $\mathbb{E}(\mu_s | \mathcal{F}_t^P)$  so we can replace the Brownian motion  $W$  above with any sufficiently well behaved martingale  $M$ , and the choice of martingale does not affect the optimal trading strategy (unless we start using non-linear utility functions), and  $P$  will be a generalized bridge process under  $\mathcal{F}^P$ . In this case, since

$$P_0 - P_{t_0} = \mu(0 - t_0) + M_0 - M_{t_0}$$

so the natural choice of initial distribution for  $\mu$  now is the law of  $(P_0 - P_{t_0} - (M_0 - M_{t_0})) / (0 - t_0)$  with  $P_0$  and  $P_{t_0}$  taking their observed values.

## 3 The Merton problem with unknown drift

In this section, we remove the friction (i.e. the price impact) but we now allow the agent to be risk-averse by using a non-linear utility function. Assuming

$$dS_t = \mu dt + \sigma dW_t$$

for some unknown  $\mu$ , we consider the classical Merton problem with  $r = 0$ , and let  $\phi_t$  denote the agent's stock holding at time  $t$ , which we assume has to be  $\mathcal{F}_t^S$ -adapted. Then the total wealth of the agent  $X_t$  evolves as

$$dX_t = \phi_t dS_t$$

so the HJB equation for the value function  $V(S, x, t) = \sup_{\phi \in \mathcal{A}} \mathbb{E}_{S, X, t}(U(X_T))$  is

$$V_t + \frac{S - S_{t_0}}{t - t_0} V_S + \frac{1}{2} \sigma^2 V_{SS} + \sup_{\phi} [\phi \frac{S - S_{t_0}}{t - t_0} V_x + \frac{1}{2} \sigma^2 \phi^2 V_{xx} + \sigma^2 \phi V_{Sx}] = 0$$

and we can then solve for  $\phi^*$  in feedback form, and then re-write as a non-linear PDE. For the case when  $U(x) = -e^{-\alpha x}$ , using the ansatz  $V(S, x, t) = -e^{-\alpha(x + w(S, t))}$ , we find that

$$w_t + \frac{1}{2} \sigma^2 w_{SS} + \frac{1}{2} \frac{(S - S_{t_0})^2}{(t - t_0)^2 \alpha \sigma^2} = 0$$

for which the terminal condition is  $w(S, T) = 0$  if there is no liquidation penalty. This can be solved in closed-form (using Feynman-Kac) to give

$$w(S, t) = \frac{\frac{(T-t)(S^2 - 2SS_{t_0} + S_{t_0}^2 + (t_0-t)\sigma^2)}{(t-t_0)(T-t_0)\sigma^2} + \log \frac{T-t_0}{t-t_0}}{2\alpha}$$

and

$$\phi^*(S, t) = \frac{\hat{\mu}_t}{\alpha\sigma^2} = \frac{S_t - S_{t_0}}{\alpha\sigma^2(t - t_0)}. \quad (6)$$

This is of the same form as the solution  $\bar{\phi} = \frac{\mu}{\alpha\sigma^2}$  for the problem when  $\mu$  is known, but now  $\mu$  has been replaced with  $\hat{\mu}_t$ , and  $\phi^*(S, t) \rightarrow \bar{\phi}$  as  $t \rightarrow \infty$ . Even if the true  $\mu = 0$ , we see that it is optimal for the agent to trade with partial information about  $\mu$  (see simulation in Figure 2).

**Remark 3.1** For the case of log utility  $U(x) = \log x$  when  $S$  is a general semimartingale  $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$ , it is well known that  $\phi_t^* = \frac{\mu_t}{\sigma_t^2}$ , so in this case  $\phi^* = \frac{\hat{\mu}_t^*}{\sigma_t^2}$  (this is known as the growth optimal portfolio).

**Remark 3.2** In practice, one could argue that one should not start trading unless we have already rejected the null hypothesis that  $\mu = 0$ .

### 3.1 Adding small proportionate transaction costs

From (6) we see that

$$d\phi^*(S_t, t) = \frac{dS_t}{\alpha\sigma^2(t - t_0)}$$

so  $\frac{d\langle\phi^*\rangle_t}{d\langle S \rangle_t} = \frac{1}{\alpha^2\sigma^4(t-t_0)^2}$ , and (from the formal computations in section 2.1 in [KM17], or section 4.1 in [KL13]) the leading order term for the optimal trading strategy with proportional transaction costs of size  $\varepsilon \ll 1$  and fixed time horizon  $T > 0$  with exponential utility function as above is to engage in the minimal amount of trading to keep  $\phi_t$  within  $\bar{\phi} \pm \Delta\phi_t$ , where

$$\Delta\phi_t^* = \pm \left( \frac{3}{2\alpha} \frac{d\langle\phi^*\rangle_t}{d\langle S \rangle_t} S_t \right)^{\frac{1}{3}} \varepsilon^{\frac{1}{3}}$$

and we see that the no-trade region (NTR) shrinks when  $t$  goes large or when  $S_t$  goes small (see numerics below).

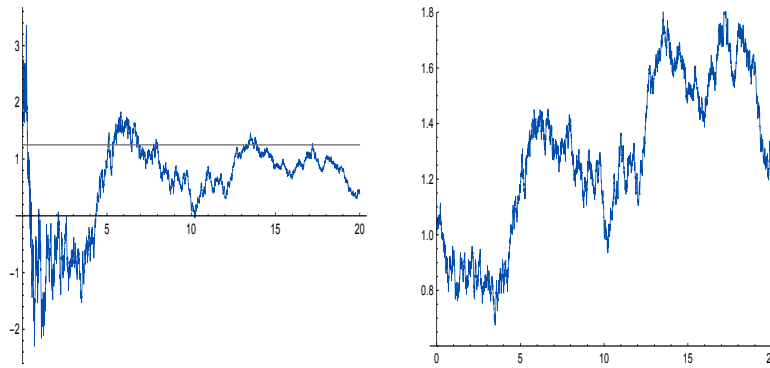


Figure 2: On the left we see a Monte Carlo simulation of the optimal stock holding  $\phi_t^*$  with  $\mu$  unknown (blue) and  $\mu$  known (grey), and the corresponding stock price process  $S_t$  (right plot) for the Merton problem with exp utility and unknown drift with  $t_0 = -1$ ,  $S_{t_0} = .95$ ,  $S_0 = 1$ ,  $T = 20$ ,  $\alpha = 1$ ,  $\sigma = .2$  and true  $\mu = 0.05$ . Since  $S_0 > S_{t_0}$  by assumption, the agent initially held a long position but went short as the stock price went down.

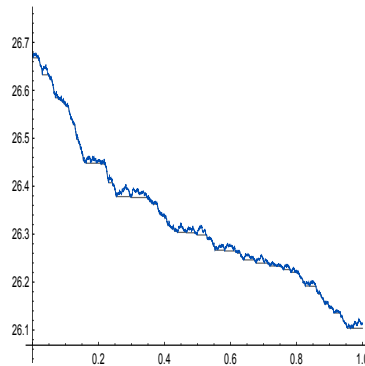


Figure 3: Here we have plotted the upper boundary for the No-Trade region (blue) with proportional transaction costs of size  $\varepsilon$ , and the optimal stock holding  $\phi_t^*$  (in grey) when the true drift is unknown, for  $t_0 = -1$ ;  $\mu = 0.05$ ,  $\sigma = .2$ ,  $\varepsilon = .005$ ;  $S_{t_0} = S_0 - \mu|t_0|$ ,  $S_0 = 1$ ;  $T = 1$ ,  $\alpha = 1$ , and  $\phi$  starting on the upper boundary. Note that a smaller investor needs to choose a larger  $\alpha$  value to ensure a smaller  $\bar{\phi}$  since we are working with exp utility.

## References

- [AJ14] Ait-Sahalia, Y. and J.Jacod, “High-Frequency Financial Econometrics”, Princeton University Press
- [AN22] Abi-Jaber, E. and Neuman E., “Optimal Liquidation with Signals: the General Propagator case”, preprint (2022)
- [CDO23] Cartea, A., F.Drissi and P.Osselin, “Bandits for Algorithmic Trading with Signals”, SSRN preprint, 2023
- [Chi] Chigansky, P., “Introduction to Nonlinear Filtering”, lecture notes
- [BMO20] Belak, C., J.Muhle-Karbe and K.Ou., “Liquidation in Target Zone Models, *Market Microstructure and Liquidity*, Vol. 4 (2020), No. 03, pp. 1950010
- [BDL10] Björk, T., M.H.A.Davis and C.Landén, “Optimal investment under partial information”, *Mathematical Methods of Operations Research*, volume 71, pages 371–399, 2010
- [FSS22] Forde, M., L.Sánchez-Betancourt and B.Smith “Optimal trade execution for Gaussian signals with power-law resilience”, *Quantitative Finance*, 22(3), 585-596, 2022
- [KS91] I.Karatzas and S.Shreve (1991), *Brownian motion and Stochastic Calculus*, Springer-Verlag.
- [KL13] Kallsen, J. and S.Li, “Portfolio Optimization under Small Transaction Costs: a Convex Duality Approach”, preprint, 2013.
- [KM17] Kallsen, J. and J.Muhle-Karbe, “The General Structure of Optimal Investment and Consumption with Small Transaction Costs”, *Mathematical Finance*, Vol. 27 (2017), No. 3, pp. 659-703
- [Kut04] Kutoyants Yu. A., “Statistical Inference for Ergodic Diffusion Processes”, Springer Series in Statistics, London, 2004, 496p.

- [LS04] Liptser, R and Shiriyayev, A., 2004. “Statistics of Random Processes”, 2nd Edition. Vol. I. Springer Verlag.
- [NV22] Neuman, E. and M.Voß, “Optimal Signal-Adaptive Trading with Temporary and Transient Price Impact”, *SIAM J. Financial Mathematics*, Vol. 13, No. 2, 551-575, 2022.
- [CDJ18] Cartea, A., R.Donnely and S.Jaimungal, “Enhancing trading strategies with order book signals”, *Applied Mathematical Finance*, 25:1, 2018
- [PRS23] Pulido, S., M.Rosenbaum and E.Sfendourakis “Understanding the least well-kept secret of high-frequency trading”, arxiv preprint, 2023
- [WW97] Whalley, A.E. and P.Wilmott, “An Asymptotic Analysis of an Optimal Hedging model for Option Pricing with transaction costs”, *Mathematical Finance*, Vol. 7, No. 3 (July 1997), 307–324

## A Computing $\mathbb{E}(\mu | (P_s)_{s \in [t_0, t]})$

If  $M_t = \mu t + \sigma W_t$ , then for a flat prior for  $\mu$  on  $\mathbb{R}$  at  $t_0$  (which is clearly an improper prior), using Bayes’ formula and Girsanov’s theorem, the posterior  $p(\mu | (P_s)_{s \in [t_0, t]})$  of  $\mu$  at  $t > 0$  (given  $(P_s)_{s \in [t_0, t]}$ ) is

$$\begin{aligned}
 p(\mu | (P_s)_{s \in [t_0, t]}) &\propto \text{Likelihood function of } (P_s)_{s \in [t_0, t]} \\
 &= \frac{1}{\sigma} e^{\int_{t_0}^t \gamma dP_s - \frac{1}{2} \int_{t_0}^t \gamma^2 ds} \mathbb{Q}_0 \left( \frac{d(P - P_0)}{\sigma} \right) \\
 &= \frac{1}{\sigma} e^{\frac{\mu}{\sigma} (P_t - P_{t_0}) - \frac{1}{2} (\frac{\mu}{\sigma})^2 (t - t_0)} \mathbb{Q}_0 \left( \frac{d(P - P_0)}{\sigma} \right) \\
 &= \text{const.} \times e^{-\frac{(\mu - \bar{\mu})^2}{2\sigma^2/(t - t_0)}} \mathbb{Q}_0 \left( \frac{d(P - P_0)}{\sigma} \right)
 \end{aligned}$$

where  $\mathbb{Q}_0$  denotes the Wiener measure on  $(C([t_0, t]), \mathcal{B}(C([t_0, t])), \mathbb{Q}_0)$ ,  $\gamma = \mu/\sigma$  and  $\bar{\mu} = \frac{P_t - P_{t_0}}{t - t_0}$ , so the posterior for  $\mu$  is  $N(\frac{P_t - P_{t_0}}{t - t_0}, \frac{\sigma^2}{t - t_0})$  as one would expect, so (formally at least)  $\mathbb{E}(\mu | (P_s)_{s \in [t_0, t]}) = \frac{P_t - P_{t_0}}{t - t_0}$ .

If we modify this analysis to instead use a prior at  $t = 0$  which is  $N(\frac{P_0 - P_{t_0}}{0 - t_0}, \sigma^2/|t_0|)$ , then we also find that  $\mathbb{E}(\mu | (P_s)_{s \in [0, t]}) = \frac{P_t - P_{t_0}}{t - t_0}$ .